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PAPERS

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ON RATIONAL SURFACES WHOSE PRIME SECTIONS ARE CANONICAL CURVES

By PATRICK DU VAL.

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1. *Generalities.*

It is familiar that the surfaces for which the arithmetical and geometrical genera and all plurigenera are alike equal to unity fall into a sequence of continuous families*, each depending on nineteen moduli, and related to the sequence of natural numbers $\pi = 2, 3, \ldots$, in such a way that the *general* surface of the family related to a number π (or, as we may say, whose characteristic genus is π) has on it no linear systems of curves save the various multiples of a particular one, whose grade, freedom, and genus are respectively $2\pi-2$, π, and π. The projective model of this system is a surface of order $2\pi-2$ in $[\pi]$ (where, as usual, $[r]$ denotes r-dimensional space), whose prime sections are canonical curves of genus π. On account, however, of the fundamental property of these surfaces†, that every linear system without base points or fundamental curves is self-adjoint, it follows that the projective model of every such system is a surface whose prime sections are canonical curves of their genus. The n-ple of the system of genus π has genus $n^2(\pi-1)+1$, and its projective model is thus a surface of the same order, and in the same space, as the standard model of the surface whose characteristic genus

* F. Severi, *Atti d. R. Ist. Veneto* (68), 2 (1909), 249.

† F. Enriques and L. Campedella, *Lezioni sulla Teoria delle Superficie Algebriche*, 1 (1930), sez. 51.

has this value; it is, however, an entirely different surface. Special surfaces (*i.e.* depending on fewer than nineteen moduli) have other systems of curves on them than that of the characteristic genus and its multiples, and are thus transformable into special cases of various of the families.

The first few families are: The double plane with sextic branch curve ($\pi = 2$); the quartic surface in [3] ($\pi = 3$); the intersection of a quadric and a cubic primal in [4] ($\pi = 4$); the intersection of three quadric primals in [5] ($\pi = 5$); and the section by a quadric primal of the quintic threefold locus in [6], which is the Grassmannian of the linear congruence of lines in [4] ($\pi = 6$)*. In [5], however, is another octavic surface, whose prime sections are likewise canonical (hyperelliptic) curves of genus 5, namely the double Veronese surface whose branch curve is 12-ic (the section of the Veronese surface by a cubic primal), this being clearly the projective model of the system double of the prime sections of the double plane with sextic branch curve; similarly in [9], as well as the family of characteristic genus 9, we have the quadric section of the "Veronese threefold" (projective model of the system of quadrics in [3]), which is clearly the projective model of the system double of the prime sections of the quartic surface in [3]; and in [10], as well as the family of characteristic genus 10, there is the quadric section of the cone projecting from a point the novenic surface of del Pezzo, this being the projective model of the system triple of the prime sections of the double plane with sextic branch curve; and so on.

The purpose of the present paper is to discover what special cases of these surfaces are rational, instead of being of genus 1. It is at once clear that the surface can only be rational in virtue of some singular point which is of such a nature as to impose a condition on the adjoint primals, which are thus reduced in number from 1 to 0. If there is one point of such character that it should virtually impose one condition it will actually impose it, since the conditions actually imposed by the singularities only fall short of the virtual number if some of them are dependent on the rest, which cannot happen where there is only one condition in question. On the other hand, if there are several such points, or a point which should virtually impose several conditions, at least one condition will actually be imposed, so that p is reduced from 1 to 0, while p_a becomes negative, and the surface is then equivalent to an irrational ruled surface. We conclude in general that

A surface whose prime sections are canonical curves is rational if and only if it has one and only one singular point of such a nature as to impose one virtual condition on the adjoint primals.

* P. Du Val, *Rend. d. R. Accad. d. Lincei* (6), 15 (1932), 346.

It is easy to find instances of such singular points. It is known*, for instance, that a conical point of genus p imposes p virtual conditions on the adjoint primals, so that an elliptic conical point is a point of the kind we seek. [By a conical point of genus p is intended a point whose first neighbourhood is a curve of genus p, containing no further singularity. By the first neighbourhood of a point is intended the curve which arises from that point when the surface is transformed by means of primals of any order having that point as a simple base point; the first neighbourhood of any point of the first neighbourhood is a second neighbourhood of the original point, and so on.] Among elliptic conical points is to be counted the ordinary tacnode, whose first neighbourhood is a double line with four pinch points.

We may extend the term unode to cover all singular points of which the first neighbourhood is a simple line, but itself contains some singular point. (Ordinarily the term is used only when the point is a double point, but from the present point of view this appears an artificial restriction.) By Enriques' principle of "unloading" (*scaricamento*)† it is clear that a unode, whose first neighbourhood contains just one point such as to impose one virtual condition on the adjoint primals, will itself impose one virtual condition on them. We have thus to add to the elliptical conical point the unode in whose first neighbourhood is an elliptic conical point, that in whose first neighbourhood is another unode having in turn an elliptic conical point in its first neighbourhood, and so on. (These singularities we shall call unodes elliptic in the second, third, etc. neighbourhood.)

2. *Two-sheeted surfaces.*

Since the double normal rational $(\pi-1)$-ic curve with $2\pi+2$ branch points is a canonical (hyperelliptic) curve of genus π, the double normal rational $(\pi-1)$-ic ruled surface with a $(2\pi+2)$-ic branch curve is a surface whose prime sections are canonical curves of genus π. Not every such surface, however, can be subsumed under our sequence of families of surfaces of genus 1; in fact, if the branch curve consists of $2\pi+2$ generators, the

* I do not find a reference for this, though it cannot be new. It is most easily seen by taking the surface to be in [3], when the point will be an s-ple point (say), through which pass $\frac{1}{2}(s-1)(s-2)-p$ branches of the double curve. The number of conditions necessary to make the adjoint surfaces pass $(s-2)$-ply through this point, over and above those necessary to make it pass through the double curve, is equal to the number of linearly independent forms the surface can have in the first neighbourhood of the point, containing the tangents to all the branches of the double curve, but only $(s-3)$-ple at the point; *i.e.* the number of ϕ-curves to the first neighbourhood of the point on the surface.

† F. Enriques and O. Chisini, *Teoria Geometrica delle Equazioni*, 2 (1918), 431.

surface is clearly a hyperelliptic ruled surface of genus π; if the branch curve meets each generator twice, the surface is rational and belongs to the known family of rational surfaces having hyperelliptic prime sections*, and having a pencil of conics (here the doubled generators of the ruled surface) which trace the canonical involution on each prime section [this surface is represented on a plane by the system of $(\pi+2)$-ic curves with a π-ple base point X say, and $2\pi+6$ simple base points which are special in being the points of contact of tangents from X to a $(\pi+4)$-ic curve with a $(\pi+2)$-ple point at X]. Finally, if the branch curve meets each generator four times, the surface is of genus 1, and such surfaces form a sub-family depending on eighteen moduli in the family of surfaces of genus 1 and characteristic genus π†. (The branch curve cannot meet each generator more than four times except in the case $\pi = 3$, when the ruled surface is a quadric, and if the branch curve, which is octavic, meets each generator of one system six times, then it meets each of the other system twice, and offers no new case.)

A common limiting case of the surface of genus 1 and one which is rational occurs when the ruled surface is a cone, and the branch curve passes four times through its vertex, and meets each generator twice further. The vertex is, in this case, clearly an elliptic conical point, since its first neighbourhood is a double normal rational curve (of order $\pi-1$) with four branch points. The system of curves representing the rational surface on the plane is now specialised so that a prime section of the surface can consist of $\pi-1$ generators of the ruled surface, which are represented by the lines through X; it is easily seen that this is achieved by making the $2\pi+6$ simple base points all lie on a cubic through X, which is then a fundamental curve of the system, and represents the vertex of the cone; this cubic with $\pi-1$ lines through X forms a $(\pi+2)$-ic passing in the required way through the base points.

The two-sheeted surface of genus 1 (in which the branch curve meets each generator four times) can, however, be made rational without making the ruled surface into a cone, simply by specialising the branch curve. For, on any two-sheeted surface, a fourfold point on the branch curve is clearly to be regarded as a tacnode of the surface, since its first neighbourhood is a double line with four branch points. Again, a point on the branch curve where three branches touch without osculating, or, as it is usually stated, where there are two consecutive triple points (what we may call a tritacnode of the curve), is to be regarded as a unode (of order 2, as

* G. Castelnuovo, *Rend. di Palermo*, 4 (1890), 73.

† F. Enriques, *Mem. d. Soc. Italiano*, 40 (3), 10 (1896), 201.

are all singular points of a two-sheeted surface) elliptic in its second neighbourhood; for the first neighbourhood consists of a line, simple since it forms part of the branch curve, and having a point in it through which pass three branches of the residual branch curve; this point, being fourfold for the total branch curve, is a tacnode of the surface.

The double plane with sextic branch curve is rational, as is well known*, if its branch curve has a fourfold point or a tritacnode, *i.e.* if the surface itself has a tacnode or quadratic unode elliptic in the second neighbourhood, and in no other case. The four rational quartic surfaces† have respectively a triple (elliptic conical) point, a tacnode, and in both the other cases quadratic unodes, which are elliptic respectively in the second and third neighbourhoods, since from these points the quartics project into the two double planes just named, whose singular points are elliptic respectively in the first and second neighbourhoods. The double quadric whose branch curve meets each generator of each system four times (and is accordingly a quartic section of the quadric) is clearly a limiting case of the quartic surface; the two cases of this in which the branch curve has a fourfold point and a tritacnode are rational, and are special cases of the quartic with tacnode and unode elliptic in the second neighbourhood respectively; the former surface is represented by the system of sextics with seven double and four simple base points, the four being two pairs in the Geiser involution determined by the seven; and the latter by septenics with one triple and nine double base points, of which the triple and eight of the double points are nine associated points, the remaining base point being the point of contact of a tangent from the triple point to a cubic through the nine associated points. The case in which the quadric is a cone, and the branch curve passes four times through its vertex, meeting each generator twice further, falls as we have seen under the heading of the double ruled quadric, branching along an octavic curve which meets each generator of one system twice and each of the other system six times, the surface being represented by quintics with a triple base point X and twelve simple base points, which are the points of contact of tangents from X to a quintic with a triple point at X, in this case further specialized in that all the base points lie on a cubic.

We shall, in general, find that the double ruled surface whose branch curve (meeting each generator four times) has a fourfold point falls under a

* M. Noether, *Sitzungsber. d. Phys.-Med. Soc. zu Erlangen*, 10 (1878), 81–86, and G. Castelnuovo and F. Enriques, *Rend. di Palermo*, 14 (1900), 290.

† M. Noether, *Math. Annalen*, 33 (1889), 546.

more general case of surface with a tacnode; that the double ruled surface whose branch curve (meeting each generator four times) has a tritacnode falls under a more general case of surface having a unode elliptic in the second neighbourhood; and that the double cone whose branch curve passes four times through the vertex, meeting each generator twice further, falls under the case of the double ruled surface whose branch curve meets each generator twice, merely by making all the base points lie on a cubic (which is then fundamental for the system, and represents the neighbourhood of the vertex).

3. *The principal theorem.*

To obtain our results in a more precise form, suppose that the surface Ξ is rational, and is represented on a plane by a linear system (ξ), and that the prime sections of Ξ are canonical curves, so that the characteristic series of (ξ) is its canonical series. Let the first adjoint system of (ξ) be (ϕ_1), so that, if m is the order of (ξ) and i_α its multiplicity at a base point A_α, the order and multiplicity of (ϕ_1) are $m-3$ and $i_\alpha-1$.

By saying that the characteristic series of (ξ) is its canonical series, we mean that through the intersections (other than the base points) of any two curves ξ, ξ' passes a curve ϕ_1, which does not meet either of them further save in the base points. Now two cases are possible: either the variable part of (ϕ_1) is irreducible, or it is composed with a pencil; the latter case leads to the conclusion that (ξ) is hyperelliptic, and that the pencil with which the variable part of (ϕ_1) is composed traces its canonical g_2^1, and gives the double ruled surface whose branch curve meets each generator twice. In the former case, however, let the fixed part of (ϕ_1), if any, be of order m' and pass with multiplicity i'_α through each A_α. Then, in the pencil determined by the curves ξ, ξ', there is one curve which contains as part the variable part of the ϕ_1 through the intersections of ξ, ξ', the residual part of this curve of the pencil being of order $m'+3$, and passing with multiplicity $i_\alpha'+1$ through each A_α, and having accordingly the fixed part of (ϕ_1) as its first adjoint system, so that it is elliptic. [If the fixed part of (ϕ_1) vanishes this curve is a cubic passing simply through all the base points.] In any case, this curve is easily seen to be fundamental for (ξ), so that it represents a single point on Ξ, prime sections through which constitute the first adjoint system to the prime sections in general, *i.e.* it represents the singular point which reduces the genus.

We now proceed to prove that this curve can in all cases be transformed (by a Cremona transformation of the whole plane) into a cubic, and that

the transform of the system (ξ) has then all its base points on this cubic, which is fundamental for it, and that the transform of the system (ϕ_1), which is first adjoint to the transform of (ξ), has no fixed part.

Todd* has shown that the fixed part (if any) of the first adjoint system of a linear system of plane curves (other than a pencil of elliptic curves) consists of one or more parts, each of which is (a) rational, and (b) just determined by the conditions imposed on it by its multiplicities at the base points, which are all independent. This is accordingly true of the fixed part of (ϕ_1), σ say. Let $\sigma_1, \sigma_2, \ldots$ be the irreducible parts of this curve, and let the order and multiplicity at A_α of σ_n be $m_n', i'_{\alpha n}$. Then the conditions (a), (b) above are expressed by two equations whose sum and difference are

$$\sum_\alpha (i'^2_{\alpha n}) = m_n'^2 + 1,$$

$$\sum_\alpha (i'_{\alpha n}) = 3m_n' - 1,$$

analogous to the familiar equations of condition for a homaloidal net†. From these equations, exactly as in the case of those for a homaloidal net, can be derived Noether's inequality‡, showing that the order of the curve σ_n can be reduced by a quadratic transformation; the transformed curve will still be rational, and will still be completely determined by its multiplicities at the base points (which are the transforms of the original base points, less those which are base points of the transformation, plus those arising from the fundamental lines of the transformation), and can thus be reduced in order again, and so on till the curve is transformed, by a product of quadratic transformations of the whole plane, to the neighbourhood of a base point. (The case of coincident base points can be dealt with by the method of Chisini§, exactly as for homaloidal nets.) Each curve σ_n is thus deprived not only of first but of r-th adjoint curves||. It follows at once, however, that this is also true of the whole curve σ, since, by an obvious repetition of Noether's "$Af+B\phi$" theorem, the r-th adjoint curves of σ are linearly dependent on those of them which break up into an r_1-th adjoint of σ_1, and r_2-th adjoint of σ_2, ..., where $r_1+r_2+\ldots = r$, and since one at least of the numbers r_n must be greater than 0 for $r > 0$, one component at least of the composite curve is impossible. But the r-th adjoint system of σ is the $(r+1)$-th of the elliptic curve fundamental for (ξ); hence this curve is without adjoint systems beyond

* J. A. Todd, *Proc. Camb. Phil. Soc.*, 27 (1931), 298.

† H. P. Hudson, *Cremona transformations* (1927), 5.

‡ *Ibid.*, 9, and M. Noether. *Math. Annalen*, 5 (1872), 635. *Cf.* my subsequent paper, p. 26 *infra.*

§ Enriques and Chisini, *op. cit.*, 3 (1924), 168.

|| Enriques and Chisini, *op. cit.*, 3 (1924), 188.

the first (which is σ), and this is precisely the condition that it may be transformable into a cubic by a Cremona transformation of the whole plane*. This elliptic curve is, however, the difference between (ξ) and the variable part of (ϕ_1); the cubic into which it transforms is consequently the difference between the transforms of these systems, and, since (ϕ_1) as a whole transforms into the first adjoint system to the transform of (ξ), the cubic passes simply at least through every base point of the transform of (ξ). Consequently, if the transform of (ϕ_1) still has any fixed parts (due to fundamental curves in the transformation), the quadratic transformations by which these may be reduced in order and ultimately eliminated have all their base points on the cubic, and accordingly leave it a cubic.

We thus arrive ultimately at a system equivalent under a Cremona transformation to (ξ), and having all its base points on a cubic J, which is fundamental for the system (*i.e.* has no variable intersection with the variable curve of the system); such, moreover, that its first adjoint system is without fixed part, and is equivalent under the same transformation to the variable part of (ϕ_1). We shall, in future, assume throughout that (ξ) is given in this form in the first instance, *i.e.* that its base points lie on a cubic, fundamental for it, and that its first adjoint system (ϕ_1) is free from fixed parts.

The nature of the multiple point on Ξ represented by the cubic J can be determined by means of the following two properties of any linear system whose base points all lie on a cubic, which summarise obvious identities between the orders and multiplicities at the base points of the curves concerned:

The genus of the first adjoint system falls short of that of the original system by the order of the series traced by the former on the cubic; and

The order of the series traced by the first adjoint system on the cubic exceeds or falls short of that of the series traced by the original system, by as much as the number of base points of the latter exceeds or falls short of 9.

Since the series traced by (ξ) on J is of order 0, and, since (ϕ_1) must meet J in at least one point not virtually absorbed in the base points, it follows that the number of base points is

$$\epsilon \geqslant 10.$$

Suppose first that $\epsilon \geqslant 11$; then (ϕ_1) traces on J a series of order $\epsilon - 9$ and freedom $\epsilon - 10$, so that on the surface Φ_1, the projective model of (ϕ_1), and

* Enriques and Chisini, *op. cit.*, 3 (1924), 190.

the projection of Ξ from its singular point, J represents a normal elliptic curve of order $\epsilon-9$; this is the first neighbourhood of the singular point of Ξ, which is accordingly an elliptic conical node of order $\epsilon-9$. The surface Φ_1 in this case is easily seen to be of order $2\pi-\epsilon+7$, in $[\pi-1]$, its prime sections being normal and non-special curves of genus $\pi-\epsilon+9$.

If, however, $\epsilon = 10$, then the curves (ϕ_1) meet J in only one point not virtually absorbed in the base points; this point is accordingly not variable but is a base point of (ϕ_1), without being one of (ξ). In this case, J is fundamental also for (ϕ_1), whose characteristic series is thus canonical, so that Φ_1 is a surface of the same kind as Ξ but of characteristic genus $\pi-1$ instead of π, *i.e.* its prime sections are canonical curves of genus $\pi-1$. The first neighbourhood of the singular point of Ξ is mapped not on J but on the first neighbourhood of the additional base point of (ϕ_1), and is thus a simple line, containing a singular point, namely the point of Φ_1 represented by the fundamental curve J. The singular point of Ξ is thus a unode, which is of order 2, since the orders of Ξ and Φ_1, the grades of (ξ) and (ϕ_1), are $2\pi-2$ and $2\pi-4$ respectively.

The further analysis of the singularity in this case merely involves the consideration of the number of base points of (ϕ_1), which are all those of (ξ) which are not simple, and one simple one in addition. It is clearly impossible for more than two of the base points of (ξ) to be simple, since if it were so (ϕ_1) would have only one further intersection with each of the cubics, ∞^2 at least, through the remaining base points. If two of the base points of (ξ) are simple, (ϕ_1) has only one further intersection with each cubic of the pencil through the eight remaining points; this further intersection is evidently at the ninth base point of the pencil, and the system (ϕ_1) can only be the pencil itself. The system (ξ) is thus that of sextics having eight double and two simple base points on a cubic, which represents the double plane whose sextic branch curve has a tritacnode.

If (ξ) has ten base points, of which just one is simple, the remaining nine are base points of (ϕ_1), which has, in addition, one simple base point not belonging to (ξ); *i.e.* (ϕ_1) has ten base points, of which at least one [just one, unless (ϕ_1) is the system just considered] is simple. Thus (ϕ_2), the second adjoint of (ξ), the first adjoint of (ϕ_1), has ten base points, of which at least one is simple, and so on till we arrive by successive adjunction at a system with more than one simple base point, and this we have seen can only be the system of sextics with eight double and two simple base points on a cubic; the reverse process, of adding the curve J to the system, neglecting the simple base point, can be repeated indefinitely. We thus obtain an endless series of surfaces, one for each value of the prime section genus π, each of which is the projection of that above it from its singular

point, the lowest member of the series being that for $\pi = 2$, the double plane branching along a sextic with a tritacnode. (That for $\pi = 3$ is one of the rational quartics mentioned above.) Clearly, since the unode on each surface is in the first neighbourhood of that on the next, and that for $\pi = 2$ has a unode elliptic in the second neighbourhood, the member of the series for any π has a unode elliptic in the π-th neighbourhood.

If, on the other hand, (ξ) has ten base points, none of which is simple, (ϕ_1) has eleven, Φ_1 has a tacnode, and the unode of Ξ is elliptic in the second neighbourhood.

We have seen that, if (ϕ_1) is composed with a pencil, so that the curves of (ξ) are hyperelliptic, it is not necessary that the base points should all lie on a cubic; there is, however, no reason why they should not do so, and the cubic is then fundamental for (ξ). This, of course, gives the case of the double rational cone, whose branch curve passes four times through the vertex and meets each generator twice further. The elliptic conical point is then non-normal, and of order $2\pi-2$ (that of the surface); in no other case, however, can an elliptic conical point be of order greater than π, since, if its order is $\mu = 9-\epsilon$, the prime sections of Φ_1 are irreducible curves of genus $\pi-\mu$. Hence

A rational surface whose prime sections are canonical curves of genus π is either

A double normal rational ruled surface of order $\pi-1$, branching along a curve of order $2\pi+2$ which meets each generator twice; or

Endowed with an elliptical conical point of order μ, $2 \leqslant \mu \leqslant \pi$; or

Endowed with a unode of order 2, elliptic in either the second or the π-th neighbourhood.

4. *Special cases.*

The following table gives the birationally distinct cases which arise for the first few values of π. It is clear that these can be completely (if somewhat laboriously) enumerated up to any required value of π, since a linear system whose base points are on a cubic has a first adjoint system of lower genus than its own, and the systems whose base points are on a cubic can thus be enumerated genus by genus, arguing to each system from its first adjoint, which must always be amongst those already enumerated.

The first column of the table gives the value of π. The second gives the classification of the surfaces under the theorem just enunciated, H indicating the surfaces whose prime sections are *hyperelliptic* canonical curves (*i.e.* the double ruled surfaces), E^n those which have elliptic

conical nodes of order n, and U_n those which have quadratic unodes, elliptic in the n-th neighbourhood. The third column indicates the order of the curves and the fourth the multiplicities of the base points of the simplest linear system which represents the surface on a plane, $r(i)$ indicating r i-ple base points.

$\pi = 2$	$H = E^2$	4	$1(2)+10(1)$
	U_2	6	$8(2)+2(1)$
$\pi = 3$	H	5	$1(3)+12(1)$
	E^3	4	$12(1)$
	E^2	6	$7(2)+4(1)$
	U_2	7	$1(3)+9(2)$
	U_3	9	$8(3)+1(2)+1(1)$
$\pi = 4$	H	6	$1(4)+14(1)$
	E^4	5	$2(2)+11(1)$
	E^3	6	$6(2)+6(1)$
	E^2 (i)	7	$1(3)+8(2)+2(1)$
	E^2 (ii)	9	$8(3)+3(1)$
	U_2	9	$7(3)+3(2)$
	U_4	12	$8(4)+1(3)+1(1)$
$\pi = 5$	H	7	$1(5)+16(1)$
	E^5	5	$1(2)+13(1)$
	E^4	6	$5(2)+8(1)$
	E^3	7	$1(3)+7(2)+4(1)$
	E^2 (i)	9	$7(3)+2(2)+2(1)$
	E^2 (ii)	7	$10(2)+1(1)$
	E^2 (iii)	8	$1(4)+10(2)$
	U_2 (i)	10	$1(4)+8(3)+1(2)$
	U_2 (ii)	12	$8(4)+2(2)$
	U_5	15	$8(5)+1(4)+1(1)$

$\pi = 6$	H	8	1(6)+18(1)
	E^6 (*i*)	6	1(3)+1(2)+13(1)
	E^6 (*ii*)	5	15(1)
	E^5	6	4(2)+10(1)
	E^4	7	1(3)+6(2)+6(1)
	E^3 (*i*)	7	9(2)+3(1)
	E^3 (*ii*)	9	7(3)+1(2)+4(1)
	E^3 (*iii*)	8	1(4)+9(2)+2(1)
	E^2 (*i*)	10	1(4)+8(3)+2(1)
	E^2 (*ii*)	12	8(4)+1(2)+2(1)
	E^2 (*iii*)	9	6(3)+4(2)+1(1)
	E^2 (*iv*)	8	2(3)+9(2)
	U_2 (*i*)	12	7(4)+2(3)+1(2)
	U_2 (*ii*)	10	10(3)
	U_6	18	8(6)+1(5)+1(1)

It may be noted that in each case the surface E^π whose elliptic conical point is of the highest order projects from this point into the normal rational ruled surface of order $\pi-2$ [except in the case of $\pi = 6$, E^6 (*ii*)], and accordingly is one of the surfaces lying on rational scrolls of planes to which I have called attention elsewhere*; the speciality in this case consists in the fact that the scroll of planes is a cone, and that the cubic primal which cuts out the surface on the scroll (residually to $\pi-4$ of its planes) has a double point at the vertex of the cone. Similarly, the surface $\pi = 6$, E^6 (*ii*) lies on the cone projecting a Veronese surface from a point, since the projection of the surface from its node is a Veronese surface; it is traced on this cone (residually to a quadric cone) by a cubic primal which, in this case also, has a double point at the vertex of the cone. In all other cases (so long as $\pi > 4$, and apart from the case H) the surface is the complete intersection of all the quadrics containing it.

A further point of note is that, since the surfaces whose conical nodes are of order $\pi-p$ project from these points into rational surfaces whose prime sections are normal and non-special curves of genus p, and since the order of

* P. Du Val, *Rend. d. R. Accad. d. Lincei* (6), 15 (1932), 346.

a rational surface whose prime sections have genus 1, 2, 3, 4 (and probably any value of $p > 1$) has an upper bound, so has the value of π for which surfaces with $E^{\pi-1}$, $E^{\pi-2}$, $E^{\pi-3}$, $E^{\pi-4}$, ... occur. Surfaces with $E^{\pi-1}$, for instance, occur only for $\pi \leqslant 10$, since each gives rise to the del Pezzo surface of order $\pi-1$, and the order of these surfaces does not exceed nine.

A more important remark is that, when $\pi-1$ has a square factor n^2, some of the surfaces for this value will be represented by systems of plane curves in which the order and all multiplicities at the base points have n as factor. For in $[\pi]$, as well as the surface of characteristic genus π, we have another, projective model of the n-ic sections of that of characteristic genus $(\pi-1)/n^2+1$, and every special case of this latter surface, in particular the cases which are rational, will lead to a corresponding special case of the surface in $[\pi]$. If, for instance, one of the rational surfaces for the characteristic genus $(\pi-1)/n^2+1$ is represented by a linear system of m-ic curves having an i_α-ple base point at A_α, then the system of nm-ics having an ni_α-ple base point at A_α represents the surface in $[\pi]$, derived from this. Now, if u_α is the elliptic parameter on J of the base point A_α of this system, we have clearly the congruence

$$\Sigma (i_\alpha u_\alpha) \equiv 0 \text{ (mod periods)}$$

as the condition that the system may exist. If, however, we choose a new set of base points B_α, such that their elliptic parameters v_α satisfy $n\Sigma (i_\alpha v_\alpha) \equiv 0$, but $\Sigma (i_\alpha v_\alpha) \not\equiv 0$ (mod periods), these are the base points of a system of curves, of the same order and multiplicities nm and ni_α, which is not the n-ple of any other system, and which consequently represents a surface in $[\pi]$ belonging as special case to the family of characteristic genus π.

The above table shows instances of this for $\pi = 5$, the surfaces of characteristic genus π being the complete intersection of three quadrics, while the double Veronese surface with 12-ic branch curve represents the quadric sections of the surface of characteristic genus 2. The cases E^2 (*iii*) and U_2 (*ii*) are represented by plane systems in which the order and all multiplicities are even, and hence the choice of the base points on J may be such that the system is double of that representing the surface E^2 or $U_2 (\pi = 2)$, in which case we have the double Veronese surface with fourfold point or tritacnode on its branch curve; or else such that the system is not double of any other, in which case it represents a proper intersection of three quadrics.

PLANE PARTITIONS

By T. W. CHAUNDY.

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MacMahon*, by imposing on a partition the condition of arrangement of its parts in monotonic order, was able to define many-rowed partitions, or "plane" partitions, which are not mere obvious combinations of ordinary, *i.e.* linear or one-rowed, partitions. Thus

$$\begin{array}{lll} 2\ 1\ 1, & 2\ 1, & 2, \\ & 1 & 1 \\ & & 1 \end{array}$$

in which both rows and columns run in non-ascending order, are three distinct plane partitions of the number 4. By his algebra of "lattice functions", MacMahon was able to obtain generating functions for plane partitions of various types, culminating in the notable formula†

$$\prod_{r=1}^{\infty} (1-x^r)^{-r} \tag{1}$$

for the generating function of unrestricted plane partitions.

The analogy of this formula with Euler's formula

$$\prod_{r=1}^{\infty} (1-x^r)^{-1} \tag{2}$$

for the generating function of unrestricted linear partitions has naturally encouraged the hope that some direct or intuitive proof of (1) might be found similar to the intuitive proof of (2), but so far this hope has been

* MacMahon, *Combinatory analysis*, 2 (1916).

† MacMahon, *op. cit.*, 2 (1916), 243, §495, the last formula.

vain*. I have recently† sketched an algebraic treatment of plane partitions, based on the recurrence-formulae satisfied by their generating functions, which seems simpler than MacMahon's method of "lattice functions". In this Note I describe a more "intuitive" procedure based on the partitions themselves rather than on the algebra of their generating functions.

1. It is convenient to widen a little the condition of monotonic arrangement of the columns‡, while retaining the monotonic property of the rows. Thus, if

$$\begin{matrix} r_1 & r_2 & r_3 & \dots \\ s_1 & s_2 & s_3 & \dots \\ t_1 & t_2 & t_3 & \dots \\ \dots & \dots & \dots & \end{matrix}$$

is a plane partition of k rows, I prescribe arbitrarily $k-1$ positive§ integers $a_{12}, a_{23}, \dots, a_{k-1,k}$. Then the condition of monotonic ordering of the columns is replaced by the conditions

$$r_p > s_p - a_{12} > t_p - a_{12} - a_{23} > \dots \quad (p = 1, 2, 3, \dots), \tag{3}$$

together with

$$s_p - a_{12} \geqslant 0, \quad t_p - a_{23} \geqslant 0, \quad \dots \quad (p = 1, 2, 3, \dots), \tag{4}$$

since the partition is into positive parts. I shall call the plane partition *proper*, if the conditions (3) are satisfied between every pair of rows. The simpler case of non-ascending order in the columns corresponds, of course, to the case in which $a_{12}, a_{23}, \dots$ are all unity.

Again, I prescribe that the parts in the r-th row do not exceed i_r in magnitude or j_r in number, where the integers i_r satisfy the inequalities

$$i_1 \geqslant i_2 - a_{12} \geqslant i_3 - a_{12} - a_{23} \geqslant \dots. \tag{5}$$

* See, however, my later paper, *Quart. J. of Math.* (Oxford), 3 (1932), 76–80.

† *Quart. J. of Math.* (Oxford), 2 (1931), 234–40.

‡ We might similarly widen the monotony of the rows, but this complicates rather than simplifies the discussion.

§ I exclude negative integers and zero not of necessity, but for simplicity.

The plane partition so defined, together with its generating function, I shall represent by the symbol

$$\begin{bmatrix} i_1 & i_2 & i_3 & \dots \\ & a_{12} & a_{23} & \dots \\ j_1 & j_2 & j_3 & \dots \end{bmatrix}.$$

The linear partition of j parts, none of which exceeds i, I shall denote, together with its generating function, by the symbol (i, j).

2. I begin by considering the two-rowed partition

$$\begin{bmatrix} i_1 & i_2 \\ & a \\ j_1 & j_2 \end{bmatrix}.$$

Write down, first of all, any pair of partitions (i_1, j_1), (i_2, j_2), thus

$$(6) \qquad \begin{matrix} (i_1) & r_1 & r_2 & r_3 & \dots & (j_1) \\ (i_2) & s_1 & s_2 & s_3 & \dots & (j_2), \end{matrix}$$

where the symbols in brackets indicate the greatest value of a part and the number of parts respectively. If this two-rowed partition is not proper, suppose that the last "fault" is in the m-th column, *i.e.* that

$$(7) \qquad \begin{cases} s_m - a \geqslant r_m, \\ r_p > s_p - a \quad (p > m), \end{cases}$$

while no assertion is made about columns preceding the m-th. Taking $m = 3$, we may represent this in the form

$$(8) \qquad \begin{matrix} (i_1) & r_1 & r_2 & r_3 & r_4 & \dots & (j_1) \\ & & & \wedge & \vee & & \\ (i_2) & s_1 & s_2 & s_3 & s_4 & \dots & (j_2). \end{matrix}$$

Out of this partition we can construct the partition

$$(9) \qquad \begin{matrix} (i_2 - a) & s_1 - a & s_2 - a & s_3 - a & r_3 & r_4 & \dots & (j_1 + 1) \\ & & & & \wedge & \vee & & \\ (i_1 + a) & & & r_1 + a & r_2 + a & s_4 & \dots & (j_2 - 1). \end{matrix}$$

If the last "fault" in (8) is in the m-th column, then the last "fault" in (9) is in the $(m-1)$-th column reckoned from the bottom row: if $m=1$, then (9) is "proper". Moreover, out of any partition (9) a partition (8) can be constructed*. There is thus an exact (1, 1) correspondence of "improper" partitions (8) with *all* partitions (9) whether proper or not, *i.e.* with every pairing of the two linear partitions

$$(i_2-a,\ j_1+1),\quad (i_1-a,\ j_2-1)$$

Now, if (8) is a partition of n, then (9) is a partition of $n-a$. The generating function of "improper" partitions (8) is thus

$$x^a(i_2-a,\ j_1+1)\,.\,(i_1+a,\ j_2-1).$$

Subtracting this from the generating function of all partitions (8), we get at once

$$(10)\qquad \begin{bmatrix} i_1 & & i_2 \\ & a & \\ j_1 & & j_2 \end{bmatrix} = (i_1,\ j_1)\,.\,(i_2,\ j_2)-x^a(i_2-a,\ j_1+1)\,.\,(i_1+a,\ j_2-1)$$

$$= \begin{vmatrix} (i_1,\ j_1) & (i_1+a,\ j_2-1) \\ x^a(i_2-a,\ j_1+1) & (i_2,\ j_2) \end{vmatrix}.$$

For partitions in more than two rows the procedure must be developed somewhat; the details are sufficiently shown by taking the three-rowed partition

$$\begin{bmatrix} i_1 & & i_2 & & i_3 \\ & a & & b & \\ j_1 & & j_2 & & j_3 \end{bmatrix}.$$

We begin by writing down any linear partition (i_1, j_1) above a proper two-rowed partition

$$\begin{bmatrix} i_2 & & i_3 \\ & b & \\ j_2 & & j_3 \end{bmatrix},$$

* In the case $m=1$ we need the conditions (5) to ensure that $r_1 \leqslant i_1$, etc., in (8).

thus:

$$(11)\qquad \begin{cases} i_1) & r_1 & r_2 & r_3 & \dots & (j_1) \\ i_2) & \lceil s_1 & s_2 & s_3 & \dots & (j_2) \\ i_3) & \lfloor t_1 & t_2 & t_3 & \dots & (j_3), \end{cases}$$

where the bracket indicates that the pair of rows so connected is "proper". Now (11) considered as a three-rowed partition may turn out to be either proper or improper. If, for the moment, we write $[I]$ for the generating function of the improper three-rowed partitions, we have the equation between generating functions

$$(12)\qquad (i_1, j_1)\begin{bmatrix} i_2 & i_3 \\ & b \\ j_2 & j_3 \end{bmatrix} = \begin{bmatrix} i_1 & i_2 & i_3 \\ & a & b \\ j_1 & j_2 & j_3 \end{bmatrix} + [I].$$

We consider the improper partitions $[I]$. The faults in (11) must lie between rows 1, 2. Suppose that the last fault is as shown in (13):

$$(13)\qquad \begin{cases} i_1) & r_1 & r_2 & r_3 & \dots & (j_1) \\ & & \wedge & \vee & & \\ i_2) & \lceil s_1 & s_2 & s_3 & \dots & (j_2) \\ i_3) & \lfloor t_1 & t_2 & t_3 & \dots & (j_3). \end{cases}$$

Then we can construct the partition

$$(14)\qquad \begin{cases} i_2-a) & s_1-a & s_2-a & r_2 & r_3 & \dots & (j_1+1) \\ & & & \wedge & \vee & & \\ i_1+a) & & & r_1+a & s_3 & \dots & (j_2-1) \\ & & & & \vee & & \\ i_3) & & t_1 & t_2 & t_3 & \dots & (j_3). \end{cases}$$

Here the slant bracket indicates that rows 1, 3 are proper with the convention that they be shifted relatively one place clockwise, for we have

$$s_1-a > t_1-a-b, \quad s_2-a > t_2-a-b, \quad r_2 > t_3-a-b, \quad \dots;$$

the third of these follows, since, in (13),

$$r_2 \geqslant r_3 > s_3-a > t_3-a-b.$$

Now in (14) row 2 is presumably improper as regards both of the rows 1, 3, and we see that the last fault (if any) between rows 2, 3 lies

below or to the left of the last fault between rows 1, 2. If we add to (14) the similar partitions

$$(15)\qquad \begin{Bmatrix} i_2-a) \\ i_1+a) \\ i_3) \end{Bmatrix} \begin{matrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 & \dots & (j_1+1) \\ & & & \vee & & & \\ & & \sigma_2 & \sigma_3 & \sigma_4 & \dots & (j_2-1) \\ & & & \wedge & \vee & & \\ \tau_1 & \tau_2 & \tau_3 & \tau_4 & & \dots & (j_3), \end{matrix}$$

in which the last fault between rows 2, 3 is *to the right* of the last fault between rows 1, 2, then we get the complete combination of a linear partition (i_1+a, j_2-1) with a proper two-rowed partition

$$\begin{bmatrix} i_2-a & & j_1+1 \\ & a+b & \\ i_3 & & j_3 \end{bmatrix}.$$

Now (15) can be correlated with the partition

$$(16)\qquad \begin{Bmatrix} i_2-a) \\ i_3-b) \\ i_1+a+b) \end{Bmatrix} \begin{matrix} \rho_1 & \rho_2 & \rho_3 & \rho_4 & \dots & (j_1+1) \\ \tau_1-b & \tau_2-b & \tau_3-b & \sigma_3 & \dots & (j_2+1) \\ & & & \wedge & & \\ & & & \sigma_2+b & \dots & (j_3-2), \end{matrix}$$

in which rows 1, 2 are now proper, and the last row is independent.

If (13) is a partition of n, then (14) and so (15) are partitions of $n-a$, and therefore (16) is a partition of $n-a-2b$. Thus (16), and therefore (15), has the generating function

$$x^{a+2b}(i_1+a+b, j_3-2) \begin{bmatrix} i_2-a & & i_3-b \\ & a & \\ j_1+1 & & j_2+1 \end{bmatrix},$$

while (14) and (15) between them have the generating function

$$x^a(i_1+a, j_2-1) \begin{bmatrix} i_2-a & & i_3 \\ & a+b & \\ j_1+1 & & j_3 \end{bmatrix}.$$

Thus $[I]$, the generating function of (14), is

$$x^a(i_1+a,\ j_2-1)\begin{bmatrix} i_2-a & & i_3 \\ & a+b & \\ j_1+1 & & j_3 \end{bmatrix} - x^{a+2b}(i_1+a+b,\ j_3-2)\begin{bmatrix} i_2-a & i_3-b & \\ & a & \\ j_1+1 & j_2+1 & \end{bmatrix},$$

and so finally, from (12), we have, for the generating function of the three-rowed partition

$$\begin{bmatrix} i_1 & i_2 & i_3 \\ & a & b & \\ j_1 & j_2 & j_3 \end{bmatrix},$$

the expression

$$(17)\qquad (i_1,\ j_1)\begin{bmatrix} i_2 & i_3 \\ b & \\ j_2 & j_3 \end{bmatrix} - x^a(i_1+a,\ j_2-1)\begin{bmatrix} i_2-a & & i_3 \\ & a+b & \\ j_1+1 & & j_3 \end{bmatrix}$$

$$+x^{a+2b}(i_1+a+b,\ j_3-2)\begin{bmatrix} i_2-a & i_3-b & \\ & a & \\ j_1+1 & j_2+1 & \end{bmatrix},$$

which, by aid of (10), may be cast into the determinant form

$$(18)\qquad \begin{vmatrix} (i_1,\ j_1) & (i_1+a,\ j_2-1) & (i_1+a+b,\ j_3-2) \\ x^a(i_2-a,\ j_1+1) & (i_2,\ j_2) & x^{-b}(i_2+b,\ j_3-1) \\ x^{2a+2b}(i_3-a-b,\ j_1+2) & x^{2b}(i_3-b,\ j_2+1) & (i_3,\ j_3) \end{vmatrix}.$$

The argument and the form of the result can be extended without difficulty to many-rowed partitions.

4. In the special case in which the lengths of the rows are unrestricted, and for every r

$$i_{r+1} = i_r + a_{r,\,r+1},$$

the result simplifies. Thus in the three-rowed partition

$$\begin{bmatrix} i & i+a & i+a+b \\ & a & b & \\ \infty & \infty & & \infty \end{bmatrix}$$

the partitions (13), (14), (15), (16) are all of the same type; the parts in rows 1, 2, 3 may not exceed $i, i+a, i+a+b$ respectively. The formula (17) then reduces to the product of

$$(i,\ \infty),\quad (i+a,\ \infty),\quad (i+a+b,\ \infty),$$

and

$$(19)\qquad (1-x^b)-x^a(1-x^{a+b})+x^{a+2b}(1-x^a).$$

But (19) itself reduces to the product

$$(1-x^a)\,(1-x^b)\,(1-x^{a+b}).$$

Thus we have

$$\begin{bmatrix} i & i+a & i+a+b \\ a & b & \\ \infty & \infty & \infty \end{bmatrix} = \frac{(1-x^a)\,(1-x^b)\,(1-x^{a+b})}{\prod_{r=1}^{i}(1-x^r)\prod_{r=1}^{i+a}(1-x^r)\prod_{r=1}^{i+a+b}(1-x^r)}$$

and similarly in the many-rowed partition

$$\begin{bmatrix} i_1 & i_2 & i_3 & \cdots \\ i_2-i_1 & i_3-i_2 & \cdots & \\ \infty & \infty & \infty & \cdots \end{bmatrix} = \frac{\prod_{r,s}(1-x^{i_r-i_s})}{\prod_{i}\prod_{r=1}^{i}(1-x^r)}.$$

In particular

$$(20)\qquad \begin{bmatrix} i & i+1 & i+2 & \cdots & i+k \\ 1 & 1 & \cdots & 1 & \\ \infty & \infty & \infty & \cdots & \infty \end{bmatrix} = \frac{(1-x)^k\,(1-x^2)^{k-1}\cdots(1-x^k)}{\prod_{t=0}^{k}\prod_{r=1}^{i+t}(1-x^r)}$$

$$= \prod_{r=1}^{k}(1-x^r)^{-r}\prod_{r=k+1}^{i}(1-x^r)^{-(k+1)}\prod_{r=i+1}^{i+k}(1-x^r)^{-(i+k+1-r)}.$$

Since $a=1=b=\ldots$, the columns of the partition are monotonic (non-ascending) and thus *every* part is restricted not to exceed i. The generating function (20) is that which MacMahon* writes $GF(i;\ k:\ \infty)$. If we take first i and then k to infinity, we obtain MacMahon's concise formula

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* *Loc. cit.*, 2 (1916), 243, §495, last formula but two.

for the unrestricted plane partition

$$(1) \qquad GF(\infty;\ \infty;\ \infty) = \prod_{r=1}^{\infty} (1-x^r)^{-r}.$$

Had this formula (1) been our sole objective, we could, of course, have attained it directly by the methods of this Note, the only departure from pure partition arguments being the use of an algebraic identity* for the reduction of (19). This, too, can be replaced (rather artificially) by a pure partition argument†.

The shape of the partitions (14), (15), (16) suggests that the study of plane partitions no longer straight-edged on the left may be a fruitful source of new formulae, but difficulties arise if the left-hand edge overhangs at any stage.

* This identity, oddly enough, recurs in the partitioning of a number into specified parts, $i_1, i_2, i_3, \ldots$, the number of parts being also specified.

† T. W. Chaundy, *Quart. J. of Math.* (Oxford), 3 (1932), 76–80.

ON THE DIRECTRICES OF A SET OF POINTS IN A PLANE

By PATRICK DU VAL.

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1. *Introduction.*

It was remarked over twenty years ago by Schoute† that the twenty-seven lines on a cubic surface can be put into one-one correspondence with the vertices of a certain polytope in six-dimensional space, in such a manner that all the incidence relations between the lines are mirrored in metrical properties of the polytope, and conversely; so that the famous "Group of the twenty-seven lines" is simply isomorphic with the group of symmetry of the polytope. More recently it was pointed out by Coxeter‡ that an essentially similar relation holds between the figure of the twenty-eight bitangents of the general plane quartic and a polytope in seven dimensions; and Todd and myself perceived a similar but less complete relation between the tritangent planes of the canonical curve of genus 4 and a polytope in eight dimensions.

These relations are not haphazard, but depend on the facts that the three figures considered reduce in their essentials to the figures of six, seven, and eight points respectively in a plane (the cubic surface being represented by the system of cubics with six base points, and the other two figures studied by means of the Geiser and Bertini involutions). It appears in general that a set of n points in a plane defines a discrete set of curves (which we shall call its directrices) which can be put into one-one correspondence in a convenient manner with the vertices of a certain polytope in n-dimensional space; this correspondence it is the chief purpose of the present paper to establish, illustrate, and apply.

The points will generally be supposed to be general in position, and are to be regarded as potentially the set of base points of a complete linear

† P. H. Schoute, *Proc. Kon. Akad. Wet. Amsterham* (1910), 375–383.

‡ H. S. M. Coxeter, *Proc. Camb. Phil. Soc.*, 24 (1928), 7–9.

system of plane curves. Certain specialisations of their position are not likely to affect the conclusions of the present investigations; *e.g.* if several of the points (more than would generally be possible) are restricted to lie on a curve whose genus is greater than 0, or if nine of them are the base points of a pencil of elliptic curves, the properties now under discussion will be found to be unaffected; if, on the other hand, a rational curve exists containing more than it normally ought of the points (*e.g.* if three lie on a line or six on a conic, or if there is a cubic having a node at one and passing through seven others), or if several of them are adjacent, so that the conditions which they impose as base points of a system of curves involve contact of some order, then it will be found that the curves which we are to consider will not all be distinct, but otherwise the case will not be very materially changed. We shall not further consider cases of special position.

We shall call the number of the points n, and give them the symbols $K_1, \ldots, K_n$, a general one being K_α. The complete linear system of curves of order m which have K_α as i_α-ple base point ($\alpha = 1, \ldots, n$) will be indicated by $m(i_1, \ldots, i_n)$, or simply $m(i)$. A system such as this, whose base points are all among the points of our set (K), will be said to be based on (K).

We shall speak of the intersections of two curves, other than those absorbed by the multiplicities assigned to the curves at the base points (K), as their *free* intersections, though in any particular instance they may not chance to be variable. If undashed and dashed symbols refer to the two curves respectively, their number is

$$[CC'] = mm' - \Sigma(ii'),$$

and in particular the grade of the system $m(i)$ is $m^2 - \Sigma(i^2)$.

Any Cremona transformation of the plane which is based on the set of points (K), *i.e.* which makes the lines of a new plane correspond to a homaloidal net of curves in the old plane which is based on (K), transforms those of the base points (K) which are not base points (or, if we like, are base points for which $i = 0$) for the homaloidal net, into points, to which we may continue to give the same names as they had before; and the remainder it transforms into curves, which are called fundamental curves of the reverse transformation; while the reverse transformation is by means of a homaloidal net in the new plane (made to correspond to the lines of the old) having the same number of base points as that in the direct transformation, which in turn correspond to curves in the old plane, called fundamental curves in the direct transformation; these points in the new plane may be given the names of the base points in the old plane, in any convenient permutation, since they are of the same number. We thus

have a group of operations, every one of which is either a Cremona transformation of the plane, based on the set of points (K), or merely a permutation of the names assigned to these points, and every one of which leaves us with a set of n points $K_1, \ldots, K_n$ as before. These operations we shall call briefly the transformations based on (K). It is manifest that a linear system based on the points (K) is transformed by any of these transformations into a linear system based on the new points (K), of the same grade, freedom, and genus as the original system, and complete if that was complete; the numbers $m(i)$ defining the new system can be calculated from those defining the old system, with those defining the transformation (together, of course, with the specification of the particular fundamental curves which correspond to the new base points).

2. *Directrices of the base points.*

We proceed now to consider what conditions a curve must satisfy in order to be fundamental in a Cremona transformation on the points (K). In the first place, since it corresponds point for point to the neighbourhood of a point K_α in the second plane, it is rational; and a point of it which corresponds to more than one point in the neighbourhood of the point K_α is a base point of the transformation, *i.e.* all its singular points are points of (K). Further, its multiplicities at the base points are such as to deprive it of free intersections with the net defining the transformation. But if the freedom of the system defined by the symbol $m(i)$ of the curve in question was greater than 0, all its members, which would form at least a pencil, and hence cover every point of the plane, would be fundamental curves, since they would all be without free intersections with the net, and every point of the plane would correspond to K_α. Since this is not the case, we conclude that the multiplicities of the curve at the base points are such as to reduce its genus and freedom alike to zero.

A directrix of the points (K) is a curve whose multiplicities (i) at the points (K) are such as to reduce its genus and its freedom alike to zero.

This is the definition of the curves with which the investigation is concerned.

The conditions that the genus and freedom may vanish are

$$\Sigma \tfrac{1}{2} i(i-1) = \tfrac{1}{2}(m-1)(m-2), \qquad \text{(i)}$$

$$\Sigma \tfrac{1}{2} i(i+1) = \tfrac{1}{2} m(m+3), \qquad \text{(ii)}$$

respectively; adding and subtracting which we obtain two others, namely

$$\Sigma(i) = 3m-1, \tag{iii}$$

$$\Sigma(i^2) = m^2+1, \tag{iv}$$

which we shall call the equations of condition. These relations between $m(i)$, or any two of them, serve to characterise a directrix of the points (K).

Since a transformation based on the points (K) does not alter either the freedom or the genus of a linear system based on them, it transforms every directrix into another, *i.e.* it permutes the directrices among themselves; provided that we count among the directrices the neighbourhoods of the base points. This is simply achieved by conventionally attributing to the neighbourhood of K_α the symbol $0(i)$, where $i_\beta = -\delta_{\alpha\beta}$ ($\delta_{\alpha\beta}$ being the ordinary Kronecker symbol). This is seen to satisfy the equations of condition, and also to give j_α as the number of free intersections of a curve $m(j)$ with the neighbourhood of K_α, which is evidently desirable.

We can prove Noether's inequality† for the symbol $m(i)$ of a directrix in the same way as for that of a homaloidal net. If $i_1 \geqslant i_2 \geqslant i_3 \geqslant i_\alpha$ ($\alpha = 4, 5, \ldots, n$), by multiplying (iii) by i_3 and subtracting from (iv), we obtain an identity which can be written

$$\begin{aligned}(i_1+i_2+i_3-m-\tfrac{1}{3})(m-\tfrac{1}{3}) \\ = (i_1-i_3)(m-\tfrac{1}{3}-i_1)+(i_2-i_3)(m-\tfrac{1}{3}-i_2)+\sum_4^n (i_3-i_\alpha)i_\alpha+\tfrac{10}{9}\end{aligned}$$

in which every term on the right, and the second factor on the left, are positive; from this we infer that

$$i_1+i_2+i_3 > m,$$

so that a transformation based on the points (K) exists (namely that by the homaloidal net of conics through K_1, K_2, K_3) by which the directrix in question is transformed into one of lower order. By successive reduction it is evident that every directrix of the base points can be transformed ultimately into a line, or into the neighbourhood of a base point, as we choose. This is, of course, on the assumption that the base points are at least three in number. We have thus the theorem,

The transformations based on the set of points (K) permute the directrices of the points among themselves, in a way which is transitive upon them, i.e. such that there exists at least one transformation of the group which transforms any one into any other, if the points are at least three in number.

† H. P. Hudson, *Cremona transformations* (1927), ch. 1, § 10.

On the other hand, it is clear that

The only transformation, other than identity, based on two points is the interchange of them; and, while this interchanges the two directrices which are their neighbourhoods, it leaves unchanged the line joining them; the three directrices of two points form accordingly an unsymmetric system.

Another property which is obvious is that

The permutation of the directrices among themselves effected by any transformation based on the base points leaves invariant the number of free intersections of any two of them.

Conversely, we shall show that any permutation having this property is effected by a transformation based on the points; for this we first need to prove that any set of n directrices, no two of which have a free intersection, can be simultaneously transformed into the neighbourhoods of the n base points. This is easily seen to be true for any value of n if it is true for the next lower; since by what has been shown (supposing, as we obviously may, that $n \geqslant 3$) any directrix of the set can be transformed into the neighbourhood of K_n, and the remaining $n-1$, since they have no free intersection with that transformed into this neighbourhood, go into curves not passing through K_n, namely directrices of the remaining $n-1$ points; but by hypothesis there is a transformation based on these $n-1$ points (consequently leaving K_n unchanged), which transforms the $n-1$ directrices into the neighbourhoods of the base points. Thus, since it is definitely true in the case $n = 3$, where the only such triads of directrices are the neighbourhoods of the base points, and the lines joining them by pairs, and these triads are interchanged by the transformation by conics through the base points, we may conclude generally that

Every set of directrices, in number equal to the base points, of which no two have a free intersection, can be transformed simultaneously into the neighbourhoods of the base points, by a transformation based on these points.

It is easily seen, however, that a permutation of the directrices amongst themselves such as to leave invariant the number of free intersections of any two is uniquely determined when it is specified what directrices replace the neighbourhoods of the base points, since the numbers of free intersections of a directrix with these are the quantities (i) for the curve which replaces it, which is thus completely determinate. Thus we see that

Every permutation of the directrices among themselves leaving invariant the number of free intersections of every two is effected by a unique transformation based on the base points, and conversely.

It is an immediate corollary that

The maximum number of directrices of which no two have any free intersections is n.

3. *Representation by polytopes. First method.*

We shall now represent the directrices of the base points by points in space of n dimensions, in which we shall use non-homogeneous coordinates, as we intend ultimately to attribute metrical properties to the space, and represent the transformations based on the points by symmetry operations in the space. We shall for convenience call the base points $K_0, \ldots, K_{n-1}$ instead of $K_1, \ldots, K_n$. We take in the space $[n]$ coordinates $x_1, \ldots, x_{n-1}, y$, defined in terms of the symbol $m(i)$ of any directrix by

$$x_a = m - i_0 - 2i_a, \quad y = m - i_0, \tag{i}$$

as the coordinates of a point representing the directrix in question. It is easily seen that, in consequence of the equations of condition, these satisfy identically

$$\Sigma (x^2) + (9-n) y^2 = 4(y+1), \tag{ii}$$

so that all the representing points lie on a quadric. Further, if dashed and undashed symbols refer to two different directrices and their representing points,

$$\Sigma (x-x')^2 + (9-n)(y-y')^2 = 8 [mm' - \Sigma (ii') - 1]. \tag{iii}$$

In view of these relations it is natural to attribute to the space the metric (Euclidean, degenerate, or Minkowskian according as $n < 9$, $n = 9$, or $n > 9$) in which the absolute quadric is given by

$$\Sigma (x^2) + (9-n) y^2 = 0 \tag{iv}$$

in the prime at infinity. On this understanding the identities (ii), (iii) can be stated in words:

The points of $[n]$ representing the directrices all lie on a sphere.

The square of the distance between any two of these points (in terms of a convenient unit, namely $2\sqrt{2}$ times that in terms of which the coordinates are measured) exceeds by unity the number of free intersections of the directrices corresponding to them.

The special unit of length used for this measurement, which is clearly the least distance that can occur between two points representing directrices, we shall call the metrical unit, as opposed to the coordinate

unit, which may, of course, be changed with a change of coordinate system, but which in the present coordinate system is $1/(2\sqrt{2})$ times the metrical unit.

The equation of the circumscribing sphere can be written

$$\Sigma(x^2)+[\sqrt{(9-n)}\,y-2\sqrt{2}B]^2=8R^2, \tag{v}$$

where R and B are respectively in metrical units the radius of the sphere, and the perpendicular from the centre on to the prime $y=0$. They are given by

$$2R^2=(10-n)/(9-n), \tag{vi}$$

$$2B^2=1/(9-n). \tag{vii}$$

We may remark that those directrices which do not pass at all through K_{n-1}, namely the directrices of the remaining $n-1$ base points, are singled out from the rest by the condition $i_{n-1}=0$, *i.e.* $y=x_{n-1}$, which on being substituted in (iv) gives exactly the same equation, save that $n-1$ is substituted for n; from which it appears that the representation of the directrices of $n-1$ points obtained as part of those of n points is the same as that obtained directly. It may be obtained in a slightly different manner by remarking that the directrices which have no free intersections with the neighbourhood of K_{n-1} are represented by points at (metrical) unit distance from the point representing this particular directrix; and that the unit sphere about this point meets the sphere (iv) in a sphere in a prime (namely, of course, the prime $y=x_{n-1}$) which contains all the points in question.

A permutation of the directrices among themselves which leaves invariant the number of free intersections of any two corresponds clearly to a permutation of the representing points among themselves which leaves invariant the distance between any two, *i.e.* to a symmetry operation on the set of points as a figure in metrical space. The final results of the previous section give accordingly that

Every transformation based on the base points corresponds to a unique symmetry operation on the set of points representing the directrices, and conversely. There is one such symmetry operation which transforms any regular simplex of (metrical) unit edge and n vertices among the points into that which represents the neighbourhoods of the base points, the vertices of one simplex being made to correspond to the vertices of the other in any required permutation.

These simplexes, being of n vertices, are in primes, which (being all equivalent under the symmetry operations of the set of points) are

equidistant from the centre of the sphere on which all the points lie. We may take as typical of these primes that containing the n points representing the neighbourhoods of the base points; its equation is obtained by combining (i) with § 2 (iii), to give

$$-4m = \Sigma(x) - (n-3)y - 2, \tag{viii}$$

whence (since $m = 0$ is the condition for a directrix to be the neighbourhood of a base point) the equation of the prime in question is

$$\Sigma(x) - (n-3)y = 2, \tag{ix}$$

so that the distance of all these primes from the centre is found to be A metrical units, where

$$2A^2 = \frac{9}{n(9-n)}. \tag{x}$$

We shall adopt in general Coxeter's nomenclature† and notation for polytopes; in particular the simplex, cross polytope, and measure polytope, in $[n]$ will be called α_n, β_n, γ_n respectively; we conclude that

There are among the points representing directrices the vertices of α_{n-1}'s of unit edge, each of which corresponds to a homaloidal net based on the base points, the n vertices of such an α_{n-1} corresponding to n directrices which are transformed into the neighbourhoods of n base points in the plane whose lines correspond to the curves of the homaloidal net. Conversely, every homaloidal net based on the base points corresponds to such an α_{n-1}.

There are other regular polytopes to be picked out from among the points. Consider the pencil of lines through K_0; this linear system is given by the symbol $1(i)$, where $i_\alpha = \delta_{0\alpha}$. There are $n-1$ curves of the system which break up into a pair of directrices, those namely which consist of a line K_0K_α, with the neighbourhood of the point K_α. Each of these $2(n-1)$ directrices has one free intersection with that which forms with it a curve of the pencil, and none with any of the others; so that its representing point is distant $\sqrt{2}$ metrical units from that paired with it and one unit from each of the others, *i.e.* the points form a β_{n-1} of unit edge, opposite vertices of which correspond to directrices forming together a curve of the pencil. In fact, the points representing these directrices are just those for which $y = 0$, $x_\beta = \pm 2\delta_{\alpha\beta}$, for some value of α, which are just the vertices of a β_{n-1} in the prime $y = 0$, distant B metrical units from the centre. Since any other pencil of rational curves based on the base points can be transformed into this by a transformation based on

† H. S. M. Coxeter, "The polytopes with regular-prismatic vertex figures", *Phil. Trans. Royal Soc.* (A), 229 (1930), 329–425, hereinafter referred to as P.R.V.F.

them†, we see that

There are among the points representing directrices the vertices of β_{n-1}'s of unit edge, each of which corresponds to a pencil of rational curves based on the base points, in such a way that each pair of opposite vertices of such β_{n-1} corresponds to a pair of directrices which together form a curve of the pencil, and to every such pencil corresponds such a β_{n-1}.

All the α_{n-1}'s corresponding to homaloidal nets are equivalent with regard to the symmetry group of the system of points, and are perpendicularly distant A metrical units from the centre. Similarly all the β_{n-1}'s corresponding to pencils of rational curves are equivalent, and are perpendicularly distant B metrical units from the centre.

To prove that the points representing the directrices are themselves the vertices of a polytope in $[n]$, whose boundaries are these α_{n-1}'s and β_{n-1}'s, it is only necessary to show that every bounding α_{n-2} of either belongs also to just one other α_{n-1} or β_{n-1}. Now the α_{n-2} which corresponds to the neighbourhoods of all the base points but one (which is a typical bounding α_{n-2} of an α_{n-1}) belongs also to the β_{n-1} corresponding to the pencil of lines through this base point, and to no other α_{n-1} or β_{n-1}, since the appropriate homaloidal net or pencil respectively must have this point as sole base point. Let this point be K_0, then the β_{n-1} is that already considered, in the prime $y = 0$; the α_{n-2} in the secundum traced on it by $\Sigma(x) = 2$ is that which it shares with α_{n-1}. But all those obtained from this by the change of sign of an even number of the coordinates (x) are equivalent to it under the symmetry group of the figure, since (as is easily seen) the change of sign of x_α, x_β, a rotation of π about the secundum $x_\alpha = x_\beta = 0$, corresponds to the transformation by conics through K_0, K_α, K_β, the new K_α, K_β being those which arise from the lines K_0K_α, K_0K_β respectively, contrary to the usual notation; and in general the change of sign of any even number $2r$ of coordinates corresponds to the transformation by $(r+1)$-ic curves passing r-ply through K_0 and simply through the $2r$ base points corresponding to the coordinates affected. Thus

All the bounding α_{n-2}'s of every α_{n-1}, and half those of every β_{n-1}, are equivalent, and belong to just one α_{n-1} and just one β_{n-1}.

The remaining α_{n-2}'s of the β_{n-1} in $y = 0$ are all equivalent under the operations just explained, and we take as typical that corresponding to the neighbourhoods of the $n-2$ base points $K_2, \ldots, K_{n-1}$ and the line K_0K_1.

† F. Enriques, *Teoria geometrica delle equazioni*, 3 (1915), 194.

These do not belong to any set of n directrices without mutual free intersections, since, if they did, the line K_0K_1 and the n-th directrix of the set could be transformed into the neighbourhoods of base points by a Cremona transformation having K_0, K_1 as sole base points. Hence the corresponding α_{n-2} does not belong to any α_{n-1}; on the other hand, it belongs to just two β_{n-1}'s, those namely corresponding to the pencils of lines through K_0, K_1 respectively; if it belonged to any other, the corresponding pencil would be transformable into a pencil of lines through a base point by a Cremona transformation having K_0, K_1 as sole base points. Thus

Every bounding α_{n-2} of a β_{n-1} which does not belong also to an α_{n-1} belongs to just one other β_{n-1}, and all such α_{n-2}'s are equivalent.

It may be remarked that the linear system which is the sum of the two pencils corresponding to these β_{n-1}'s is of freedom 3, grade 2, and genus 0, and accordingly represents the prime sections of a quadric surface in [3] (in the standard case to which we have reduced the α_{n-2} and two β_{n-1}'s, it consists of the system of conics through K_1, K_0). The set of directrices which correspond to the vertices of α_{n-2} are then evidently just those which have no free intersection with the general curve of this system, and which correspond accordingly to the neighbourhoods of points, $L_1, \ldots, L_{n-1}$ say, on the quadric. The two pencils of rational curves corresponding to the two β_{n-1}'s correspond on the quadric to the two systems of generators, and the $n-1$ curves in each such pencil which break up into a pair of directrices (represented by a pair of opposite vertices of β_{n-1}, one belonging to the α_{n-2} in question, and one not) correspond on the quadric to the neighbourhood of a point L_a and the generator through it.

We may sum up these results as follows:

The points representing the directrices of n base points are the vertices of a uniform polytope in $[n]$, whose prime boundaries are α_{n-1}'s and β_{n-1}'s and whose secundum boundaries are α_{n-2}'s of two kinds; some belonging to an α_{n-1} and a β_{n-1}, and some to two β_{n-1}'s.

It is easily seen that all sets of $n-2$ directrices, without mutual intersections, are equivalent to the neighbourhoods of $n-2$ base points, so that all α_{n-3}'s of the polytope are equivalent; each evidently belongs to two α'_{n-2}'s of the first kind and one of the latter, and to two β_{n-1}'s and one α_{n-1}.

It is further evident that the polytope of the series in $[n-1]$ is the vertex figure of that in $[n]$, since its vertices are those adjacent to a given

vertex of the former. Moreover, the polytope in [3] corresponding to the directrices of three points is easily seen to be the triangular prism $[\alpha_1, \alpha_2]$ in [3], so that, by taking inverse vertex figures of this, we see that

For $n \leqslant 8$, the polytope representing the directrices of n points is Coxeter's "Pure Archimedean" $(PA)_n$, or $(n-4)_{21}$†.

The polytope representing the directrices of nine points is the same as Coxeter's $(PA)_9$, or 5_{21}, as regards the mutual distances of its vertices, and as regards the topological relations of its elements, but it is inscribed in a paraboloid of spherical revolution, instead of falling entirely into [8].

This last is, of course, possible only on account of the degeneracy of the metric in the case $n = 9$, since (the term y^2 being absent altogether) it is immaterial how the figure is distended or contracted parallel to the y-axis. If, in fact, a new coordinate $x_n = y\sqrt{(9-n)}$ is introduced to make the metric Euclidean in all cases, then, when $n = 9$, this coordinate is zero throughout, and the effect is to project the paraboloid given by equation (iv) parallel to its axis on to the tangent prime at the vertex, when the polytope becomes the ordinary $(PA)_9$, which is in [8].

The polytope whose vertices represent the directrices of n base points, even when $n > 9$, we shall accordingly call $(PA)_n$.

4. *Case of nine base points. Second method.*

We shall now consider the important limiting case $n = 9$, and establish the representation of the directrices by $(PA)_9$, in a slightly different way. We shall call the base points $K_1, \ldots, K_9$, and take ordinary Cartesian coordinates in [9], $x_1, \ldots, x_9$, given by

$$x_a = 3i_a - m, \tag{i}$$

which satisfy of themselves

$$x_\alpha \equiv x_\beta \pmod 3, \tag{ii}$$

and, in virtue of § 2 (iii),

$$\Sigma(x) = -3, \tag{iii}$$

which two equations (or rather one equation and one congruence) Coxeter gives‡ as defining the system of vertices of 5_{21} or $(PA)_9$. It is easy to find

† P.R.V.F., 8.

‡ P.R.V.F., 10.1 (save for change of origin).

expressions for $m(i)$ in terms of (x), and to verify that these give positive and integral values for all integral solutions of (ii), (iii). The edge of the polytope in this form is $3\sqrt{2}$ times the coordinate unit, and this accordingly is the metrical unit. By substituting for the coordinates from (i), and replacing $\Sigma(i)$ and $\Sigma(i^2)$ as they occur by means of § 2 (iii) and § 2 (iv), we have, in fact, no difficulty in proving the identity

$$\Sigma(x-x')^2 = 18\,[mm' - \Sigma(ii') + 1], \qquad \text{(iv)}$$

which corresponds to § 3 (iii), the change of unit being apparent in the replacement of the coefficient 8 by 18.

This form is convenient for the exhibition of many important properties of 5_{21}, regarded as a polytope or space filling in [8]. It is, for instance, immediately obvious that the vertices form a lattice; that is to say, that the translation which transforms any vertex into any other is a symmetry operation of the whole system. If we transfer the origin to any point of the lattice, the defining conditions take the same form, namely (ii) remains unchanged, and (iii) becomes

$$\Sigma(x) = 0. \qquad \text{(v)}$$

(It is actually in this form that the conditions are given by Coxeter.) The vertices adjacent to the origin are then 240 in number, namely those given by every permutation of the coordinates

(2, 2, 2, −1, −1, −1, −1, −1, −1)	of which	there	are 84,
(3, −3, 0, 0, 0, 0, 0, 0, 0)	,,	,,	,, 72,
(−2, −2, −2, 1, 1, 1, 1, 1, 1)	,,	,,	,, 84,
			240.

We thus find that the edges of the polytope are parallel to 120 directions, which are given by the joins of diametrically opposite pairs among these vertices. It is seen that thirty-six of these directions are perpendicular to the primes $x_\alpha = x_\beta$, which are primes of symmetry of the whole figure, the reflection in such a prime merely interchanging the values of the coordinates x_α, x_β, and being accordingly the symmetry operation which corresponds to the interchange of the base points K_α, K_β. The remaining eighty-four directions are perpendicular to the primes

$$x_\alpha + x_\beta + x_\gamma = \Sigma^*(x) = 0,$$

where the summation Σ^* extends over the six suffixes other than α, β, γ. If the polytope is restored to the form (ii) (iii), so that the vertices are

directly in correspondence with the directrices of the base points, then the reflection in the prime of symmetry

$$x_\alpha + x_\beta + x_\gamma = \Sigma^*(x) + 3 = 0$$

corresponds to the transformation of the plane by conics through K_α, K_β, K_γ, and that in the prime

$$x_\alpha + x_\beta + x_\gamma + 3 = \Sigma^*(x) = 0$$

to the transformation by quintics passing twice through the remaining six base points. Since the vertex figure of 5_{21} is uniform, the 120 pairs of vertices are all equivalent under the symmetry of the figure, so that the 120 primes of symmetry which we have found to pass through a vertex are equivalent, and the reflections in them are conjugate operations in the symmetry group. We may summarise thus:

5_{21} has 120 systems of parallel edges, and perpendicular to each of these a system of parallel primes of symmetry; the distance between consecutive primes in each system is half a metrical unit, a prime of the system passing through every vertex of the polytope and bisecting every edge of the corresponding (perpendicular) system. The reflections in all the primes are conjugate operations in the symmetry group, and so are the unit translations parallel to all the edges.

We can now make the equations of condition somewhat more general by considering more general curves than the directrices. We have seen that every β_{n-1} of $(PA)_n$ corresponds to a pencil of rational curves, every α_{n-1} to a homaloidal net, and every α_{n-2} which belongs to two β_{n-1}'s to a linear system of rational curves of grade 2 and freedom 3, in each case based on the given base points; for these systems the equations of condition are

$$\Sigma(i) = 3m - \epsilon, \tag{vi}$$

$$\Sigma(i^2) = m^2 - (\epsilon - 2), \tag{vii}$$

where $\epsilon = 1, 2, 3, 4$ for the directrix, pencil, net, and system of freedom 3 respectively, *i.e.* $\epsilon - 1$ is the freedom of the system. It is at once evident, if we represent these systems also by points in [9], by the rule (i), that the points corresponding to systems of freedom $\epsilon - 1$ will all lie in the prime

$$\Sigma(x) = -3\epsilon, \tag{viii}$$

and it can be found by methods similar to those already used that, if dashed

and undashed letters refer to different systems, the expression

$$\Sigma\left(\frac{x}{\epsilon}-\frac{x'}{\epsilon'}\right)^2$$

is equal to an expression involving the number of free intersections of the general curves of the two systems. This expression clearly gives the square of the distance, in terms of a suitable unit, between the projections from the origin of the two representing points on any of the parallel primes

$$\Sigma(x) = \text{const.},$$

so that the points representing pencils, nets, and systems of freedom 3, project from the origin into the mid-points of the corresponding β_8's, α_8's, and α_7's respectively of 5_{21}. The systems of points in the primes

$$\Sigma(x) = -3,\ -6,\ -9,\ -12$$

are accordingly the systems of vertices of the four polytopes 5_{21}, 2_{15}, 1_{52}, and $t_1 2_{15}$ (since the mid-points of the α_7's belonging to two β_8's of 5_{21} correspond to the mid-points of the edges of the 2_{15} semi-reciprocal to it†).

We can, in fact, find sets of defining congruences for the vertices of these polytopes, analogous to the relations (ii), (iii), or (ii), (v) for 5_{21}. It is only necessary to examine what further restrictions are to be imposed on a solution in integers of (ii), (viii) in order that it may lead to integral values of $m(i)$, and then to detect and exclude those solutions of (vi), (vii) which correspond to improper or degenerate pencils, nets, etc. Every directrix, counted twice, together with the cubic through the base points, satisfies the conditions for a pencil, and the corresponding point in [9] is the projection on to $\Sigma(x) = -6$ of the vertex of 5_{21} representing the directrix. The vertices which we obtain in this prime will therefore be those, not only of 2_{15}, but also of the 5_{21} semi-reciprocal to it, which is of edge $6\sqrt{2}$ in the coordinate units. Now it is easily seen that the condition for integral values of $m(i)$ corresponding to a point whose coordinates satisfy (ii), (viii) (with $\epsilon = 2$) is that all, five, or one only of the nine coordinates shall be even; and it is at once evident that the points of this system whose coordinates are all even are precisely the vertices of the 5_{21} of edge $6\sqrt{2}$, and do not correspond to proper pencils. Further, of the remaining vertices, those for which the parity of each coordinate is determined in a particular way form another 5_{21}, congruent and similarly placed to the former, inscribed in 2_{15}; of such 5_{21}'s inscribed in 2_{15} there are accordingly just 135, corresponding to the nine principal axes in [9], and their 126 combinations by fives. Such a lattice 5_{21}, being just those

† P.R.V.F., 5.1, explains the notation for simple truncations.

vertices of 2_{15} which are obtained from a given one by the *translational* symmetry operations of 2_{15} (which are, of course, the symmetry translations of the 5_{21} semi-reciprocal to it), are the centres of a system of similarly placed β_8's of 5_{21}, which are likewise equivalent under the translational symmetry of the polytope.

To sum up:

The polytope 2_{15} *has* 135 *inscribed lattices, corresponding to* 135 *systems of similarly oriented* β_8*'s of* 5_{21}.

[We may note in passing that it is a natural proceeding to consider, of any degenerate polytope, the number of its inscribed lattices, *i.e.* the number of sets into which its vertices are divided by the condition that the subgroup of the symmetry group which consists simply of translations shall be transitive on each set separately. For instance, in the cases of the degenerate regular polyhedra, the tesserohedron (plane of squares) and the trigonohedron (plane of triangles) have each one inscribed lattice, but the hexagonohedron (plane of hexagons, like wire netting) has two, corresponding to two sets of similarly oriented triangles in the trigonohedron reciprocal to it.]

We pass on to the case of the homaloidal nets. In this case, it can easily be seen that the combination of the cubic through the base points with a directrix counted twice and another counted once (the two having no free intersections) provides an improper solution of the equations of condition; such a set of values of $m(i)$ is evidently represented by a trisector of the joining line of the projections on $\Sigma(x) = -9$ of the points representing the two directrices, *i.e.* by a trisector of an edge of the projection of 5_{21} on to this prime (which is, of course, a 5_{21} of edge length $9\sqrt{2}$). The totality of vertices representing these improper solutions are accordingly the vertices of the polytope $t_{01}5_{21}$†. This has 240 inscribed lattices, two corresponding to each system of parallel edges of 5_{21}.

The condition that we may obtain integral values of $m(i)$ from a solution of (ii), (viii) (with $\epsilon = 3$) can be thus stated:

If the residues (mod 9) of the coordinates are 0, 3, 6, the number of each occurring is congruent to 1 (mod 3), but if they are 1, 4, 7, or 2, 5, 8, the number of each is divisible by 3.

An inscribed lattice is given by three sets of residues of the coordinates (mod 9), obtained from any one of themselves by adding 0, 3, 6,

† The notation employed here for truncations is part of the notation used by Coxeter in some unpublished work; the only cases other than the simple (*i.e.* single-suffix) truncations with which we shall be concerned are those of the form t_{01}, which is explained in *Proc. Camb. Phil. Soc.*, 27 (1931), 211, and some to which we shall refer later.

respectively to all the coordinates simultaneously. We can tabulate the possibilities as under:

Residues (mod 9)									*Number of lattices inscribed in*	
0	1	2	3	4	5	6	7	8	$t_{01}5_{21}$	1_{52}
1	.	.	4	.	.	4	.	.		
4	.	.	1	.	.	4	.	.		630
4	.	.	4	.	.	1	.	.		
7	.	.	1	.	.	1	.	.		
1	.	.	7	.	.	1	.	.	72	
1	.	.	1	.	.	7	.	.		
.	9	.	.	.	.	.	.	.		
.	.	.	.	9	.	.	.	.		1
.	.	.	.	.	.	.	9	.		
.	6	.	.	3	.	.	.	.		
.	.	.	.	6	.	.	3	.		84
.	3	.	.	.	.	.	6	.		
.	6	.	.	.	.	.	3	.		
.	3	.	.	6	.	.	.	.	84	
.	.	.	.	3	.	.	6	.		
.	3	.	.	3	.	.	3	.		560
.	.	9	.	.	.	.	.	.		
.	.	.	.	.	9	.	.	.		1
.	.	.	.	.	.	.	.	9		
.	.	6	.	.	3	.	.	.		
.	.	.	.	.	6	.	.	3	84	
.	.	3	.	.	.	.	.	6		
.	.	6	.	.	.	.	.	3		
.	.	3	.	.	6	.	.	.		84
.	.	.	.	.	3	.	.	6		
.	.	3	.	.	3	.	.	3		560
									—	———
									240 +	1920 = 2160,

2160 being the total number of lattices given by the rule; from a knowledge of the properties of 5_{21} it is easy to sort out as above those vertices which trisect the edges of 5_{21}. (That the remainder all belong to 1_{52}, *i.e.* that there

are no other degenerate "homaloidal nets" based on the given base points, besides those which we have found, follows from the fact, which can be established by other means, depending only on polytope theory, and not on the system of points in a plane, that 1_{52} has actually 1920 inscribed lattices.)

The 1920 *inscribed lattices of* 1_{52} *correspond to* 1920 *systems of similarly oriented* a_8*'s of* 5_{21}. *They fall into* 960 *pairs which are mutual images by reflection in any of the vertices of* 5_{21}, *and correspond to* 960 *pairs of oppositely oriented systems of* a_8*'s.*

The members of such a pair are obtained from each other in the table by the interchange of every pair of residues whose sum is 9.

A similar treatment of $t_1 2_{15}$ is possible, but it is laborious and not particularly interesting. The degenerate solutions to be discarded correspond to combinations of the cubic through the base points with a directrix doubled, and two others counted once, the three directrices having no free intersections among themselves, and give the vertices of the polytope $t_{02} 5_{21}$.

5. *Third method. Case of eight base points on a quadric, and in space.*

A quadric surface Q can be represented on a plane by projection from a point of itself, a curve of order $f+g$, which meets each generator of one system f times, and of the other g times, and which passes j times through the projecting point, being projected into a curve of order $m=f+g-j$, passing $i_1=f-j$, $i_2=g-j$ times respectively through the points of the plane which arise from the generators of Q through the projecting point. It follows that a linear system of curves on Q, whose base points $L_0, \ldots, L_n$ include the projecting point (say L_0), and given by a symbol $f, g(j)$, where $j_0, \ldots, j_n$ are the multiplicities at $L_0, \ldots, L_n$, corresponds to a complete system in the plane, of the same grade, freedom, and genus, based on the set of points $K_1, \ldots, K_{n+2}$, where K_a is the projection of L_a ($a=1, \ldots, n$), and K_{n+1}, K_{n+2} are the projections of the two generators of Q through L_0, and are given by a symbol $m(i)$ in terms of these base points, which is rationally determined by the symbol $f, g(j)$. In particular, every directrix of the points (K), or linear system based on them, of zero genus and freedom, corresponds to a perfectly definite directrix of the points (L), defined as also meaning a complete linear system of zero genus and freedom. Moreover, the free intersections of any two curves in the plane are the projections of the free intersections of the corresponding curves on Q, regarded as based on these two sets of points

respectively; it follows that the entire system of directrices, pencils, nets, and higher systems of rational curves in the plane whose relations we have been considering corresponds to a system with perfectly similar relations on Q, the number of base points being fewer by unity on the quadric than the plane. In fact, the quadric Q is the projective model of one of the systems of freedom three, grade two, and genus zero, which we have seen to correspond to the secundum α's belonging to two prime β's in the appropriate (PA) polytope; its two systems of generators are the pencils corresponding to the two prime β's, and the neighbourhoods of the base points themselves are the directrices corresponding to the vertices of the secundum α.

The interesting case of nine points in the plane corresponds to that of eight points on Q; the cubic through the nine points corresponds to the elliptic quartic through the eight points, and the case in which the nine points are associated corresponds to that in which the eight points are associated.

The conditions that a curve f, $g(j)$ may be a directrix are

$$\Sigma(j) = 2(f+g)-1, \tag{i}$$

$$\Sigma(j^2) = 2fg+1, \tag{ii}$$

and these are easily seen to be satisfied if we represent the curve f, $g(j)$ by the point (x) in [8], where

$$x_\alpha = 2j_\alpha - f, \tag{iii}$$

the conditions satisfied by the representing points being

$$x_\alpha \equiv x_\beta \pmod 2, \tag{iv}$$

$$\Sigma(x) \equiv 2 \pmod 4, \tag{v}$$

which, save for a change of origin, are Coxeter's second form of the defining conditions for 5_{21}†. In this form, the primes of symmetry are parallel, and the edges perpendicular, to the primes

$$x_\alpha \pm x_\beta = 0,$$

$$\Sigma(\pm x) = 0,$$

where, in the latter case, an even number of the eight ambiguities are positive, and an even number are negative.

In particular, the reflection in the prime

$$\Sigma(x)+2 = 0 \tag{vi}$$

† P.R.V.F., 9.1.

corresponds to the interchange of the two systems of generators of Q, for it is easily seen that, in consequence of (i), (iii),

$$4(g-f) = \Sigma(x)+2;$$

but it is clear that the interchange of the two systems of generators corresponds to reflection in a prime of symmetry, not only in this case, but in the cases of the polytopes representing the directrices of any number of points; for evidently the interchange of the two systems of generators of Q, and of f, g for every curve on it, is an operation of the kind which corresponds to a symmetry operation of the polytope; and it leaves invariant the circumcentre of the polytope (if this can be inscribed in a sphere, as is always the case for the polytopes corresponding to other numbers of base points) and every vertex of the secundum α which corresponds to the neighbourhoods of the base points, *i.e.* leaves invariant every point of a prime; and the only symmetry operation which leaves fixed every point of a prime is the reflection in that prime.

Clearly, the vertices of the polytope which lie in the prime in question are just those representing directrices for which $f = g$; but this, of course, is precisely the condition that the curve on Q may be its complete intersection with a surface of order f. We are thus led to consider, with regard to the set of points (L), regarded now as points in space rather than as on Q, the surfaces of order f and multiplicities (j) at the base points, where

$$\Sigma(j) = 4f-1, \qquad \text{(vii)}$$

$$\Sigma(j^2) = 2f^2+1, \qquad \text{(viii)}$$

these being what (i), (ii) become on putting $f = g$. These surfaces we shall call director surfaces of the base points; they have the property of meeting every quadric (there is at least the quadric Q, and there may be others) through the base points in directrices.

There are, in the prime of symmetry of the polytope corresponding to $f = g$, other secundum α's than that representing the neighbourhoods of the base points; each of these represents a set of directrices, having no free intersections among themselves, transformable into the neighbourhoods of the base points by a Cremona transformation of Q into itself, which is defined by a certain system of curves on the quadric, of grade 2 and freedom 3. This system of curves manifestly likewise has $f = g$, and is the trace on Q of a system of surfaces of order f. There will, of course, be many such surfaces tracing on Q each curve of the system, and having with Q at each point L_a contact of order j_a; but we can consider the system, actual or virtual, of surfaces of order f having j_a-ple points at L_a; and we propose to show that this system is of freedom 3, and is a homaloidal web

determining a Cremona transformation of the space containing Q, in which Q is transformed into a quadric, Q' say, in accordance with the transformation just considered, and the director surfaces which trace on Q the directrices represented by the secundum α in question are transformed into the neighbourhoods of the base points.

It is evident that the surfaces of this system satisfy the following conditions, analogous to (vii), (viii):

$$\Sigma(j) = 4(f-1), \tag{ix}$$

$$\Sigma(j^2) = 2(f^2-1), \tag{x}$$

which are those which the characteristic of every *punctual* Cremona transformation in space is known to satisfy†. It does not seem to have been explicitly proved, though it is natural to assume, that every characteristic satisfying these conditions determines a Cremona transformation; we proceed, therefore, to show that this is the case. It is a known consequence of these conditions that the multiplicities of the four highest base points are so great that a cubic involution based on these points (*i.e.* the transformation by the homaloidal net of cubics having nodes at the four points) reduces the order f of the system‡; we have consequently only to show that the conditions themselves are invariant under such a transformation to be sure that, by a succession of such transformations, the system can be transformed into one of as low order as we like, still satisfying the conditions, *i.e.* it can be transformed into the system of planes with no base points at all, and is consequently a homaloidal web. But, by considering the intersection of the surfaces with lines through each of the base points of the transformation, we see that $f-j_\alpha$ is invariant (L_α being one of these four base points), whence by summation it follows that $4f-\Sigma(j)$ is invariant, the multiplicities at the remaining base points being unaffected; and, if the freedom of the system be at least unity, so that there is at least one characteristic curve, by considering the intersections of this (in its entirety, *i.e.* without subtracting any fixed parts, etc.) with a quadric through the four base points of the transformation, it follows that $2f^2-\Sigma(f^2)$ is likewise invariant. But, if the system is not of sufficient freedom (even if it is virtual, having no actual members satisfying the conditions), we can give it sufficient freedom by abolishing the conditions at some of the base points other than those of the transformation which we are applying; and this will affect $\Sigma(j)$ and $\Sigma(j^2)$ to

† Hudson, *op. cit.*, ch. 14, § 30.

‡ *Ibid.*

the same extent before and after the transformation, so that the invariance is unaffected. The only case in which this breaks down is when there are no base points other than the four used for the transformation; and it is easily seen that the only characteristic satisfying the conditions and having not more than four base points (apart from the system of all planes) is that of the homaloidal web of cubics of the transformation itself. It is evident, moreover, that the argument which we have just expounded applies to the case of any number of base points in space, even if they do not all lie on a quadric, since it is unaffected if the base points are so specialized in position that there is a quadric through them all. We conclude that

Every characteristic satisfying (vii), (viii) (*the range of suffixes of the j's being n*) *determines a unique director surface of a given set of n points, the director surfaces being those and all those surfaces which can be transformed into the neighbourhoods of points by punctual Cremona transformations based on the given set of points. These surfaces are represented by the vertices of an n-dimensional polytope, lying in a prime of symmetry of the* $(n+1)$*-dimensional polytope corresponding to the directrices of n points on a quadric, in such a way that the square of the distance between any two vertices exceeds by unity the invariant* $ff'-\Sigma(jj')$ *associated with the two surfaces; and the vertices of an* a_{n-1} *of this polytope represent a set of director surfaces which can be simultaneously transformed into the neighbourhoods of n base points, each* a_{n-1} *being associated with a definite homaloidal web based on the given set.*

From this also it is clear that the whole group of symmetries of the polytope represents the complete group of punctual Cremona transformations of space based on the given set of points, in which two transformations are to be regarded as distinct if the new base points are named in a different order even though they are the same set of points in the two cases, and mere permutations of the base points amongst themselves are to be regarded as Cremona transformations.

In the case $n = 8$, with which we are chiefly concerned, the polytope whose vertices are those of 5_{21} lying in one of its primes of symmetry is known to be 3_{31}. The vertices of this are in fact given by Coxeter† in the form

$$x_a \equiv x_\beta \pmod 2, \qquad \text{(iv)}$$

$$\Sigma(x) = 0, \qquad \text{(xi)}$$

which are the same as (iv), (vi), save for the change of the origin to a point of the lattice. The system of vertices of 3_{31} also forms a lattice, as is

† P.R.V.F., 9.6.

evident from the defining conditions; the polytope has sixty-three systems of parallel edges, which are those edges of 5_{21} which lie in the prime (vi); and sixty-three systems of parallel primes of symmetry, each perpendicular to a system of edges, which are the traces on (vi) of those other primes of symmetry of 5_{21} which are perpendicular to it; one prime of each system goes through each vertex, and one bisects each edge, of the corresponding system exactly as in the case of 5_{21}. The primes of symmetry are parallel, and the edges perpendicular, to the primes

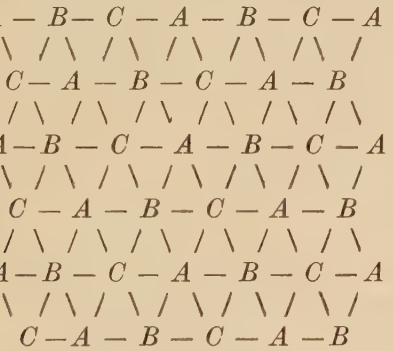
$$x_\alpha = x_\beta \quad (28 \text{ systems}),$$
$$\Sigma'(x) = \Sigma''(x) = 0 \quad (35 \text{ systems}),$$

where the summation Σ' refers to any four of the eight coordinates, and Σ'' to the remaining four.

The remaining primes of symmetry of 5_{21} are all inclined at an angle $\frac{1}{3}\pi$ to the prime (vi). Now the presence of two primes of symmetry of any figure meeting at this angle implies a third, inclined to both at the same angle, and concurrent with them in their common secundum, which is a secundum of threefold rotational symmetry. Thus the fifty-six systems of parallel primes of symmetry of 5_{21} which are oblique to the prime (vi) meet it by pairs in twenty-eight systems of parallel [6]'s, which we shall call *principal primes* of 3_{31}; they are not primes of symmetry, but, like the primes of symmetry, have the property that one of each system goes through each vertex of the polytope.

In addition to its bounding α_7's, which as we saw are α_7's belonging to two β_8's of 5_{21}, 3_{31} is bounded by 3_{21}'s, each of which has just those vertices of 5_{21} which are adjacent to both ends of one of the edges perpendicular to and bisected by the prime (vi). Hence the centres of the 3_{21}'s are just the mid-points of these edges, and are the projections on to (vi) of the vertices in one of the primes of symmetry parallel to it and adjacent in the system, *i.e.* another 3_{31}; this is, of course, in accordance with the known fact that one of the polytopes semi-reciprocal to 3_{31} is 3_{13}, which is the same thing. Since their centres form a lattice with regard to the symmetry group of the polytope, all the bounding 3_{21}'s are similarly placed; in fact, it is easy to see that the twenty-eight diagonals of each of them (*i.e.* joins of diametrically opposite vertices) are respectively perpendicular to the twenty-eight systems of principal primes of 3_{31}, since the vertices of any such 3_{21} are joined to either end of the edge of 5_{21} which is perpendicularly bisected by the prime (vi) in the centre of the 3_{21} in question, by fifty-six edges all inclined to this edge at an angle $\frac{1}{3}\pi$, and lying by pairs in twenty-eight planes through it, which planes are accordingly perpendicular to principal primes of 3_{31}, and contain the diagonals of 3_{21}.

The system of vertices lying in one of these principal primes is of some interest. Whereas in a system of parallel primes of symmetry of 5_{21} the systems of vertices in alternate primes project into each other orthogonally, and into the centres of the 3_{21}'s in the remaining primes of the system, in the case of a system of parallel principal primes of 3_{31} the systems of vertices, not in alternate primes, but in every third prime, project on to each other orthogonally. The system of vertices is known to be that of the polytope 2_{22}†, which is bounded entirely by 2_{21}'s, falling into two systems, those of one system being similarly placed, and oppositely placed to those of the other; the centres of these two systems of 2_{21}'s form the vertices of two other 2_{22}'s, semi-reciprocal to each other and to the given 2_{22}; and the vertices in a system of parallel principal primes of 3_{31} project orthogonally on to any parallel prime into the vertices of three semi-reciprocal 2_{22}'s in regular rotation as we go through the system of primes. This can be made clearer by considering that the principal prime of 3_{31} is (with regard to 5_{21}) a secundum of threefold rotational symmetry through which go three primes of symmetry meeting at equal angles. The whole systems to which these primes and secunda belong will trace on a plane perpendicular to them the lines and points shown in the annexed figure, where the points marked A, B, C indicate the traces of three sub-



systems of secunda, the vertices in all those of one sub-system projecting orthogonally into each other, and into the centres of one of the systems of similarly placed 2_{21}'s in a secundum of either of the other sub-systems; *i.e.* the systems of vertices in a secundum A, a secundum B, and a secundum C project on to any parallel secundum into the vertices of three semi-reciprocal 2_{22}'s.

† P.R.V.F., 9.8.

The centres of the bounding α_7's of either of two semi-reciprocal 3_{31}'s form the vertices of the third semi-reciprocal, 1_{33}; this represents the homaloidal webs based upon eight points in space, exactly as 1_{52} represents the homaloidal nets based upon nine points in a plane. This means, of course, that the centres of the α_7's in two adjacent parallel primes of symmetry of 5_{21} project orthogonally on to each other; the pairs of α_7's thus opposed to each other are opposite boundaries of β_8's, whose centres are midway between the two primes of symmetry, and which are, in fact, just those β_8's of 5_{21} whose centres are so placed; *i.e.*

The vertices of the polytope 1_{33} *are those vertices of* 2_{15} *which lie in a prime parallel to a system of primes of symmetry, and midway between two consecutive primes of the system.*

Now, by methods exactly similar to those of the previous section, we can obtain the defining conditions for 2_{15} in the present form, which can most conveniently be stated in terms of the residue and quotients of the coordinates (mod 2); *i.e.* we put

$$x_\alpha = s + 2t_\alpha, \tag{xii}$$

where $s = 0$ or 1 and does not involve α. The points corresponding to pencils of rational curve on Q, proper and improper, based on the given eight points, are then found to be those satisfying

$$s\Sigma(t) + \Sigma(t^2) \equiv 0 \pmod 4; \tag{xiii}$$

this gives the following possibilities:

		Parity of (t).	*Lattices inscribed in* 5_{21}	*Lattices inscribed in* 2_{15}
$s = 0$	$\Sigma(t) \equiv 2 \pmod 4$	All odd or all even	1	
		4 odd, 4 even		35
	$\Sigma(t) \equiv 0 \pmod 4$	All odd or all even		1
		4 odd, 4 even		35
$s = 1$	$\Sigma(t) \equiv 0 \pmod 4$	All odd or all even		1
		4 odd, 4 even		35
	$\Sigma(t) \equiv 2 \pmod 4$	6 odd, 2 even, or 2 odd, 6 even		28
Total number of lattices,			1	+ 135.

By singling out now those for which $\Sigma(x) = 0$, which is the equation of the prime parallel to and midway between the consecutive parallel primes of symmetry $\Sigma(x) = \pm 4$, we obtain the following:

		Parity of (t)	*Lattices inscribed in* 1_{33}
$s = 0$,	$\Sigma(t) = 0$	All odd or all even	1
		4 odd, 4 even	35
$s = 1$,	$\Sigma(t) = -4$	All odd or all even	1
		4 odd, 4 even	35
Total number of lattices,			72

These seventy-two lattices are, of course, 3_{31}'s, and fall into thirty-six pairs of semi-reciprocals, each such pair having the same parity for each t_a, but different values of s.

6. *The finite polytopes and the representation of θ-functions.*

(i) *The eight-dimensional polytopes.*

The vertex figure of 5_{21} is 4_{21}; this has 120 pairs of diametrically opposite vertices, the diagonals joining them being respectively parallel to (and in fact part of) the 120 systems of parallel edges of 5_{21}, and it has 120 primes of symmetry, namely that through the centre of the polytope in each of the 120 systems of parallel primes of symmetry of 5_{21}; these are accordingly perpendicular respectively to the 120 diagonals joining opposite vertices. Successive zones in primes parallel to a prime of symmetry are an isolated vertex, the 3_{21} of vertices adjacent to this, the 2_{13} semi-reciprocal to this, lying in the prime of symmetry itself, a second 3_{21}, reflection of the first, and finally a second isolated vertex. Next, consider the form of the vertices in the eight-coordinate form of the preceding section (the origin being at the centre of the polytope); they are either of the form $(\pm 2, \pm 2, 0, 0, 0, 0, 0, 0)$, which are the vertices of the polytope $t_1\beta_8$, or of the form $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$, an even number of the eight ambiguities being positive and an even number negative, which are the vertices of $h\gamma_8$. The $t_1\beta_8$ has sixteen bounding β_7's, which are bounding β_7's of 4_{21}, and each of which, together with the centre of 4_{21} and another vertex, forms a bounding β_8 of 5_{21} having its axes parallel to the coordinate axes; and these are the only β_8's thus oriented which have the origin as a vertex; consequently there are 135 equivalent

ways of introducing the coordinate system into 4_{21}, each giving a partition of the vertices of the polytope into those of $t_1\beta_8$ and $h\gamma_8$, corresponding respectively to the 135 systems of similarly oriented β_8's of 5_{21}, and the 135 lattices inscribed in 2_{15}. We shall refer to these 135 equivalent configurations as the β_8 configurations, and to the 120 configurations consisting of a prime of symmetry, a diagonal perpendicular to it, and the two 3_{21}'s and 2_{13} in parallel zones, as the diagonal configurations. Finally, consider the forms assumed by the vertices in the nine-coordinate representation of § 4; it is easily seen that these fall into a $t_{07}\alpha_8$, a $t_2\alpha_8$, and a $t_5\alpha_8$, all derived from the same α_8 (the t_2 and t_5 are, of course, identical except in their orientations, which are opposed). The $t_{07}\alpha_8$† has bounding α_7's, which are bounding α_7's of 4_{21}, and which fall into two sets; those of one set form with the centre of 4_{21} α_8's similarly placed to that from which the t_{07}, t_2, and t_5 are supposed to be derived; the others form with the centre α_8's oppositely oriented to this; and these are bounding α_8's of 5_{21}, and the only ones thus oriented which have the origin as a vertex; thus there are 960 ways of assigning coordinates in this form, giving 960 partitions of the vertices into those of $t_{07}\alpha_8$, $t_2\alpha_8$, and $t_5\alpha_8$, corresponding respectively to the 960 pairs of oppositely oriented systems of similarly oriented α_8's of 5_{21}, and 960 pairs of lattices inscribed in 1_{52}. These we shall call the 960 α_8 configurations.

4_{21} represents the directrices of eight points. Now the system of sextics having eight base points, all double, is well known, and traces on the plane the pairs of the Bertini involution‡; it represents the prime sections of a quartic surface in [3] consisting of an ordinary quadric cone, doubled or two-sheeted, a pair of companion points corresponding to a pair in the involution; the surface branches at the vertex of the cone, which is represented by the ninth point associated with the base points, and along a cubic section of the cone (which is a sextic of genus 4, and canonical), corresponding to the united curve of the involution. The cubics through the eight base points and the ninth point represent the generators of the cone, each of which has four branch points, the vertex and three points on the sextic, and is accordingly an elliptic curve. The 240 directrices of the base points represent conics on the surface, the only conics which it has; these occur by companion pairs in the 120 tritangent planes of the branch curve, other than the tangent planes of the cone; and it is easily seen that two companion conics correspond to directrices represented by diametrically opposite vertices of 4_{21}. Further, a symmetry

† $t_{0,n-1}\alpha_n$ is Coxeter's later (more systematical) nomenclature for what in P.R.V.F., 8, is called $e\alpha_n$.

‡ E. Bertini, *Annali di Mat.* (2), 7 (1877), 244.

operation of 4_{21} corresponds to a permutation of these conics among themselves which does not alter their geometrical relations; now a permutation among themselves of the tritangent planes of the general canonical sextic of genus 4, which leaves invariant all geometrical relations amongst them, corresponds, of course, to a permutation among themselves of the θ-characteristics† of genus 4 which leaves invariant the parity of each one, and all relations expressed by the vanishing (mod 2) of the sum of any even number of them; the odd θ-characteristics being in one-one correspondence with the tritangent planes. These operations constitute what we call the θ-characteristic group of genus 4. In the special case under consideration, where the quadric on which the curve lies is a cone, and where consequently one even θ-function vanishes at the origin, corresponding to the ∞^1 tritangent planes which are the tangent planes of the cone, the further geometrical relations between the planes introduced by the specialisation are such that the permutations which leave them invariant are those which keep fixed the particular even θ-characteristic corresponding to the function which vanishes at the origin, which may be taken without loss of generality to be [0]. There is, of course, one symmetry operation of 4_{21} which corresponds to the identical operation on the θ-characteristics, namely the central inversion or interchange of the vertices in diametrically opposite pairs; this leaves fixed each of the 120 diagonal configurations, 135 β_8 configurations, and 960 α_8 configurations; it corresponds in the plane to the interchange of the points in every pair of the Bertini involution, leaving fixed every pair as a whole, and on the two-sheeted cone to the interchange of the two sheets at every point. But every other symmetry operation of 4_{21} corresponds to a permutation of the tritangent planes leaving invariant their geometrical relations, and to an operation of the θ-characteristic group leaving fixed the θ-characteristic [0]. That is to say,

The factor group of the central inversion in the symmetry group of 4_{21} *is simply isomorphic with that sub-group of the* θ*-characteristic group of genus* 4 *which leaves fixed the even* θ*-characteristic* [0], *the* 120 *diagonal configurations being invariantly in one-one correspondence with the* 120 *odd* θ*-characteristics.*

One effect of leaving fixed the characteristic [0] is to obscure the distinction between θ-characteristics and period characteristics, since there is now an invariant one-one relation between the θ-characteristic $[\epsilon]$ and the period characteristic $(\epsilon) = [\epsilon] + [0]$. The group in question (iso-

† For nomenclature in connection with θ-functions, sets of characteristics, etc., see, *e.g.*, Krazer and Wirtinger, *Encyk. Math. Wiss.*, II, B 7.

morphic with the factor group of the central inversion in the symmetry of 4_{21}) accordingly permutes the characteristics of either kind amongst themselves in such a way as to leave invariant

the even characteristic [0],

the parity of each one,

the mutual syzygy or asyzygy of every two,

and all relations expressed by the vanishing (mod 2) of the sum of any number of characteristics.

Now the tracing in detail of the configurations inscribed in 4_{21} which correspond to various known configurations of tritangent planes, and sets of θ-characteristics, is laborious and straightforward, and could be expatiated upon almost without end. We shall confine ourselves to stating some of the more essential results without proof.

The 120 diagonal configurations correspond, as we have said, to the 120 odd characteristics. The 135 β_8 configurations correspond to the 135 even characteristics, other than [0]. Three odd characteristics whose sum is zero correspond to three tritangent planes of the curve whose points of contact are the nine intersections of the curve with a rational cubic lying on the cone, and to three diagonals of 4_{21}, which are the diagonals of a regular plane hexagon inscribed in it, the corresponding primes of symmetry being a triad such as we have described, meeting at equal angles in a secundum of threefold rotational symmetry. Corresponding to each period characteristic there is a system of twenty-eight pairs of odd θ-characteristics having the given characteristic as sum; if the period characteristic is odd, this system is represented by twenty-eight hexagons having the corresponding diagonal in common, two characteristics which form a pair in the system corresponding to the two diagonals, other than the given one, of such a hexagon; the remaining vertices of these hexagons are those of the two 3_{21}'s of the diagonal configuration, and their planes contain respectively the twenty-eight diagonals of each of these 3_{21}'s; if, on the other hand, the period characteristic is even, the fifty-six odd θ-characteristics forming these pairs correspond to the diagonals of the $t_1\beta_8$ in the corresponding β_8 configuration, two which form a pair corresponding to the two mutually perpendicular diagonals which lie in one of the twenty-eight principal planes of β_8†. The sum of the odd character-

† Of course, by principal axes, planes, etc., of a polytope β_n or γ_n, are meant those of the appropriate rectangular Cartesian frame, in which the vertices of β_n have the form $(\pm 1, 0, \ldots, 0)$ and those of γ_n $(\pm 1, \pm 1, \ldots, \pm 1)$.

istics corresponding to two diagonals is even or odd, *i.e.* the corresponding period characteristics are syzygetic or asyzygetic, according as the diagonals are mutually perpendicular or oblique (and according as the rational cubic through the points of contact of the corresponding tritangent planes does not or does lie on the quadric cone containing the curve). Further, to every period characteristic corresponds an operation of the θ-characteristic group, consisting of the interchange of any two θ-characteristics of like parity which have the given characteristic as sum, the rest being left fixed; if the period characteristic is odd, this operation corresponds to the reflection in the prime of symmetry of the corresponding diagonal configuration; if it is even, the operation does not leave [0] fixed, and consequently corresponds to no symmetry operation of 4_{21}.

The 960 a_8 configurations correspond to 960 *fundamental systems* of ten even θ-characteristics of which [0] is one; where by a fundamental system is understood a maximum set of θ-characteristics of which every three are asyzygetic, and whose sum is zero, and such that every θ-characteristic is expressible in just two ways as the sum of an odd number of characteristics of the system, and every period characteristic in two ways as the sum of an even number. In the case in which the characteristics of the set are all even, and [0] is one of them, every odd characteristic is the sum of three of the ten, *i.e.* of either three or two of the nine other than [0]. This is just what is expressed by the nine-coordinate form of the polytope, since, if the nine coordinates and nine vertices of the a_8 are made to correspond to the nine characteristics of the system, other than [0], then each vertex of $t_2 a_8$, with the diametrically opposite vertex of $t_5 a_8$, is seen to involve three of the nine, and the eighty-four diagonals so arising represent the sums of threes of the nine characteristics; while each diagonal of $t_{07} a_8$ involves two of the nine, and the thirty-six diagonals of 4_{21} so arising represent the sums of pairs of the nine characteristics.

It is natural to investigate whether any such simple coordinate form of the polytope can be found related to a fundamental system of which [0] is not a member. In this case, if [1], ..., [X] are the ten characteristics of the system (where X is merely introduced as a single symbol in place of 10, to avoid confusion), there is a relation

$$[12345] = [6789\text{X}] = [0]$$

(sums being, in accordance with the usual notation for θ-characteristics, written as though they were products). We can take coordinates $x_1, \ldots, x_\text{X}$ in ten-dimensional space, and (letting $\alpha\beta\gamma\delta\epsilon$ be a permutation of 12345, *ijklm* of 6789X) represent the odd characteristic $[\alpha\beta\gamma]$ by the pair

of vertices

$$x_\alpha = x_\beta = x_\gamma = 0, \quad -x_\delta = x_\epsilon = 1; \qquad x_6 = x_7 = x_8 = x_9 = x_X = 0;$$

$$x_\alpha = x_\beta = x_\gamma = 0, \quad x_\delta = -x_\epsilon = 1; \qquad x_6 = x_7 = x_8 = x_9 = x_X = 0;$$

$[ijk]$ by the corresponding thing with the two pentads of coordinates interchanged; $[\alpha ij]$ by the pair

$$x_\alpha = 4, \quad x_\beta = x_\gamma = x_\delta = x_\epsilon = -1; \qquad x_i = x_j = 3, \quad x_k = x_l = x_m = -2;$$

$$x_\alpha = -4, \quad x_\beta = x_\gamma = x_\delta = x_\epsilon = 1; \qquad x_i = x_j = -3, \quad x_k = x_l = x_m = 2;$$

and $[\alpha\beta i]$ by the corresponding thing obtained by interchanging the two pentads of coordinates, and changing the signs of all those in one pentad. The points having these coordinates are the vertices of 4_{21}. By adding all combinations of these (since 5_{21} is a lattice) we obtain the following simple defining relations for 5_{21} in terms of the given coordinate system:

$$\left.\begin{aligned} \Sigma_\alpha(x) = \Sigma_i(x) &= 0, \\ 2x_\alpha \equiv x_i \pmod 5 &\end{aligned}\right\} \qquad \text{(i)}$$

(where $\alpha = 1, 2, 3, 4, 5$; $i = 6, 7, 8, 9, X$).

It is also possible to have fundamental systems consisting of two even and eight odd characteristics. In this case, when one of the two even characteristics is [0], the eight odd characteristics correspond to diagonals of 4_{21} whose ends form a pair of diametrically opposite α_7's; these α_7's belong to a unique $h\gamma_8$ inscribed in 4_{21}, that namely in the β_8 configuration corresponding to the even characteristic, other than [0], of the system. If the coordinate system proper to this β_8 configuration is adjusted by the change of sign of some of its coordinates (an odd number of them), the two α_7's take the forms

$$(1, -1, -1, -1, -1, -1, -1, -1), \quad (-1, 1, 1, 1, 1, 1, 1, 1)$$

in all possible permutations of the coordinates. These we may call the odd characteristics [1], ..., [8]. Every other characteristic is expressible in two ways as sums of some of the nine characteristics of the system, other than [0], and one of these two sums will include the other even characteristic, $[\epsilon]$ say, corresponding to the β_8 configuration; thus every characteristic is expressible uniquely as a sum of the eight odd characteristics; the sum of three, four, seven, or all of them being even, and the sum of one, two, five, or six of them being odd. Now we can take one of the α_7's to be that corresponding to the neighbourhoods of the eight base points, and let the numerical symbol assigned to the base point correspond

to that assigned to the θ-characteristic and the coordinate in the space of the polytope. We then find that the directrices correspond to the vertices as follows:

Directrices of eight points.	*Vertices of* 4_{21}.
0 (−1, 0, 0, 0, 0, 0, 0, 0)	(1, −1, −1, −1, −1, −1, −1, −1)
1 (1, 1, 0, 0, 0, 0, 0, 0)	(−2, −2, 0, 0, 0, 0, 0, 0)
2 (1, 1, 1, 1, 1, 0, 0, 0)	(−1, −1, −1, −1, −1, 1, 1, 1)
3 (2, 1, 1, 1, 1, 1, 1, 0)	(−2, 0, 0, 0, 0, 0, 0, 2)
4 (1, 1, 1, 1, 1, 2, 2, 2)	(1, 1, 1, 1, 1, −1, −1, −1)
5 (1, 1, 2, 2, 2, 2, 2, 2)	(2, 2, 0, 0, 0, 0, 0, 0)
6 (3, 2, 2, 2, 2, 2, 2, 2)	(−1, 1, 1, 1, 1, 1, 1, 1)

which can be summed up in the relation

$$x_a = m - 1 - 2i_a. \qquad \text{(ii)}$$

And it is then easily seen that, whereas the vertices in the first and last lines (corresponding to directrices of order 0 and 6) have been made to correspond to the individual odd characteristics of the fundamental system, those of the middle line (corresponding to cubic directrices) correspond to the sums by pairs of these characteristics, those of the third and fifth lines (corresponding to conic and quartic directrices) correspond to their sums by fives, and those of the second and sixth lines (corresponding to linear and quintic directrices) to their sums by sixes.

A Göpel group of sixteen period characteristics [that is, a set of sixteen, including (0), which contains the sum of any two of its members, and of which every two are syzygetic] consists either entirely of even characteristics or of eight odd and eight even. In the latter case, the eight odd characteristics correspond to eight mutually perpendicular diagonals and primes of symmetry, and it is obviously proper to inquire if the polytope assumes a simple form when referred to these diagonals as axes of coordinates. If $[1], \ldots, [8]$ are the eight odd characteristics and $[a], \ldots, [g]$, $[0]$ the eight even, then there are seven relations of which

$$[a] = [bd] = [ef] = [gc] = [18] = [24] = [56] = [73] \qquad \text{(iii)}$$

is typical, the other six being obtained from it by cyclic permutation of the symbols $[a], \ldots, [g]$, and of the symbols $[1], \ldots, [7]$. As regards the even characteristics only, this can be expressed by the vanishing of the triad

sum $[abd]$ and of the six others that can be obtained from it by cyclic permutation, each such triad occurring three times in the seven continued equations. As regards the odd characteristics, we can eliminate the even characteristics from (iii) and obtain three times over each of seven relations of which

$$[1248] = [3567] = [0] \qquad \text{(iv)}$$

is typical, the others again being obtained by cyclic permutation of the first seven coordinates.

If then we take the diagonals of 4_{21} corresponding to the odd characteristics $[1], \ldots, [8]$ as axes of coordinates $x_1, \ldots, x_8$, and the ends of these diagonals to be of the form $(\pm 2, 0, 0, 0, 0, 0, 0, 0)$, etc., we find that the remaining vertices fall into fourteen γ_4's, namely

$$(\pm 1, \pm 1, 0, \pm 1, 0, 0, 0, \pm 1) \quad \text{and} \quad (0, 0, \pm 1, 0, \pm 1, \pm 1, \pm 1, 0),$$

and those derived from them by cyclic permutation of the first seven coordinates.

Finally, this gives the defining relations for 5_{21}:—eight coordinates, either all of like parity or four odd and four even, the possible divisions into tetrads being those appearing in the seven relations of which (iv) is typical.

7. *The finite polytopes and the representation of θ-functions.*

(ii) *The seven-dimensional polytopes.*

Turning next to the seven-dimensional finite polytopes, we have to deal with 3_{21} as vertex figure of 4_{21}, as boundary of 3_{31}, and as representing the directrices of seven points in a plane; and with its semi-reciprocal 2_{13} as vertex figure of 3_{31}, as lying in a prime of symmetry of 4_{21}, and as representing the director surfaces of seven points in space. In both 4_{21} and 3_{31} the two polytopes occur in a manner which makes clear that their symmetry is of the same kind; for those symmetry operations of 4_{21} which leave fixed the ends of a particular diagonal are clearly the symmetry operations of the 2_{31} and two 3_{21}'s in primes perpendicular to it; while the symmetry operations of 3_{31} which leave fixed a particular vertex are clearly the symmetry operations of the 2_{31} of vertices adjacent to this, and of the bounding 3_{21} of the semi-reciprocal 3_{31} of which this point is the centre. In each case the two polytopes are placed concentrically, and rigidly combined into a single configuration, the vertices of 2_{31} projecting from the centre into the centres of bounding β_6's of 3_{21}, and the vertices of 3_{21} into the centre of bounding 2_{21}'s of 2_{31}. The remaining boundaries

of each figure are α_6's, those of the one being (or projecting from the centre of the whole figure into α_6's which are) respectively concentric with and oppositely oriented to those of the other, their centres being the vertices of the third semi-reciprocal 1_{32}. This, of course, determines a one-one correspondence between the homaloidal nets based on seven points in a plane and the punctual homaloidal webs based on seven points in space. The α_6 of 3_{21} whose vertices correspond to the neighbourhoods of the base points in the plane may be taken to be that which is concentric and oppositely oriented to that of 2_{31} whose vertices represent the neighbourhoods of the base points in space, so that there is a one-one correspondence between the two sets of base points. We can, in fact, relate the homaloidal net $m(i)$ in the plane to the homaloidal web $f(j)$ in space by the relations

$$\left.\begin{aligned} j_\alpha &= 2(m-i_\alpha-1) \\ f &= 2m-1 \end{aligned}\right\}, \qquad \text{(v)}$$

when it is at once found that the equations of condition for the two systems, namely § 4 (vi) and (vii) (with $\epsilon = 3$) and § 5 (ix) and (x), translate directly into each other, so that every homaloidal web corresponds to a homaloidal net, and conversely. (*This is the case only when the number of base points in each configuration is seven.*) We have also the relation

$$2ff'-\Sigma(jj') = 4\,[mm'-\Sigma(ii')]-2,$$

showing that the number of free intersections of the curve of intersection of the general members of two of the webs with the general quadric through the base points is a definite function of the number of free intersections of the general members of the corresponding nets; as we should expect, since these are the invariants which are represented in the two cases by the distance between the corresponding vertices of 1_{32}.

We have, in the combined figure of 3_{21}, 2_{31}, and 1_{32}, twenty-eight lines joining opposite vertices of 3_{21}, and perpendicular to a pair of opposite 2_{21}'s of 2_{31}; perpendicular to each such line, through the centre, passes a prime which, regarded as in the space of 4_{21}, 2_{31} being in a prime of symmetry, is a secundum of threefold rotational symmetry, through which go three primes of symmetry at equal angles; if we regard 2_{31} as vertex figure of 3_{31}, this prime is that one through the vertex at which the vertex figure is taken, of a pencil of parallel principal primes; we shall call it a principal prime of 2_{31} or of 3_{21}. The vertices of 2_{31} lying in it are the vertex figure of 2_{22}, namely 1_{22}, the third semi-reciprocal to the two 2_{21}'s in primes parallel to it on either side (which are, of course, oppositely oriented). The zones of 3_{21} in primes parallel to this are a vertex, a 2_{21} of vertices

adjacent to this, another oppositely oriented to it, and a second single vertex. The configuration of parallel primes containing these semi-reciprocal figures, and the diagonal of 3_{21} perpendicular to them, we shall call a principal configuration of 3_{21} or of 2_{31}.

A diagonal of 2_{31}, on the other hand, is perpendicular to a prime of symmetry of the figure; the successive zones in primes perpendicular to it are β_6, $h\gamma_6$ (in the prime of symmetry), and β_6 (reflection of the first) in 3_{21}, and in 2_{31} a vertex, $h\gamma_6$ (residual to that of 3_{21}, *i.e.* such that the two placed together concentrically make up γ_6), $t_1\beta_6$ (in the prime of summetry), $h\gamma_6$ (reflection of the former), and a second isolated vertex. This group of figures in parallel primes, and the diagonal of 2_{31} perpendicular to them, we shall call a symmetry configuration of the polytope.

The manner in which the principal configurations and symmetry configurations occur in 3_{21} and 2_{31}, and their relations to the primes of symmetry of 4_{21}, are illustrated by the following figures, in which the plane of the paper is supposed to be that of a hexagon inscribed in 4_{21}, and a principal plane of one of its β_8 configurations respectively:

$$\begin{array}{ccccc} & & V & & \\ & & & & \\ V & & 2_{21} & & 2'_{21} & & V \\ & & & & \\ & 2'_{21} & & 1_{22} & & 2_{21} & \\ & & & & \\ V & & 2_{21} & & 2'_{21} & & V \\ & & & & \\ & & V & & \end{array} \qquad \begin{array}{ccccccc} & & & V & & & \\ & & & & & & \\ & & \beta_6 & & h\gamma_6 & & \beta_6 \\ & & & & & & \\ V & & h'\gamma_6 & & t_1\beta_6 & & h'\gamma_6 & & V \\ & & & & & & \\ & & \beta_6 & & h\gamma_6 & & \beta_6 \\ & & & & & & \\ & & & V & & & \end{array}$$

In each case V indicates a vertex of 4_{21} lying in the plane; the remaining symbols mark the traces on the plane of secunda—[6]'s, that is—absolutely perpendicular to it, containing the polytopes named; 2_{21} and $2'_{21}$ indicate two semi-reciprocal (oppositely oriented) 2_{21}'s; $h\gamma_6$ and $h'\gamma_6$ two residual $h\gamma_6$'s. In each case a diagonal of 4_{21} is vertical, so that the third horizontal row contains a 2_{31} (being the trace of a prime of symmetry) while those on either side contain 3_{21}'s divided into zones in the first figure in accordance with a principal configuration, in the other in accordance with a symmetry configuration.

A further type of configuration, of importance in the study of these polytopes, arises if 5_{21} is taken in the form § 5 (iv) and (v), with the origin moved to a point of the lattice, and if the 3_{21} considered be that consisting of vertices adjacent to the origin and to (1, 1, 1, 1, 1, 1, 1, 1); these are a $t_1\alpha_7$ given by the coordinates (1, 1, 1, 1, 1, 1, −1, −1) in every

permutation, and another given by (0, 0, 0, 0, 0, 0, 2, 2) in every permutation; by a further change of origin to the centre of this, and clearing of fractions by doubling throughout, these assume the forms (1, 1, 1, 1, 1, 1, −3, −3) and (−1, −1, −1, −1, −1, −1, 3, 3), and are seen to be oppositely oriented, *i.e.* to be t_1 and t_5 of the same α_7. Referred to the same origin and axes, 2_{31} breaks up into a $t_{06}\alpha_7$ whose vertices are of the form (0, 0, 0, 0, 0, 0, 2, −2) in every permutation, and a $t_3\alpha_7$ whose vertices are of the form (1, 1, 1, 1, −1, −1, −1, −1). These figures inscribed in 3_{21} and 2_{31} we shall speak of as an α_7 configuration. As in the case of the $t_{07}\alpha_8$ in the α_8 configuration of 4_{21}, the $t_{06}\alpha_7$ here has two sets each of eight bounding α_6's, which with the centre of the polytope make up α_7's similarly oriented to the two reciprocal α_7's from which the figure is obtained by the t operation. They are bounding α_7's of 3_{31}, and are the only ones oriented in these two opposite ways which have the centre of 2_{31} for a vertex; thus the α_7 configuration corresponds to a pair of oppositely oriented systems of similarly oriented α_7's of 3_{31}, and to a pair of semi-reciprocal 3_{31}'s inscribed in 1_{33}.

The configurations of seven points in a plane and of seven points in space both lead to the consideration of θ-functions of three variables, by means of the general plane quartic curve; the former, of course, gives the Geiser involution†, in which each pair together with the seven points are nine associated points; the cubics through the seven points represent the prime sections of what we may call the Geiser surface, the projection of the general cubic on to a plane from a general point of itself, a double plane branching along a general quartic curve; companion points of the double plane correspond to a pair of the Geiser involution; and two directrices residual with respect to cubics through the base points represent a pair of companion lines on the Geiser surface; there are twenty-eight such pairs, lying in the twenty-eight bitangents of the quartic branch curve, seven represented by the neighbourhood of a base point and the cubic passing twice through that and once through every other base point, and twenty-one represented by the line joining two base points and the conic through the remaining five. In the projection of the cubic surface, one pair of lines comes from the neighbourhood of the projecting point and the section by the tangent plane there, each of the others from a line on the surface and the conic coplanar with it through the projecting point. Companion lines on the surface correspond to diametrically opposite vertices of 3_{21}, so that the pairs of opposite vertices of 3_{21}, *i.e.* the twenty-eight principal configurations of the polytopes, correspond to the

† Geiser, *Journal für Math.*, 67 (1867), 78.

twenty-eight bitangents of the general plane quartic and to the twenty-eight odd θ-characteristics of genus 3. The central inversion of the polytope corresponds to the interchange of the two points in every pair of the Geiser involution, *i.e.* to the interchange of companions on the double plane, leaving the branch curve fixed; so that this symmetry operation on the polytopes corresponds to the identical permutation of the β-characteristics. The polytopes being this time in space of an odd number of dimensions, the central inversion is of the nature of a reflection, and the factor group of the central inversion in the whole symmetry group is the group of rotational symmetry, the symmetry group being the direct product of the sub-groups consisting respectively of identity and the central inversion, and of the rotational symmetries; each of these sub-groups is self-conjugate, and the factor group of the other. Thus we see that

The group of rotational symmetry of the polytopes 3_{21}, 2_{31}, *and* 1_{32} *is simply isomorphic with the* θ*-characteristic group of genus* 3, *the twenty-eight principal configurations corresponding to the twenty-eight odd* θ*-characteristics.*

A period characteristic corresponds to the linear series of points of contact of ∞^1 quadritangent conics of the quartic; these occupy the place of a pair of companion pencils of conics on the Geiser surface, and correspond to a pair (residual with respect to sextics passing twice through all the base points) of pencils of rational curves based on the base points, and hence to a pair of opposite β_6's of 3_{21}, to a pair of opposite vertices of 2_{31}, and to a symmetry configuration of the compound figure. That is, as we should expect,

The sixty-three symmetry configurations of 3_{21}, 2_{31}, *and* 1_{32} *correspond to the sixty-three period characteristics of genus* 3. *The operation corresponding to each period characteristic, which interchanges every two* θ*-characteristics of like parity which have the period characteristic as sum, and leaves the rest fixed, corresponds to the reflection*† *in the prime of symmetry in the corresponding symmetry configuration: and the Steinerian complex, or six pairs of bitangents which form conics in the corresponding quadritangent system (whose odd* θ*-characteristics have the period characteristic as sum), correspond to the six planes joining the diagonal of* 2_{31} *to the principal axes of the* β_6 *on*

† Strictly speaking, the operation in the group of *rotational* symmetry of the polytope, corresponding to this operation in the θ-characteristic group, is the product of the reflection in the prime of symmetry with the central inversion, *i.e.* it is the inversion in the axis perpendicular to the prime.

which the successive zones in this configuration are based, each of which planes contains two diagonals of 3_{21} *equally and oppositely inclined to that of* 2_{31}.

The operation represented by reflection in a prime of symmetry has the effect on the period characteristics of interchanging every two which are asyzygetic, and which have the given one as sum, leaving the rest fixed. It follows that

Two primes of symmetry which are perpendicular correspond to syzygetic period characteristics; two which are inclined at the angle $\frac{1}{3}\pi$ *represent asyzygetic period characteristics; and three which meet at this angle in a secundum of threefold rotational symmetry (perpendicular to the plane of a hexagon inscribed in* 2_{31}*) correspond to three period characteristics whose sum is zero, and of which every pair is asyzygetic.*

The similarity of these results (being only somewhat more general in form, owing to the generality of the quartic curve in question) to the relations found for 4_{21} is obvious.

It remains to find an interpretation for the thirty-six a_7 configurations. It is natural to expect that these will correspond to the thirty-six even θ-characteristics, and this is, in fact, easily shown to be the case. The notation for the odd characteristics by the pairs of eight symbols, which singles out one even θ-characteristic and assigns to the rest the thirty-five bifid symbols $(\alpha\beta\gamma\delta : \epsilon\zeta\eta\theta)$, or partitions of the eight symbols into tetrads, is associated with a set of eight fundamental systems each containing seven odd and one even θ-characteristics. The seven odd are those of the form $a\mu$, where μ is a fixed one of the eight symbols, and a is each of the others in turn; and the even is that which is singled out by the notation; and the relation to the notation is that, if the symbol $[a]$ is given to the odd θ-characteristic $a\mu$, and $[\mu]$ to the even characteristic of the system, then the triple sum $[\alpha\beta\mu]$ gives the odd characteristic called by the pair symbol $\alpha\beta$, whether μ be one of $\alpha\beta$ or not, and the pentad sum

$$[\alpha\beta\gamma\delta\mu] = [\epsilon\zeta\eta\theta\mu]$$

gives the even characteristic associated with the bifid $(\alpha\beta\gamma\delta : \epsilon\zeta\eta\theta)$, to whichever tetrad μ may belong. The sixty-three period characteristics are obtained by adding $[\mu]$ to each of these θ-characteristics except $[\mu]$ itself, and are consequently twenty-eight of the form $(\alpha\beta)$ and thirty-five of the form $(\alpha\beta\gamma\delta) = (\epsilon\zeta\eta\theta)$. Now the a_7 configuration clearly gives the pairs of opposite vertices of 3_{21} in a form depending on the pairs of the eight suffixes $1, \ldots, 8$ of the coordinates; and the pairs of opposite vertices of 2_{31} fall into those of $t_{06}\, a_7$, depending on pairs of the eight suffixes, and

those of $t_3 a_7$, depending on the separation of them into tetrads. This arrangement evidently is just what is required to make the a_7 configuration correspond completely to the eight symbol notation for the characteristics, which, as we have seen, depends on the arbitrary selection of a particular even θ-characteristic. The Aronhold seven-systems (or heptads of odd characteristics which, with the given even one, make up fundamental systems), consisting of all the odd θ-characteristics corresponding to pairs having a fixed symbol in common, are represented by the pairs of opposite a_6's of the $t_1 a_7$ and $t_5 a_7$. To sum up:

The thirty-six a_7 configurations correspond to the thirty-six even θ-characteristics, the eight Aronhold seven-systems of odd characteristics which have a given even characteristic as sum being represented by the eight pairs of opposite bounding a_6's in the a_7 configuration.

Finally, it is natural to inquire whether the figure can have as many as seven mutually perpendicular primes of symmetry; clearly, being in seven dimensions, it cannot have more. The answer is evidently in the affirmative, the seven symmetry configurations corresponding to the seven period characteristics, other than (0), of a Göpel group of order 8; such a Göpel group divides the θ-characteristics into eight group-sets (or sets in each of which the eight characteristics are obtainable from any one of themselves by addition of the eight period characteristics of the group), of which one consists entirely of even characteristics, and the other seven half of even and half of odd; the four odd characteristics in such a group-set forming a group-set for one of the seven sub-groups of order 4 in the Göpel group, the seven group-sets which have some odd members being thus related respectively to the seven sub-groups of order 4 in the group. Now a group-set of four odd characteristics, corresponding to a Göpel group of order 4, is what is sometimes called a syzygetic tetrad, or set of four odd θ-characteristics of zero sum, and corresponds to a set of four bitangents whose points of contact are on a conic. It is easily seen that such a tetrad is represented by the four diagonals of a cube of (metrical) unit edge, concentric with and inscribed in 3_{21}; thus the Göpel group corresponds to an analysis of the vertices of 3_{21} into those of seven equal and concentric cubes. This is exactly what is achieved by taking the vertices of 4_{21} in the form referred to eight mutually perpendicular primes of symmetry, and taking the 3_{21} of vertices adjacent to $(0, 0, 0, 0, 0, 0, 0, 2)$; these all have $x_8 = 1$, so that the eighth coordinate can be neglected; it is easily seen that the vertices are those of the cube $(\pm 1, \pm 1, 0, \pm 1, 0, 0, 0)$, and six others obtained from it by the cyclic permutation of the suffixes $1, \ldots, 7$, and that the coordinate primes are primes of symmetry.

The relation of the figure of seven points in space to the quartic curve and the θ-functions of three variables can be established independently by means of the involution of Kantor† and the figure of Hesse‡. The former is analogous to the involution of Bertini in the plane, in the same way as the Geiser involution in space is to that in the plane; it is, in fact, easy to see that the surfaces $4(2, 2, 2, 2, 2, 2, 2)$ determine an involution, the free intersections of any three of them being four pairs in the involution variable independently. The projective model of this system of surfaces is accordingly a quartic threefold locus counted twice, and is, in fact, the cone projecting a Veronese surface from a point outside its space, doubled and branching along a certain surface, which is its intersection with a suitable cubic primal, and having also an isolated branch point at the vertex of the cone. This locus we may call the Kantor threefold, and the surface we shall call the Kantor surface§. The ∞^2 quadrics through the seven points represent the ∞^2 quadric cones on the cone projecting the Veronese surface, which, doubled and branching along cubic sections of themselves, form Bertini surfaces on the Kantor locus; and the elliptic quartic curves, which are the characteristic curves of this system of quadrics, represent the generators of the cone, doubled and branching in four points, the vertex and three points of the branch surface. The eighth associated point L_8 to the base points, through which all these pass, of course corresponds to the vertex of the cone. The analogy in all this to the Bertini involution is evident. Now of the elliptic quartics through eight associated points, as is well known, twenty-eight break up into the line joining two of the points and a rational cubic through the remaining six; and these are related in a simple way to the twenty-eight bitangents of a plane quartic, which is birationally equivalent to the sextic locus of vertices of cones amongst the ∞^2 quadrics, which is also the locus of nodes of nodal (and hence rational) quartics amongst the ∞^2 quartics; this curve, in fact, goes through the two intersections of the line and cubic which form such a degenerate quartic, and when the curve is represented by a plane quartic these two points correspond to the points of contact of a bitangent. Of the two parts of such a degenerate quartic, one (that which does not pass through L_8) represents the neighbourhood of a point only on the Kantor locus, which is a singular point of the branch surface; the other part represents the generator of the cone which passes through this node of the branch surface, two of its four branch points coinciding in the

† Kantor, *American J. of Math.*, 19 (1897), 1, 382.

‡ O. Hesse, *Journal für Math.*, 49 (1855), 279.

§ Some account of this surface, or more strictly of that related to it in the same way as the Weddle surface is to that of Kummer, will be found in Coble, *Algebraic geometry and theta-functions* (1929), §§ 44, 47.

node. The sextic curve of Hesse (the locus of vertices of cones) represents the locus of points in which the branch surface is touched by generators of the cone, and is thus the apparent contour of the branch surface from the vertex of the cone; from the vertex then, the branch surface projects into the Veronese surface which is the base of the cone, three-sheeted, and branching along the curve which corresponds to the sextic of Hesse; if the Veronese surface is represented in the ordinary way on a plane, it is easily seen that the branch curve is of the fourth class, and is, in fact, the dual of the quartic curve which is its canonical representative. The twenty-eight nodes of the class quartic, like the general points of the curve, are branchings affecting only two sheets of the triple plane, and (like nodes on the branch curve of a double plane) give conical nodes of the surface, corresponding to the twenty-eight singular points which we have seen to exist on the surface in six dimensions; at the twenty-four cusps of the class-quartic, however, the three sheets are all involved in the branching.

A directrix of the base points corresponds to a Veronese surface on the Kantor threefold locus; two directrices which are residual with respect to the quartics determining the involution (and which correspond to diametrically opposite vertices of 2_{31}) represent a pair of companion Veronese surfaces, forming together the section of the locus by a prime which touches the branch surface all along the common curve of the two surfaces, a normal elliptic sextic. This prime and its curve of contact pass through twelve of the twenty-eight nodes of the surface, which form six pairs in an involution on the curve; and the six pairs of bitangents of the plane quartic corresponding to these six pairs of nodes are the six pairs of a Steinerian complex. It thus appears that the sixty-three diagonals of 2_{31} (the symmetry configurations) correspond to the sixty-three Steinerian complexes of bitangents, *i.e.* to the period characteristics, and the relation between the polytopes and the θ-functions is the same as that derived from the plane figure of seven points.

The branch surface of the Kantor locus is, in fact, seen to be the section by one of its singular primes of the Wirtinger θ-locus† of genus 3, *i.e.* the locus related to θ-functions of three variables in the same way as the Kummer surface is to those of two variables. This locus has sixty-four singular points and sixty-four singular primes, to which characteristics can be attributed in such a manner that a point lies in a prime if the sum of their characteristics is odd, *i.e.* the two sets of characteristics are such that the sum of two out of different sets behaves (under the invariant transformations of the locus) like a θ-characteristic, and the sum of two

† W. Wirtinger, *Untersuchungen über Thetafunktionen* (1896).

of the same set (consequently) like a period characteristic. The order of the locus is 24; each singular prime touches it along a surface of order 12; this has twenty-eight nodes, which are those of the sixty-four singular points whose characteristics have odd sums with that of the prime; and sixty-three singular primes, touching it along sextic curves, which are the traces on it of the other sixty-three primes; thus the surface has, as we have seen, twenty-eight nodes corresponding to the twenty-eight odd θ-characteristics, and sixty-three singular primes corresponding to the sixty-three period characteristics, other than (0).

It is clear that the figure in three dimensions is rendered more symmetrical if we consider the system of director surfaces of the eight *associated* points $L_1, \ldots, L_8$, which, like those of eight general points, are represented by the infinite polytope 3_{31}. The twenty-eight principal configurations correspond to the twenty-eight quartics through the base points which break up into a line and a cubic. Now every director surface meets the general quartic through the base points once outside them, and consequently meets one and not the other of the two parts into which one of the twenty-eight quartics breaks up. It can easily be found that vertices of 3_{31}, representing surfaces which meet the same one of such a pair of curves, lie in alternate members of the corresponding system of principal primes, the vertices in the other primes of the system representing director surfaces which meet the other component curve. The translational symmetry operations of the polytope keep each principal and symmetry configuration fixed as a whole, but may interchange among themselves the two sets of alternate primes in a principal configuration; for instance, the unit translation along a system of parallel edges (which are perpendicular to a symmetry configuration) interchanges the two sets of primes in each principal configuration oblique to the symmetry configuration and leaves fixed those in each of the others. This corresponds to a Cremona transformation based on the eight points, which transforms into itself as a whole each of the twenty-eight degenerate quartics, but interchanges the two components in twelve of them (those, namely, which correspond to the twelve bitangents of a Steinerian complex, corresponding to the symmetry configuration). This can conveniently be illustrated by the product of the transformations by means of the homaloidal nets 3 (2, 2, 2, 2, 0, 0, 0, 0) and 3 (0, 0, 0, 0, 2, 2, 2, 2), which correspond to the reflections in two consecutive primes of the symmetry in the system corresponding to the symbol (1234 . 5678); their product corresponds accordingly to the unit translation perpendicular to these primes; and it is easily seen to interchange the two members of each of the degenerate quartics such as [12] or [56], while leaving unchanged those such as [15].

By combining operations of this kind we see that

Any translational symmetry operation of 3_{31} *corresponds to a Cremona transformation based on the eight associated points which leaves unaltered, as a whole, each of the twenty-eight degenerate quartics, but interchanges the two components, either in twelve of them, corresponding to a Steinerian complex of bitangents, or in sixteen, corresponding to all the bitangents except a Steinerian complex, or in all or in none of them.*

It may be remarked that these operations, from this point of view, form an Abelian group of order 2^7, which is just right, since they form a translation group in space of seven dimensions, and each translation is regarded from this point of view as cyclic of period 2.

8. *The finite polytopes and the representation of θ-functions.*

(iii) $h\gamma$ *and the hyperelliptic functions.*

The director surfaces of six points in space are represented by the vertices of $h\gamma_6$, which is the vertex figure of 2_{31}, and is what lies in a prime of symmetry of 3_{21}. Based on the six points is the Geiser involution, each of whose pairs makes up with the six base points eight associated points. The projective model of the system of quadrics 2 (1, 1, 1, 1, 1, 1) is a double three-dimensional space, branching along the Kummer surface. The pairs of directrices which together form quadrics of the system—namely the neighbourhood of a base points and a quadric cone 0 (−1, 0, 0, 0, 0, 0) and 2 (2, 1, 1, 1, 1, 1), and a pair of planes 1 (1, 1, 1, 0, 0, 0) and 1 (0, 0, 0, 1, 1, 1)—represent plane sections of the double locus which break up into two separate planes, meeting along a conic along which each of them touches the branch surface; there are sixteen such planes, or tropes of the Kummer surface, corresponding to the sixteen pairs of opposite vertices of $h\gamma_6$; these sixteen diagonals accordingly correspond to the sixteen characteristics of genus 2 in such a way that any one of them can be made to correspond to (0), and the rest then correspond to the fifteen period characteristics. The correspondence is very simple; the six odd θ-characteristics form a fundamental system [1], ..., [6], the sums by threes of these are the ten even θ-characteristics $[\alpha\beta\gamma] = [\delta\epsilon\zeta]$, and the sums by pairs or fours are the fifteen period characteristics $(\alpha\beta) = (\gamma\delta\epsilon\zeta)$; of two residual $h\gamma_6$'s, referred to the principal axes, one has six pairs of opposite vertices of the form (1, −1, −1, −1, −1, −1), (−1, 1, 1, 1, 1, 1), and ten of the form (1, 1, 1, −1, −1, −1), (−1, −1, −1, 1, 1, 1), which evidently correspond to the sixteen θ-charac-

teristics, while the other has one pair

$$(1, 1, 1, 1, 1, 1), \quad (-1, -1, -1, -1, -1, -1),$$

and fifteen of the form $(1, 1, -1, -1, -1, -1)$, $(-1, -1, 1, 1, 1, 1)$; these correspond to the period characteristic (0) and the other fifteen period characteristics respectively. The interchangeability of the two $h\gamma_6$'s represents the duality of the Kummer surface, whose nodes and tropes correspond to the diagonals of the two polytopes. The operation of adding a given characteristic to all simultaneously corresponds to a projective or correlative self-transformation of the Kummer surface, according as the added characteristic is a period characteristic or θ-characteristic, and to the transformation of the γ_6 by changing the sign of those coordinates which correspond to the symbols appearing in the bracket representing the added characteristic, which appropriately does not or does interchange the two $h\gamma_6$'s according as the number of these symbols is even or odd. When the vertices of one $h\gamma_6$ represent the planes on the double three-space, the pairs of opposite vertices of the other (that is to say, the pairs of opposite α_5's of the first) correspond to the nodes of the Kummer surface; in fact, as is clear, a pair of opposite α_6's represents just such a set of planes as can be represented by the neighbourhoods of the base points, and the quadric cones corresponding to them; and these are just the director surfaces which have a free intersection with (or contain) the cubic through the base points, which represents the neighbourhood of a particular node. The singling out of this node by the representation causes it to be represented by the period characteristic (0), when the other nodes correspond to the other period characteristics, that represented by the line $L_\alpha L_\beta$, in fact, to the period characteristic $(\alpha\beta)$; the six tropes which pass through it correspond to the six odd θ-characteristics (that represented by the neighbourhood of L_α and the cone of vertex L_α to the characteristic $[\alpha]$) and the other ten tropes to the even θ-characteristics (that represented by the planes $L_\alpha L_\beta L_\gamma$, $L_\delta L_\epsilon L_\zeta$, to the characteristic $[\alpha\beta\gamma] = [\delta\epsilon\zeta]$).

The six principal axes of the γ_6 configuration, which correspond to the six odd θ-characteristics, correspond also, in a way that is independent of which node or trope is made to correspond to the characteristic (0), *i.e.* is invariant under the sixteen projective and sixteen correlative self transformations of the Kummer surface (the transformations of γ_6 which correspond to these being such as to leave the principal axes fixed) to the six congruences of bitangents of the Kummer surface. This is most clearly exhibited when $h\gamma_6$ is considered as the central zone of 3_{21}, in the symmetry configuration. This corresponds to considering the Geiser surface

represented on one of the quadrics through the six base points by the traces on it of the others, *i.e.* the plane quartic which is a general prime section of the Kummer surface. The vertices of $h\gamma_6$ represent sixteen bitangents of this quartic which are the traces of the sixteen tropes of the Kummer surface, those of the two β_6's the remaining twelve bitangents (which form a Steinerian complex), being two out of each of the six quadro-quadric congruences. The six pairs which make up the Steinerian complex correspond to the six diameters or principal axes of β_6.

The relation to an actual curve of genus 2 is most easily seen by letting the plane touch the Kummer surface, when the curve of section has a node, and is of genus 2, and hyperelliptic. The two bitangents of each congruence which lie in the plane coincide, and give the six tangents to the nodal quartic from its node, *i.e.* the six united points of its canonical g_2^1, which, as is known, correspond to the six odd θ-characteristics. The sixteen sections of the tropes of the Kummer surface give proper bitangents of the curve corresponding to the sixteen characteristics symmetrically. This special Geiser surface can be represented on the plane by cubics through seven points $K_0, \ldots, K_6$, of which six, $K_1, \ldots, K_6$, lie on a conic; this is just the condition that there may be a quartic having a node at K_0 and touching K_0K_α at K_α $(\alpha = 1, \ldots, 6)$; and this quartic is the actual united curve of the Geiser involution, the conic separating out. The involution, in fact, is the particular case (for united curve of genus 2) of the de Jonquières involution, in which the united curve is hyperelliptic and of any genus.

This leads to a generalization of some of the present results. If $K_1, \ldots, K_{2p+2}$ are the points of contact of tangents from K_0 to a $(p+2)$-ic curve having a p-ple point at K_0, the system of $(p+1)$-ics having $(p-1)$-ple points at K_0 and passing simply through the remaining base points defines the de Jonquières involution having the $(p+2)$-ic as united curve, and represents the prime sections of a double normal rational ruled surface of order $p-1$ in p-fold space, branching along a normal hyperelliptic $2p$-ic curve, which meets each generator of the ruled surface twice. The generators are represented by the pencil of lines through K_0, and those of them which touch the branch curve, $2p+2$ in number, by those lines of the pencil which break up into two directrices (the neighbourhood of a base point and the line joining that point to K_0), which correspond to the pairs of opposite vertices of a β_{2p+2} of the appropriate (PA) polytope, which is $(2p+3)$-dimensional. This is exactly the β_{2p+2} which lies in the prime $y = 0$ of § 3; in fact, every vertex of the polytope represents a curve of order $1+(p-2)y$, where y is as defined in § 3. In particular, the vertices in the prime $y = 1$, which clearly form a $h\gamma_{2p+2}$ whose principal

axes are parallel to those of β_{2p+2}, represent rational $(p-1)$-ic curves on the double surface; two of these curves which correspond to diametrically opposite vertices of $h\gamma_{2p+2}$ are companions on the surface, and form together the section of the surface by a prime which touches the branch curve in p points instead of meeting it in $2p$ points. It is known, of course, that a normal non-special curve of order $2p$, and genus p, in space of p dimensions, has just 2^{2p} such contact primes, which correspond to the 2^{2p} characteristics of genus p, in a symmetrical manner, so that any one can be made to correspond to a given θ-characteristic or period characteristic, and the rest are then determinate, the character of the correspondence being unchanged if any period characteristic or θ-characteristic is added to all simultaneously. We thus have the 2^{2p} characteristics of genus p, in the hyperelliptic case, represented by the 2^{2p} diagonals of $h\gamma_{2p+2}$, whose $2p+2$ principal axes correspond to the $2p+2$ united points in the canonical involution on the hyperelliptic curve. The meaning of this representation is, of course, quite simple to see; for, taking the hyperelliptic curve in canonical form, as a double rational $(p-1)$-ic in p-fold space, with $2p+2$ branch points, certain odd θ-characteristics correspond to the primes joining $p-1$ branch points, certain even θ-characteristics to the primes through $p-3$ branch points and touching the curve, certain further odd θ-characteristics to primes through $p-5$ branch points and touching the curve twice, and so on; while the remaining even θ-characteristics which do not correspond to any contact prime, proper or improper, can be made to correspond to the partitions of the branch points into two sets of $p+1$. The principal axes of a Cartesian coordinate frame in $2p+2$ dimensions being made to correspond to the branch points, the θ-characteristic associated with any combination of the branch points is represented by the point having the corresponding coordinates equal to $+1$ and the rest to -1, and by that in which these coordinates are -1 and the rest $+1$. All the points thus obtained are clearly the vertices of $h\gamma_{2p+2}$, odd or even according as p is even or odd. The successive zones given by $\Sigma(x) = \text{const.}$ correspond to θ-characteristics which are alternately odd and even, the central zone $\Sigma(x) = 0$ corresponding to even θ-characteristics. If p is even the extreme zones are bounding α_{2p+1}'s, formed of vertices having only one positive and only one negative coordinate respectively; these correspond to systems of primes passing through one branch point only, and touching the curve at all its further intersections, and the corresponding θ-characteristics are odd or even according as $p/2$ is odd or even. If p is odd, the extreme zones are single vertices, having all their coordinates of one sign; these correspond to primes touching the curve in $(p-1)/2$ points, and to an odd or even θ-characteristic

according as $(p-1)/2$ is even or odd. In either case, the period characteristics can be likewise represented by the even $h\gamma_{2p+2}$, the two vertices for which all the coordinates have the same sign representing the period characteristic (0).

In every case with which we have had to deal, a pair of opposite bounding (prime) α's of the polytopes represents a fundamental system; in the case of $h\gamma_6$, every fundamental system is thus represented; in the case of 3_{21}, every fundamental system which consists of seven odd and one even characteristic; in the case of 4_{21}, every fundamental system which consists of eight odd, [0], and one other even characteristic; in the case now before us, when p is even, every fundamental system which can be transformed, by adding a fixed characteristic to all its members, into that which consists of all the characteristics represented by single symbols; when p is odd, all those which can be transformed in the same way into one of the fundamental systems which consists of the characteristic associated with no branch point of the curve, together with those associated with all the pairs of branch points which contain a particular one; *e.g.* in the case $p=3$, a fundamental system such as that consisting of the even characteristic representing the tangents to the double conic which is the canonical curve, with the odd characteristics [18], ..., [78], where $[\alpha\beta]$ refers to the join of the branch points U_α, U_β, regarded as an improper bitangent.

9. *The remaining finite polytopes.*

The most interesting of the finite polytopes in six dimensions is 2_{21}, whose twenty-seven vertices correspond to the directrices of six points, and to the lines on the general cubic surface. This relation need not, however, detain us long, since it has been treated by Coxeter†, and was, in fact, the beginning from which the present investigation has sprung. Each vertex has opposite to it a β_5 which represents the five pairs of lines which meet the line represented by the vertex, and each other; forty-five triangles of edge $\sqrt{2}$ metrical units inscribed in the figure correspond to the forty-five triads of lines in the forty-five tritangent planes; and thirty-six symmetry configurations correspond to the thirty-six double sixes. The latter lead each to a partition of the vertices into those of a $t_1\alpha_5$ in the prime of symmetry, whose vertices can conveniently be named $C_{\alpha\beta}$, and of two α_5's in primes parallel to it, images of each other in the prime of symmetry, and forming together an $[\alpha_5, \alpha_1]$ prism; these may be given the names A_α, B_α ($\alpha, \beta = 1, \ldots, 6$). This corresponds to the usual notation for the lines. 1_{22}, whose vertices are the centres of the α_5's, has a diagonal perpendicular to the prime of symmetry; its vertices in the

† H. S. M. Coxeter, *Proc. Camb. Phil. Soc.*, 24 (1927), 1.

prime of symmetry are thirty in number, being the ends of the fifteen diagonals of $t_{04}\alpha_5$; the rest form $t_2\alpha_5$'s in parallel zones on either side. This shows that each prime of symmetry is perpendicular to fifteen others, and meets the remaining twenty by pairs in ten secunda of threefold rotational symmetry. (This clearly represents the known relations between the double sixes.)

The remaining finite polytopes are of the greatest simplicity, and need not detain us. $h\gamma_5$ represents the directrices of five points in a plane, and the lines on the quartic del Pezzo surface (the intersection of a general pencil of quadrics in four dimensions). A vertex (1, 1, 1, 1, 1) corresponds to the conic through the base points, ten such as (1, 1, 1, −1, −1) to the joins of the base points by pairs, such as K_4K_5, and five such as (1, −1, −1, −1, −1) to the neighbourhoods of the base points, such as K_1. The surface lies on five cones, and has a group of 2^4 projective self transformations, five of which consist of harmonic inversion in a vertex of the simplex formed of the vertices of these cones (the common self polar simplex of the pencil of quadrics) and the opposite face, and are represented by the change of sign of all but one of the five coordinates of $h\gamma_5$; the other ten (apart from identity) consist of harmonic inversion in an edge and the opposite plane of the simplex, and are represented by the change of sign of two of the coordinates.

$t_1\alpha_5$ represents the director surfaces of five points in space; the quadrics through these represent the prime sections of the ten-nodal cubic primal in four dimensions, and the director surfaces represent the fifteen planes on the locus; the representation of these planes by the pairs of six symbols is familiar†, and this is just what $t_1\alpha_5$ gives.

$t_1\alpha_4$ in the same way represents the ten lines on the quintic del Pezzo surface‡, which are ordinarily represented by the pairs of five symbols.

The prisms $[\alpha_3, \alpha_1]$ and $[\alpha_2, \alpha_1]$, which are the vertex figures of these truncations, represent respectively the director surfaces of four points in space and the directrices of three points in a plane, *i.e.* the double four of planes in [5] which lies on the quartic threefold represented by quadrics through four points; and the skew hexagon of lines on the sextic del Pezzo surface. The fact that in each case the edges of the figure are of two kinds indicates that two planes of the double four which do not meet in a line may be related in two ways (they may meet in a point or not at all), and that two skew lines of the hexagon may be related in two ways, being either opposite sides of the hexagon or not. This leads in each case to a non-uniform vertex figure, representing in an obvious way the unsymmetrical

† H. F. Baker, *Principles of geometry*, 4 (1925), ch. 5.

‡ *Cf.*, *e.g.*, Bath, *Proc. Camb. Phil. Soc.*, 24 (1927), 48.

systems of director surfaces of three points in space, and of directrices of two points in a plane.

There remain a series of polytopes, the so-called Isohedral Archimedean, successive vertex figures of 2_{22}, each of which has just those vertices which lie in a secumdum of threefold rotational symmetry of a Pure Archimedean. Now there exists a sextic fourfold locus in [8]†, whose section by a general tangent prime is the sextic threefold locus represented by quadrics through two points; the fourfold is generated by two systems of ∞^2 planes, either of which is definable as the whole system of planes meeting four planes in general position (which are four general planes of the other system). The sections of these two plane systems give the two line systems generating the threefold, and the plane of each system through the point of contact lies in the secant prime, and gives the two planes of the threefold, which have in common the node of the locus, arising from the point of contact. By projecting the fourfold from one, two, three, and four points, we obtain a quintic locus (Grassmannian representative of the lines meeting two planes in four dimensions‡), a quartic (base of a special pencil of quadrics in six dimensions), the cubic primal of Perazzo§ in five, and double fourfold space, branching along the quartic primal with fifteen double lines, dual to the ten-nodal cubic. All of these have nodal lines, meeting by three (if they meet at all) in points which are bi-nodes of the locus; the nodes on the section by a tangent prime arise from these nodal lines, save one which is the point of contact. They have solids, finite in number, and doubly infinite systems of planes generating the locus completely (the solids on the double [4] are those touching the branch locus along quadrics, the planes generating it are those touching the branch locus along conics). The line systems generating the corresponding threefold loci are the sections of the plane systems on the fourfold, while, of the planes on the threefold, some arise as sections of the solids on the fourfold, while those through the particular node which is the point of contact of the secant prime are the planes of each generating system that pass through the point (since such a plane evidently lies in the tangent prime at each point of it). The loci are all of class 3, *i.e.* through a general secundum go three tangent primes; consider now the section by a general secundum, which is a del Pezzo surface, having on it lines represented by the vertices of a polytope of the (PA) series; such of these lines as are the sections of planes of the threefold in one of the three tangent primes through the

† C. Segre, *Rend. di Palermo*, 5 (1891), 192.

‡ J. A. Todd, *Proc. London Math. Soc.* (2), 30 (1929), 534–535.

§ U. Perazzo, *Atti R. Accad. Torino*, 36 (1901), 891.

secundum are represented by vertices in a particular prime of symmetry of the polytope; these include the lines which are sections of solids on the fourfold, as well as the section of the plane of any system which passes through the point of contact of the prime in question ; the sections, however, of the planes of this system which pass through the points of contact of the other two tangent primes from the same secundum, are (from the point of view of the threefold in the first prime) just the two lines of the corresponding generating line system which lie in the secundum, and are appropriately represented by a pair of vertices of the polytope, images of each other in the prime of symmetry. Thus the three tangent primes through the secundum correspond to three primes of symmetry of the polytope, meeting in a secundum of threefold rotational symmetry, and the vertices which lie in this secundum correspond to the solids on the fourfold locus, while the plane systems generating the fourfold correspond to triangles in planes absolutely perpendicular to the secundum, having their centres in it and a vertex in each of the three primes of symmetry. Thus the ten pairs of opposite vertices of the polytope $t_2\alpha_5$, which lies thus in 3_{21}, represent the ten pairs of companion solids on the double [4], that is the ten tropes of the quartic primal along which it branches, which are dual to the ten nodes of the cubic primal dual to this. (The interpretation of the usual notation for these tropes or nodes, the partitions of six symbols into triads, is obviously proper to $t_2\alpha_5$.) In the same way the prism $[\alpha_2, \alpha_2]$ in 2_{21} represents the nine solids on the Perazzo cubic primal in [5], and the nine lines in a pair of Steiner trihedrals on a cubic surface; and the vertex figure of this, a tetrahedron having a pair of opposite unit edges and the other four edges of $\sqrt{2}$ units, represents four solids on the particular quartic fourfold in [6] which belongs to the series.

10. *The infinite polytopes, and the generalization of* n_{pq}.

The existence in Minkowskian space of an indefinitely continued series of polytopes, each the vertex figure of the one above, of which Coxeter's (PA) series are the lower members, suggests that there may be in Minkowskian space an infinite polytope satisfying Coxeter's definition of the polytope n_{pq}, and having an infinite number of vertices, in those cases (namely when $npq > n+p+q+2$) in which no such polytope exists in Euclidean space.

The proof of the possibility of such a polytope reduces to the construction of the fundamental simplex of its symmetry group†, which is in a

† H. S. M. Coxeter, *Journal London Math. Soc.*, 5 (1931), 132.

non-Euclidean space of one fewer dimensions, namely $n+p+q$, on which the unit sphere in the Minkowski world may be supposed to be mapped. A simplification is obtained by considering the polar simplex of the required one, the dihedral angles being replaced by the edge lengths in natural measure; and we have to construct a simplex in which all edges are either $\frac{1}{2}\pi$ or $\frac{1}{3}\pi$ natural units, those of length $\frac{1}{3}\pi$ forming three chains, of n, p, and q edges respectively, radiating from one vertex, but having no other vertex in common, nor any vertex repeated within a chain, so that the whole of the $n+p+q+1$ vertices are used up.

Now in elliptic space of $n+p+q-1$ dimensions, take an $[n-1]$, a $[p-1]$, and a $[q-1]$, all absolutely perpendicular, and in each construct the dual simplex to the fundamental region of a regular simplex, *i.e.* a simplex in which a chain of edges just using up all the vertices are of length $\frac{1}{3}\pi$, all the other edges being of length $\frac{1}{2}\pi$. Every vertex of each of these figures is at distance $\frac{1}{2}\pi$ from every vertex of each of the others. Let N, P, Q be end vertices of the three chains respectively. Then there is a plane, locus of points which are distant $\frac{1}{2}\pi$ from all the vertices of these simplexes, other than N, P, Q; and there is a secundum, locus of points equidistant from N, P, Q; and this plane and this secundum meet in a point X. Now let the elliptic space $[n+p+q-1]$ be embedded in a space $[n+p+q]$, to which we attribute an elliptic or hyperbolic metric including that of the prime; *i.e.* if the absolute quadric of the elliptic space is $x_1^2+\ldots+x_{n+p+q}^2=0$, we attribute to the whole space a metric based on the absolute quadric $x_1^2+\ldots+x_{n+p+q}^2 \pm y^2=0$, where Y is a point outside the given prime. Then all points in the line XY are distant $\frac{1}{2}\pi$ from all the vertices of the simplexes other than N, P, Q, and are equidistant from these three vertices; also, if the metric is elliptic, the distance from N, P, or Q increases along the line XY from its value at X to $\frac{1}{2}\pi$ at Y; while, if the metric is hyperbolic, the distance decreases from its value at X to the value 0 at a point between X and Y; so that, attributing the elliptic or hyperbolic metric according as the distance XN is less or greater than the required value $\frac{1}{3}\pi$, we in either case obtain a point in XY which is distant $\frac{1}{3}\pi$ from each of the points N, P, Q, and $\frac{1}{2}\pi$ from each of the other points; and the simplex is complete. (The intermediate case in which $NX \doteq \ldots = \frac{1}{3}\pi$ clearly reciprocates into a fundamental simplex in Euclidean, not elliptic space, and corresponds to the known "degenerate" polytopes for which $npq = n+p+q+2$.)

We have thus found that in every case the fundamental simplex for the group of symmetry associated with the polytopes n_{pq}, p_{qn}, q_{np} exists, being in a hyperbolic instead of an elliptic space if $npq > n+p+q+2$, so

that it can be constructed on the unit sphere in Minkowskian instead of Euclidean space. Now whether we take the primes of the simplex in hyperbolic space, or those of the corresponding central sections of the sphere in Minkowskian space, as primes of reflection, we generate a group which clearly fills hyperbolic space (or covers the unit sphere) simply with replicas of the simplex, without overlapping or interstices, in such a way that every operation of the group transforms the given simplex into one such replica of itself, and conversely. If n, p, q are all distinct the simplex has no symmetry itself, and the whole group so generated is the group of symmetry of each of the polytopes n_{pq}, p_{qn}, q_{np}; if $n \neq p = q$, the simplex has itself symmetry of order 2, and the product of this into the group generated by the reflections in the primes of the simplex is the group of symmetry of n_{pq}, whereas the symmetry group of p_{qn}, q_{np} is just that generated by the reflections; if $n = p = q$, the simplex has symmetry of order 6 (trigonal dihedral), and the product of one of the three sub-groups of order 2 in this into the group generated by the reflections is the symmetry group of the corresponding one of the three semi-reciprocal n's, while the product of the whole group of order 6 into that generated by the reflections is the group of symmetry of the whole figure, consisting of the three identical semi-reciprocal polytopes; the symmetry operations of the fundamental simplex correspond to the permutations among themselves of those of n_{pq}, p_{qn}, q_{np}, which are identical. The vertices of the three polytopes are the sets of transforms (under the group generated by the reflections) of the vertices of the simplex opposite to the primes which correspond respectively to the vertices which we have called N, P, Q, respectively, while the vertices of O_{npq} are derived in the same way from the vertex which is distant $\frac{1}{3}\pi$ from each of these.

These polytopes having been proved to exist, there seems very little more to be said about them. All their properties are obtained by substituting the proper values of n, p, q in the formulae given by Coxeter. An imaginary length appearing in the formulae indicates a timelike vector, and a real length (when the whole figure is in Minkowskian space) a spacelike vector; the primes perpendicular to timelike and spacelike vectors have respectively Euclidean and Minkowskian geometry within themselves. In the same way, the angle between two vectors, given by its cosine (> 1 or < 1) or its sine or tangent (imaginary or real), indicates whether the plane of the two vectors has real or complex focoids. It is clear that the circumradius of an infinite polytope is timelike or spacelike according as the vertex figure is finite (Euclidean) or infinite (Minkowskian); and that the character of the two inradii is similarly determined by the corresponding bounding figures. It may be remarked that the polytopes

envisaged by Somerville, which correspond to partitions of hyperbolic space (*within* the absolute quadric) in the same manner as ordinary polytopes do to the partitions of a sphere, must have the bounding and vertex figures alike finite, the inradii and circumradii all timelike, which does not occur for any values of n, p, q. In fact, the only infinite polytopes which have all their bounding figures finite (*i.e.* both their inradii timelike) are the Pure Archimedeans with which we have already dealt, the "Isohedral Archimedeans" 2_{22}, 3_{22}, and 4_{22}, a polytope 4_{31}, and the known polytopes 3_{31}, 1_{33}, 1_{52}, and 2_{51}. The four latter and 2_{22} are, of course, not properly in Minkowskian space at all, while 6_{21}, 3_{22}, and 4_{31} are inscribed in cones, or spheres of zero radius. Thus the only polytope of this system which resembles the higher (PA)'s in being bounded by finite polytopes in primes whose normals are timelike, but inscribed in a sphere of spacelike radius, is 4_{22}, which is the nine-dimensional (IA), and is bounded entirely by $(PA)_8$'s.

It may be remarked that the primes of symmetry of the polytopes in Minkowskian space are all Minkowskian themselves, that is their normals are spacelike. They are all equivalent under the group generated by them.

NOTE ON THE FREE TIDAL OSCILLATIONS OF A SEA WITH SLOW ROTATION

By J. PROUDMAN.

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1. In 1903 the late Lord Rayleigh investigated the free oscillations of a slowly rotating mechanical system whose configuration can be specified by a finite number of Lagrangian coordinates*. At the same time he wrote a paper† on the free tidal oscillations of a rectangular sea of uniform depth, when the vertical component ω of the earth's angular velocity is small compared with the speed σ of the oscillations. In the case of a square sea he arrived at the formula

$$\sigma - \sigma_0 = \pm \frac{16}{\pi^2} \omega \qquad (1.1)$$

for the lowest speeds, to the first power of ω. In 1909 he returned to the problem of slowly rotating seas and oceans‡ and found a number of formulae of the type

$$\sigma - \sigma_1 = \lambda \frac{\omega^2}{\sigma_1}, \qquad (1.2)$$

where σ_1 is a non-repeated speed of the same basin when not rotating and λ is a pure number depending only on the shape of the basin.

In 1924 H. Lamb gave another approximate method of investigating the lowest free speeds§, and obtained the formula

$$(\sigma^2 - \sigma_1^2)(\sigma^2 - \sigma_2^2) = \frac{256}{\pi^4} \omega^2 \sigma^2 \qquad (1.3)$$

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* *Phil. Mag.* (6), 5 (1903), 293–297 ; *Sci. Papers*, 5, 89–92.

† *Phil. Mag.* (6), 5 (1903), 297–301 ; *Sci. Papers*, 5, 93–97.

‡ *Proc. Royal Soc.* (A), 82 (1909), 448–464 ; *Sci. Papers*, 5, 497–513. These three papers will be referred to as R_1, R_2, R_3 respectively.

§ *Hydrodynamics*, 5th ed., §§ 205a, 212a.

for the rectangular basin, pointing out that for the square sea, in which $\sigma_1 = \sigma_2 = \sigma_0$ (say), (1.3) gives

$$\sigma - \sigma_0 = \pm \frac{8}{\pi^2} \omega \tag{1.31}$$

and not (1.1).

In a recent paper* G. R. Goldsbrough has returned to the general problem of the rotating rectangular sea and has developed a method of approximation which also yields the formula (1.3).

It may be noted that, whereas all Rayleigh's formulae profess to give complete coefficients of the powers of ω retained, the formula (1.3) only does so when $\sigma_1 = \sigma_2$.

In both Rayleigh's hydrodynamical papers he uses a certain mechanical principle, and it is the object of the present note to investigate the validity of this principle. It will be shown that the principle itself is valid, but that Rayleigh's method of application is not. His error is such as to double the rotational correction to the free speed. The formula (1.1) should therefore be replaced by (1.31), and Rayleigh's values of λ in (1.2) should be halved.

I shall use a transformation of the tidal equations from the partial differential to the Lagrangian type, which I gave† in 1916. To these Lagrangian equations the methods of R_1 may be applied. It may be noted that, for the rectangular sea of uniform depth, these Lagrangian equations are equivalent to the algebraic equations used by Goldsbrough.

2. *General equations.*

Let g denote the acceleration of gravity, h the depth of the sea at any point (x, y), (ξ, η) the displacement of a particle of water, and ζ the elevation of the free surface. Then the ordinary differential equations of free tidal motion are

$$\left.\begin{aligned} \ddot{\xi} - 2\omega\dot{\eta} &= -g\frac{\partial \zeta}{\partial x} \\ \ddot{\eta} + 2\omega\dot{\xi} &= -g\frac{\partial \zeta}{\partial y} \end{aligned}\right\}, \tag{2.1}$$

* *Proc. Royal Soc.* (A), 132 (1931), 689–701.

† *Proc. London Math. Soc.* (2), 18 (1920), 21–50.

and I have shown in the paper quoted that

$$\left.\begin{aligned} \xi &= -\frac{\partial\phi}{\partial x} - \frac{1}{h}\frac{\partial\psi}{\partial y} \\ \eta &= -\frac{\partial\phi}{\partial y} + \frac{1}{h}\frac{\partial\psi}{\partial x} \\ \zeta &= \frac{\partial}{\partial x}\left(h\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial\phi}{\partial y}\right) \end{aligned}\right\}, \tag{2.2}$$

where

$$\phi = \sum_{s=1}^{\infty} p_s\phi_s, \quad \psi = \sum_{s=1}^{\infty} p_{-s}\psi_s. \tag{2.3}$$

The functions ϕ_s, ψ_s are determined by

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(h\frac{\partial\phi_s}{\partial x}\right) + \frac{\partial}{\partial y}\left(h\frac{\partial\phi_s}{\partial y}\right) + \frac{\sigma_s^2}{g}\phi_s = 0 \\ \frac{\partial}{\partial x}\left(\frac{1}{h}\frac{\partial\psi_s}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\frac{\partial\psi_s}{\partial y}\right) + \mu_s\psi_s = 0 \end{aligned}\right\} \tag{2.4}$$

over the sea, and

$$h\frac{\partial\phi_s}{\partial n} \to 0, \quad \psi_s/h \to 0 \tag{2.41}$$

at the coasts, with

$$\left.\begin{aligned} \iint h\left(\frac{\partial\phi_r}{\partial x}\frac{\partial\phi_s}{\partial x} + \frac{\partial\phi_r}{\partial y}\frac{\partial\phi_s}{\partial y}\right)dS = 0 \quad (s\neq r), \quad = 1 \quad (s=r) \\ \iint \frac{1}{h}\left(\frac{\partial\psi_r}{\partial x}\frac{\partial\psi_s}{\partial x} + \frac{\partial\psi_r}{\partial y}\frac{\partial\psi_s}{\partial y}\right)dS = 0 \quad (s\neq r), \quad = 1 \quad (s=r) \end{aligned}\right\}, \tag{2.42}$$

all integrals being taken over the total surface of the sea. The Lagrangian coordinates (p) are determined by

$$\ddot{p}_r + 2\omega \sum_{s=-\infty}^{\infty} \beta_{r,s}\dot{p}_s + \sigma_r^2 p_r = 0 \tag{2.5}$$

for $\pm r = 1, 2, 3, \ldots$, where dots denote differentiation with respect to time, and

$$\left.\begin{aligned} \beta_{r,s} &= -\iint h\left(\frac{\partial\phi_r}{\partial x}\frac{\partial\phi_s}{\partial y} - \frac{\partial\phi_r}{\partial y}\frac{\partial\phi_s}{\partial x}\right)dS \\ \beta_{r,-s} &= -\beta_{-s\,r} = \iint\left(\frac{\partial\phi_r}{\partial x}\frac{\partial\psi_s}{\partial x} + \frac{\partial\phi_r}{\partial y}\frac{\partial\psi_s}{\partial y}\right)dS \\ \beta_{-r,-s} &= -\iint\frac{1}{h}\left(\frac{\partial\psi_r}{\partial x}\frac{\partial\psi_s}{\partial y} - \frac{\partial\psi_r}{\partial y}\frac{\partial\psi_s}{\partial x}\right)dS \end{aligned}\right\}, \tag{2.6}$$

$\sigma_{-r}^2 = 0$, for $r, s = 1, 2, 3, \ldots$.

For complex periodic motion, with a time-factor $e^{i\sigma t}$, the equations (2.5) become

$$(\sigma_r^2-\sigma^2)p_r+2\omega i\sigma\sum_{s=-\infty}^{\infty}\beta_{r,s}p_s=0 \qquad (2.7)$$

for $\pm r = 1, 2, 3, \ldots$.

When $\omega = 0$, $\psi = 0$.

All infinite series, both in this section and in the sequel, are absolutely convergent.

3. *Repeated coefficient of stability.*

Suppose that $\sigma_2 = \sigma_1$ but that no other coefficient σ_s shares their common value. This common value we shall denote by σ_0, a symbol not defined in § 2.

For small values of ω/σ the equation (2.7) will possess a solution in which all coordinates other than p_1, p_2 are of the order ω/σ compared with these two. On neglecting terms of order $(\omega/\sigma)^2$, the equations (2.7) reduce to

$$(\sigma_0^2-\sigma^2)p_1+2\omega i\sigma\beta_{1,2}p_2=0,$$

$$(\sigma_0^2-\sigma^2)p_2+2\omega i\sigma\beta_{2,1}p_1=0,$$

from which, since $\beta_{2,1} = -\beta_{1,2}$, we deduce* that

$$\sigma^2-\sigma_0^2 = \pm 2\omega\sigma_0\beta_{1,2}, \qquad (3.1)$$

$$p_2 = \mp ip_1. \qquad (3.11)$$

To the order ω/σ_0 the equation (3.1) is equivalent to

$$\sigma-\sigma_0 = \pm\omega\beta_{1,2}. \qquad (3.12)$$

Now write

$$\phi_0 = \phi_1 \mp i\phi_2, \qquad (3.2)$$

so that

$$\frac{\partial}{\partial x}\left(h\frac{\partial\phi_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(h\frac{\partial\phi_0}{\partial y}\right)+\frac{\sigma_0^2}{g}\phi_0=0, \qquad (3.21)$$

and take

$$\left.\begin{aligned}\xi &= -\frac{\partial\phi_0}{\partial x}He^{i\sigma t}+\xi'\\ \eta &= -\frac{\partial\phi_0}{\partial y}He^{i\sigma t}+\eta'\\ \zeta &= -\frac{\sigma_0^2}{g}\phi_0 He^{i\sigma t}+\zeta'\end{aligned}\right\}, \qquad (3.3)$$

* *Cf.* R$_1$.

where H is an arbitrary constant and ξ', η', ζ' are of the order ω/σ_0. On substituting from (3.3) into (2.1) and neglecting terms of order $(\omega/\sigma_0)^2$, we deduce the equations

$$\left.\begin{aligned} \ddot{\xi}' &= -g\frac{\partial\zeta'}{\partial x} + X He^{i\sigma t} \\ \ddot{\eta}' &= -g\frac{\partial\zeta'}{\partial y} + Y He^{i\sigma t} \end{aligned}\right\}, \tag{3.4}$$

where

$$\left.\begin{aligned} X &= (\sigma_0^2-\sigma^2)\frac{\partial\phi_0}{\partial x} - 2\omega i\sigma_0\frac{\partial\phi_0}{\partial y} \\ Y &= (\sigma_0^2-\sigma^2)\frac{\partial\phi_0}{\partial y} + 2\omega i\sigma_0\frac{\partial\phi_0}{\partial x} \end{aligned}\right\}. \tag{3.41}$$

Now

$$\begin{aligned} \iint h\left(X\frac{\partial\phi_1}{\partial x} + Y\frac{\partial\phi_1}{\partial y}\right)dS &= (\sigma_0^2-\sigma^2)\iint h\left(\frac{\partial\phi_0}{\partial x}\frac{\partial\phi_1}{\partial x} + \frac{\partial\phi_0}{\partial y}\frac{\partial\phi_1}{\partial y}\right)dS \\ &\quad + 2\omega i\sigma_0\iint h\left(\frac{\partial\phi_0}{\partial x}\frac{\partial\phi_1}{\partial y} - \frac{\partial\phi_0}{\partial y}\frac{\partial\phi_1}{\partial x}\right)dS \\ &= (\sigma_0^2-\sigma^2)\pm 2\omega\sigma_0\beta_{1,2} \\ &= 0, \end{aligned} \tag{3.5}$$

on using (2.42), (2.6), (3.1), and (3.2). Similarly it may be shown that

$$\iint h\left(X\frac{\partial\phi_2}{\partial x} + Y\frac{\partial\phi_2}{\partial y}\right)dS = 0. \tag{3.51}$$

It follows from (3.5) and (3.51) that forces proportional to (X, Y) do no work in the two free oscillations of the non-rotating sea given by

$$\phi = \phi_1 He^{i\sigma_0 t}, \quad \phi = \phi_2 He^{i\sigma_0 t}.$$

This is the mechanical principle enunciated by Rayleigh, but in his application to the square sea* he took

$$\left.\begin{aligned} \dot{\xi} &= -i\sigma_0\frac{\partial\phi_0}{\partial x}He^{i\sigma t} + \dot{\xi}' \\ \dot{\eta} &= -i\sigma_0\frac{\partial\phi_0}{\partial y}He^{i\sigma t} + \dot{\eta}' \\ \zeta &= -\frac{\sigma_0^2}{g}\phi_0 He^{i\sigma t} + \zeta' \end{aligned}\right\}, \tag{3.6}$$

* R_2.

instead of (3.3). On substituting into (2.1) he obtained (3.4) with

$$\left.\begin{aligned} X &= \sigma_0(\sigma_0-\sigma)\frac{\partial\phi_0}{\partial x}-2\omega i\sigma_0\frac{\partial\phi_0}{\partial y} \\ Y &= \sigma_0(\sigma_0-\sigma)\frac{\partial\phi_0}{\partial y}+2\omega i\sigma_0\frac{\partial\phi_0}{\partial x} \end{aligned}\right\}, \tag{3.7}$$

instead of (3.41). On assuming

$$\iint h\left(X\frac{\partial\phi_1}{\partial x}+Y\frac{\partial\phi_1}{\partial y}\right)dS$$

to be zero, he therefore obtained

$$\sigma-\sigma_0 = \pm 2\omega\beta_{1,2},$$

instead of (3.12).

4. *Non-repeated coefficient of stability.*

Suppose that no other coefficient σ_s has the same value as σ_1. Then, for small values of ω/σ, the equations (2.7) will possess a solution in which every other coordinate is of the order ω/σ compared with p_1. On neglecting terms of the order $(\omega/\sigma)^2$, we have

$$\left.\begin{aligned} (\sigma_1^2-\sigma^2)p_1 &= 0, \\ (\sigma_r^2-\sigma^2)p_r+2\omega i\sigma\beta_{r,1}p_1 &= 0 \end{aligned}\right\}, \tag{4.1}$$

so that

$$\left.\begin{aligned} \sigma^2 &= \sigma_1^2 \\ p_r &= \frac{2\omega i\sigma_1}{\sigma_1^2-\sigma_r^2}\beta_{r,1}\,p_1 \end{aligned}\right\}, \tag{4.11}$$

for all values of r other than $r=1$. Next, retaining terms of the order $(\omega/\sigma_1)^2$ in the equation for $r=1$, but neglecting terms of higher orders, we have

$$(\sigma_1^2-\sigma^2)p_1-4\omega^2\sigma_1^2\sum_{s=-\infty}^{\infty}{}'\frac{\beta_{1,s}\beta_{s,1}}{\sigma_1^2-\sigma_s^2}p_1 = 0,$$

where Σ' denotes that $s=1$ is omitted from the summation. Since $\beta_{s,1}=-\beta_{1,s}$, we deduce that

$$\sigma^2-\sigma_1^2 = 4\omega^2\sigma_1^2\sum_{s=-\infty}^{\infty}{}'\frac{\beta_{1,s}^2}{\sigma_1^2-\sigma_s^2}, \tag{4.2}$$

which* may also be written, to the same order,

$$\sigma-\sigma_1 = \frac{2\omega^2}{\sigma_1} \sum_{s=-\infty}^{\infty}{}' \frac{\beta_{1,s}^2}{1-\sigma_s^2/\sigma_1^2}. \tag{4.21}$$

Now write

$$\left.\begin{aligned} \phi_0 &= \phi_1 + 2\omega i\sigma_1 \sum_{s=2}^{\infty} \frac{\beta_{s,1}}{\sigma_1^2-\sigma_s^2}\phi_s, \\ \psi_0 &= 2\omega i\sigma_1 \sum_{s=1}^{\infty} \frac{\beta_{-s\,1}}{\sigma_1^2}\psi_s, \\ g\zeta_0 &= -\sigma_1^2\phi_1 - 2\omega i\sigma_1 \sum_{s=2}^{\infty} \frac{\sigma_s^2}{\sigma_1^2-\sigma_s^2}\beta_{s,1}\phi_s, \end{aligned}\right\} \tag{4.3}$$

and then take

$$\left.\begin{aligned} \xi &= \left(-\frac{\partial\phi_0}{\partial x} - \frac{1}{h}\frac{\partial\psi_0}{\partial y}\right) He^{i\sigma} + \xi', \\ \eta &= \left(-\frac{\partial\phi_0}{\partial y} + \frac{1}{h}\frac{\partial\psi_0}{\partial x}\right) He^{i\sigma t} + \eta', \\ \zeta &= \zeta_0 He^{i\sigma t} + \zeta', \end{aligned}\right\} \tag{4.4}$$

where again H is an arbitrary constant, and now ξ', η', ζ' are of the order $(\omega/\sigma_1)^2$. On substituting from (4.4) into (2.1) we have equations of the form (3.4), where now

$$\begin{aligned} X = (\sigma_1^2-\sigma^2)\frac{\partial\phi_1}{\partial x} - 2\omega i\sigma_1 \sum_{s=2}^{\infty} \beta_{s,1}\frac{\partial\phi_s}{\partial x} \\ -\frac{\sigma_1^2}{h}\frac{\partial\psi_0}{\partial y} + 2\omega i\sigma_1\left(-\frac{\partial\phi_0}{\partial y} + \frac{1}{h}\frac{\partial\psi_0}{\partial x}\right), \end{aligned} \tag{4.5}$$

$$\begin{aligned} Y = (\sigma_1^2-\sigma^2)\frac{\partial\phi_1}{\partial y} - 2\omega i\sigma_1 \sum_{s=2}^{\infty} \beta_{s,1}\frac{\partial\phi_s}{\partial y} \\ +\frac{\sigma_1^2}{h}\frac{\partial\psi_0}{\partial x} + 2\omega i\sigma_1\left(\frac{\partial\phi_0}{\partial x} + \frac{1}{h}\frac{\partial\psi_0}{\partial y}\right). \end{aligned} \tag{4.51}$$

Then

$$\begin{aligned} \iint h\left(X\frac{\partial\phi_1}{\partial x} + Y\frac{\partial\phi_1}{\partial y}\right) dS &= (\sigma_1^2-\sigma^2) + 4\omega^2\sigma_1^2 \sum_{s=-\infty}^{\infty}{}' \frac{\beta_{1,s}^2}{\sigma_1^2-\sigma_s^2} \\ &= 0, \end{aligned} \tag{4.6}$$

* *Cf.* R$_1$. The formulae (1.3) may be obtained from (4.2) by neglecting all terms of the summation except that for $s = 2$, and substituting the appropriate value for $\beta_{1,2}$.

on using (2.4), (2.6), (4.2), and (4.3). In equation (4.6) we have again the mechanical principle enunciated by Rayleigh, but in his applications* he took

$$\left.\begin{aligned} \dot{\xi} &= i\sigma_1\left(-\frac{\partial\phi_0}{\partial x}-\frac{1}{h}\frac{\partial\psi_0}{\partial y}\right)He^{i\sigma t}+\dot{\xi}', \\ \dot{\eta} &= i\sigma_1\left(-\frac{\partial\phi_0}{\partial y}+\frac{1}{h}\frac{\partial\psi_0}{\partial x}\right)He^{i\sigma t}+\dot{\eta}', \\ \zeta &= \zeta_0 He^{i\sigma t}+\zeta', \end{aligned}\right\} \tag{4.7}$$

instead of (4.4). He then obtained equations of the form (3.4), but with $\sigma_1(\sigma_1-\sigma)$ instead of $\sigma_1^2-\sigma^2$ in (4.5). Hence Rayleigh obtained results equivalent to

$$\sigma-\sigma_1 = \frac{4\omega^2}{\sigma_1}\sum_{s=-\infty}^{\infty}{}' \frac{\beta_{1,s}^2}{1-\sigma_s^2/\sigma_1^2}, \tag{4.8}$$

instead of (4.21).

* R_3.

ASYMPTOTIC EXPANSIONS OF THE FUNCTION

$$F_k(x) = \int_0^\infty e^{-u^k + xu}\, du.$$

By N. G. BAKHOOM.

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1. In their original discussion of the "major arcs" in Waring's problem, Hardy and Littlewood investigate the asymptotic behaviour of a certain function, which, by a change of variable, can be reduced to the form

$$F_k(x) = \int_0^\infty e^{-u^k + xu}\, du,$$

where k is an integer greater than 1.

For large values of $|x|$, the function $F_k(x)$ has an algebraic asymptotic expansion holding uniformly in the angle

$$|\arg x| \geqslant \tfrac{1}{2}\pi - \frac{\pi}{2k} + \delta,$$

where δ is a fixed, but arbitrary, positive number. It has also an exponential asymptotic expansion holding uniformly in the angle

$$|\arg x| \leqslant \tfrac{1}{2}\pi - \frac{\pi}{2k} - \delta.$$

For their application to Waring's problem, however, the authors required an approximation to $F_k(x)$ holding uniformly in an angle extending right up to one of the critical lines $\arg x = \pm\{\frac{1}{2}\pi - \pi/(2k)\}$, namely the angle

$$\tfrac{1}{2}\pi - \frac{\pi}{2k} < \arg x < \tfrac{1}{2}\pi + \frac{\pi}{2k}.$$

This is not provided by either of the expansions mentioned above; but, by an application of the saddle point method, Hardy and Littlewood* obtained an approximation sufficient for their application.

In this paper we show, by appropriate modification of the saddle-point argument, that a combination of the algebraic and the exponential asymptotic expansions gives an expansion holding uniformly in an angle containing the critical line in its interior. If we retain only the dominant terms in each expansion, our result takes the form

$$F_k(x) = -\frac{1}{x}\left\{1+O\left(\frac{1}{|x|^k}\right)\right\}$$
$$+\sqrt{\left(\frac{2\pi}{k(k-1)}\right)}\, e^{(k-1)(x/k)^{k/(k-1)}}\left(\frac{k}{x}\right)^{(k-2)/[2(k-1)]}\{1+O(|x|^{-k/(k-1)})\}$$

uniformly in $|\arg x| \leqslant \frac{1}{2}\pi$ as $|x| \to \infty$, a formula from which (in conjunction with the algebraic expansion) the inequality given by Hardy and Littlewood can be inferred at once.

Functions more or less closely associated with $F_k(x)$ have been discussed by various writers. Burwell† investigates a set of functions of which $F_k(x)$ is a linear combination, and Brillouin‡ examines various modifications of the integral $F_k(x)$ in the special cases $k = 2, 3, 4$. These authors employ the saddle-point method, but they do not discuss in detail a range of $\arg x$ including the critical lines. A class of integrals expressible in terms of Bessel functions (and therefore possessing asymptotic expansions of the type considered here) has been investigated by Hardy§, but these integrals reduce to integrals considered in the present paper only in the special cases $k = 2$ and $k = 3$. Finally, the function $F_k(x)$ is expressible in terms of generalized hypergeometric functions, and thus belongs to a class of functions whose asymptotic expansions have been worked out in detail by the contour integration methods elaborated by Barnes‖. The

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* "Some problems of 'Partitio Numerorum'; 1: A new solution of Waring's Problem", *Göttinger Nachrichten* (Math. Phys. Klasse) (1920), 13-17.

† *Proc. London Math. Soc.* (2), 22 (1923), 57-72.

‡ *Annales Sci. de l'Ecole Normale* (3), 33 (1916), 17-69.

§ *Quart. J. of Math.*, 41 (1910), 226-240. The identities which form the subject of this paper are generalizations of the formula expressing Airy's integral in terms of Bessel functions. (See §6 below).

‖ See particularly C. Fox, *Proc. London Math. Soc.* (2), 27 (1928), 389-400 (Theorem 3), noting that

$$F_k(x) = \frac{1}{k}\sum_{n=0}^{\infty}\Gamma\left(\frac{n+1}{k}\right)\frac{x^n}{n!}.$$

direct discussion of the integral by the saddle-point method seems, however, to be of interest, and to lend itself more readily to generalization.

2. Before proceeding to determine the asymptotic expansions we prove the following

LEMMA. *If*

$$Z_1 = X_1 + iY_1, \quad Z_2 = X_2 - iY_2,$$

then

$$e^{-Z_1} = e^{-Z_2} \sum_{p=0}^{n-1} \frac{(Z_2 - Z_1)^p}{p!} + R_n(Z_1, Z_2),$$

where

$$|R_n| \leqslant \frac{|Z_2 - Z_1|^n}{n!} e^{-\min(X_1, X_2)}.$$

For, if we apply Taylor's theorem for a complex function of a real variable with exact remainder to $\phi(t) = e^{(Z_2 - Z_1)t}$, we get

$$e^{(Z_2 - Z_1)} = \sum_{p=0}^{n-1} \frac{(Z_2 - Z_1)^p}{p!} + \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} (Z_2 - Z_1)^n e^{-(Z_1 - Z_2)t} dt.$$

But

$$\left| \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} (Z_2 - Z_1)^n e^{-Z_2} e^{-(Z_1 - Z_2)t} dt \right|$$

$$\leqslant \frac{|Z_2 - Z_1|^n}{(n-1)!} \int_0^1 (1-t)^{n-1} e^{-X_2 - (X_1 - X_2)t} dt,$$

and the linear function $X_2 + (X_1 - X_2)t$ attains its minimum in the interval $0 \leqslant t \leqslant 1$ at one of the end points $t = 0$ or $t = 1$. Hence

$$X_2 + (X_1 - X_2)t \geqslant \min(X_2, X_2 + X_1 - X_2) = \min(X_2, X_1),$$

whence the result of the lemma.

3. We now determine the algebraic asymptotic expansion of the function $F_k(x)$.

Changing x to $-x$ and using Cauchy's theorem, we get

$$F_k(-x) = \int_L e^{-u^k - xu} du, \tag{3.1}$$

where L is the infinite line starting from the origin of the u-plane and making an angle $\pi/(2k) - \frac{1}{2}\delta$ with the positive real axis, where δ is fixed

and $0 < \delta < \pi/k$. The change of contour is legitimate since

$$|\exp(-u^k - xu)| \leqslant \exp(-R^k \sin \tfrac{1}{2}k\delta + |x| R)$$

$$[u = Re^{i\theta},\ 0 \leqslant \theta \leqslant \pi/(2k) - \tfrac{1}{2}\delta].$$

Substituting in (3.1) for e^{-u^k} by the result of the lemma, with $Z_1 = u^k$ and $Z_2 = 0$, we have

$$F_k(-x) = \sum_{p=0}^{n-1} \frac{(-)^p}{p!} \int_L e^{-xu} u^{pk}\, du + \int_L e^{-xu} R_n(u)\, du,$$

where

$$|R_n(u)| \leqslant \frac{|u^k|^n}{n!},$$

provided that the integrals are convergent, that is to say if $\Re(xu) > 0$ on the contour L. This is the case if

$$-\tfrac{1}{2}\pi - \frac{\pi}{2k} + \delta \leqslant \arg x \leqslant \tfrac{1}{2}\pi - \frac{\pi}{2k}.$$

Then

$$F_k(-x) = \sum_{p=0}^{n-1} \frac{(-)^p}{p!} \frac{\Gamma(pk+1)}{x^{pk+1}} + S_n(x),$$

where

$$|S_n(x)| \leqslant \int_L e^{-\Re(xu)} \frac{|u|^{kn}}{n!} |du|$$

$$\leqslant \frac{1}{n!} \int_L e^{-|xu| \sin \frac{1}{2}\delta} |u|^{nk} |du|$$

$$= \frac{1}{n!} \frac{\Gamma(nk+1)}{(|x| \sin \tfrac{1}{2}\delta)^{nk+1}}.$$

The angle may be extended to

$$-\tfrac{1}{2}\pi - \frac{\pi}{2k} + \delta \leqslant \arg x \leqslant \tfrac{1}{2}\pi + \frac{\pi}{2k} - \delta,$$

since $S_n(x)$ takes conjugate values for conjugate values of x. Changing x to $-x$, we obtain

$$(3.2) \qquad F_k(x) = -\frac{1}{x} + \sum_{p=1}^{n-1} \frac{(-)^p}{p!} \frac{\Gamma(pk+1)}{(-x)^{pk+1}} + O\left(\frac{1}{|x|^{nk+1}}\right)$$

uniformly in the angle $|\arg x| \geqslant \frac{1}{2}\pi - \pi/(2k) + \delta$ *as* $|x| \to \infty$.

4. In order to obtain an exponential asymptotic expansion of the function $F_k(x)$, it is convenient to consider an integral with both limits

infinite, namely

$$I_k(x) = \int_M e^{-u^k+xu}\,du, \tag{4.1}$$

where M is the contour consisting of the two infinite rays, $\arg u = \alpha$ and $\arg u = \beta$, where

$$-\frac{\pi}{2k} < \alpha < \frac{\pi}{2k}, \quad \pi-\frac{\pi}{2k} < \beta < \pi+\frac{\pi}{2k} \quad (k \text{ even}),$$

$$-\frac{\pi}{2k} < \alpha < \frac{\pi}{2k}, \quad \pi-\frac{3\pi}{2k} < \beta < \pi-\frac{\pi}{2k} \quad (k \text{ odd});$$

in both cases the contour is described from left to right. By Cauchy's theorem the integral is independent of the particular values of α and β; it is convenient to take $\alpha = 0$, $\beta = \pi$ when k is even and $\alpha = 0$, $\beta = \pi-\pi/k$ when k is odd.

First suppose x to be real and positive and put $x = ky^{k-1}$ and $u = yv$; we get

$$I_k(ky^{k-1}) = y \exp\{(k-1)y^k\} \int_M \exp[-y^k\{v^k-kv+k-1\}]\,dv. \tag{4.2}$$

The right-hand side of (4.2) is a regular function of y in the angle $-\pi/(2k) < \arg y < \pi/(2k)$; the formula (4.2), proved for y real and positive, therefore holds in this angle, and in particular in

$$0 \leqslant \arg y \leqslant \frac{\pi}{2k}-\delta \quad \left(0 < \delta < \frac{\pi}{2k}\right).$$

We now apply the saddle-point method to the integral (4.2); there are $k-1$ saddle points given by the stationary points of $v^k-kv+k-1$. Consider the saddle point which is real and given by $v = 1$, and put $v = 1+w$. We then have

$$\begin{aligned} I_k(ky^{k-1}) &= y \exp\{(k-1)y^k\} \int_N \exp[-y^k\{(1+w)^k-k(1+w)+k-1\}]\,dw \\ &= y \exp\{(k-1)y^k\} J \quad \text{say}, \end{aligned} \tag{4.3}$$

where N is obtained by displacing the contour M parallel to itself a distance -1. When $k = 2$,

$$J = \int_N e^{-y^2w^2}\,dw = \frac{\sqrt{\pi}}{y}.$$

Thus

$$I_2(2y) = \sqrt{\pi}\,.\,e^{y^2}. \tag{4.4}$$

We now discuss the integral J for all $k \geqslant 3$; it is convenient, however, to consider the case $k = 3$ separately afterwards.

When k is even, let $T \equiv \{(1+w)^k - k(1+w) + k - 1\}$. Then

$$\frac{d^2T}{dw^2} = k(k-1)(1+w)^{k-2} \geqslant 0$$

for all real w. Since $T = 0$ and $dT/dw = 0$ when $w = 0$, it follows that $T > 0$ for all real $w \neq 0$. Thus T/w^2, if defined to be $\frac{1}{2}k(k-1)$ when $w = 0$, is positive and continuous and tends to infinity as $w \to \pm\infty$; it therefore has a positive lower bound K say. Thus, on the contour N, we have, when k is even,

$$\Re\,[y^k\{(1-w)^k - k(1+w) + k - 1\}] \geqslant K \sin k\delta\,.\,|y|^k|w|^2 = K_1|y|^k|w|^2.$$

When k is odd, we have on the contour N from -1 to ∞ along the real axis, $\Re\,[y^k\{(1+w)^k - k(1+w) + k - 1\}] \geqslant K_1|y|^k|w|^2$ by the same argument as was used in the even case. On the other part of the contour write

$$y = |y|e^{i\theta} \quad \left(0 \leqslant \theta \leqslant \frac{\pi}{2k} - \delta\right), \qquad v = 1 + w = -re^{-\pi i/k} \quad (r > 0).$$

Then

$$\begin{aligned}
&\Re\,[y^k\{(1+w)^k - k(1+w) + k - 1\}] \\
&\qquad = \Re\,[y^k\{v^k - kv + k - 1\}] \\
&\qquad = |y|^k\left\{-r^k\cos(k\theta - \pi) + kr\cos\left(k\theta - \frac{\pi}{k}\right) + (k-1)\cos k\theta\right\} \\
&\qquad \geqslant |y|^k(r^k + k - 1)\sin k\delta \\
&\qquad \geqslant K_2|y|^k(r+1)^2 \geqslant K_2|v-1|^2|y|^k = K_2|y|^k|w|^2.
\end{aligned}$$

To the integrand of J apply the lemma with

$$Z_1 = y^k\{(1+w)^k - k(1+w) + k - 1\}, \quad Z_2 = \tfrac{1}{2}k(k-1)y^k w^2.$$

We get

$$J = \int_N \exp\{-\tfrac{1}{2}k(k-1)y^k w^2\} \sum_{p=0}^{2n} \frac{(-)^p}{p!}\left[y^k\left\{\binom{k}{3}w^3 + \binom{k}{4}w^4 + \ldots + w^k\right\}\right]^p dw$$

$$= J_1 + J_2, \qquad + \int_N R_{2n+1}(w)\,dw$$

say, where

$$|J_2| \leqslant \int_N \frac{1}{(2n+1)!} \left| y^k \left\{ \binom{k}{3} w^3 + \binom{k}{4} w^4 + \ldots + w^k \right\} \right|^{2n+1} \times e^{-\min\left[\Re y^k\{(1+w)^k - k(1+w) + k - 1\},\ \Re \frac{1}{2}k(k-1)y^k w^2\right]} |dw|$$

$$\leqslant \frac{1}{(2n+1)!} \int_N K_3 |y|^{k(2n+1)} \{|w|^{3(2n+1)} + |w|^{k(2n+1)}\} e^{-K_4 |y|^k |w|^2} |dw|$$

$$= \frac{K_3 |y|^{k(2n+1)}}{(2n+1)!} \left\{ \int_N e^{-K_4 |y^k w|^2} |w|^{3(2n+1)} |dw| + \int_N e^{-K_4 |y^k w^2|} |w|^{k(2n+1)} |dw| \right\}$$

$$< K_5(|y|^{-\frac{1}{2}k(2n+2)}) + K_6(|y|^{-\frac{1}{2}k(k-2)(2n+2)}),$$

where the K's are positive numbers depending only on k and n. This holds, provided that the integrals are convergent, that is to say if

$$\Re\{\tfrac{1}{2}k(k-1)y^k w^2\} > 0$$

on the contour N, which is the case for all $k \geqslant 4$ if $0 \leqslant \arg y \leqslant \pi/(2k) - \delta$. [The integrals do not converge for $k = 3$ throughout this angle, since on that part of the contour N for which $w = -1 - re^{-\frac{1}{3}i\pi}$ $(r > 0)$, we have

$$\Re(3y^3 w^2) = 3|y|^3\{\cos(3\arg y) + 2r\cos(3\arg y - \tfrac{1}{3}\pi) + r^2\cos(3\arg y - \tfrac{2}{3}\pi)\}$$

$$< 0 \quad \text{for} \quad r > r_0 \quad \text{in} \quad 0 \leqslant \arg y \leqslant \tfrac{1}{18}\pi - \delta.]$$

Then $$J_2 = O(|y|^{-k(n+1)}).$$

To examine J_1, let

$$\left\{ \binom{k}{3} w^3 + \binom{k}{4} w^4 + \ldots + w^k \right\}^p = \sum_{q=3p}^{kp} c_{p,q} w^q = \sum_{q=3p}^{\infty} c_{p,q} w^q,$$

where $c_{p,q} = 0$ for $q > kp$. Thus

$$J_1 = \sum_{p=0}^{2n} \frac{(-)^p}{p!} y^{kp} \sum_{q=3p}^{\infty} c_{p,q} \int_N e^{-aw^2} w^q dw, \tag{4.5}$$

where $a = \frac{1}{2}k(k-1)y^k$. The integral on the right of (4.5) is equal to zero if q is odd and to $a^{-\frac{1}{2}q-\frac{1}{2}}\Gamma(\frac{1}{2}q+\frac{1}{2})$ if q is even. Hence, writing $q = 2r$, we have

$$J_1 = \sum_{p,r} \frac{(-)^p}{p!} c_{p,2r} \frac{\Gamma(r+\frac{1}{2})}{\{\frac{1}{2}k(k-1)\}^{r+\frac{1}{2}} y^{k(r-p+\frac{1}{2})}},$$

where the range of summation is defined by $0 \leqslant p \leqslant 2n$, $2r \geqslant 3p$. Writing $r = p + s$, we deduce

$$J_1 = \sum_{p,s} \frac{(-)^p}{p!} c_{p,2p+2s} \frac{\Gamma(p+s+\frac{1}{2})}{\{\frac{1}{2}k(k-1)\}^{p+s+\frac{1}{2}} y^{k(s+\frac{1}{2})}},$$

summed over the range

$$0 \leqslant p \leqslant 2n, \quad 2p+2s \geqslant 3p,$$

that is to say $0 \leqslant p \leqslant 2n, \qquad 2s \geqslant p.$

We may restrict s to the range $s \leqslant kn$; for if $s > kn$, then

$$2p+2s > 2p+2kn > kp$$

(since $p \leqslant 2n$), so that $c_{p,\,2p+2s} = 0$. Hence

$$J_1 = \sum_{s=0}^{kn} \frac{C_s}{y^{k(s+\frac{1}{2})}} = \sum_{s=0}^{n-1} \frac{C_s}{y^{k(s+\frac{1}{2})}} + O\left(\frac{1}{|y|^{k(n+\frac{1}{2})}}\right),$$

where $$C_s = \sum_{p=0}^{2s} \frac{(-)^p}{p!}\, c_{p,\,2p+2s}\, \frac{\Gamma(p+s+\frac{1}{2})}{\{\frac{1}{2}k(k-1)\}^{p+s+\frac{1}{2}}} \quad (0 \leqslant s < n).$$

Substituting for J in (4.3) we get

$$(4.6) \qquad I_k(ky^{k-1}) = \exp\{(k-1)y^k\} \left\{ \sum_{s=0}^{n-1} \frac{C_s}{y^{k(s+\frac{1}{2})-1}} + O\left(\frac{1}{|y|^{k(n+\frac{1}{2})-1}}\right) \right\}$$

uniformly in $0 \leqslant \arg y \leqslant \pi/(2k)-\delta$ *as* $|y| \to \infty$.

5. The formula (4.6) obtained in the last article applies to a wider range of $\arg y$. For the purpose of getting the asymptotic expansion of the function $F_k(x)$, in which we are interested, it is sufficient to show that the formula applies for $0 \leqslant \arg y \leqslant \pi/(2k)+\pi/(8k^2)$, since after we pass the critical line the algebraic asymptotic expansion dominates. We may confine ourselves to the range $\pi/(2k)-\pi/(8k^2) \leqslant \arg y \leqslant \pi/(2k)+\pi/(8k^2)$, since the remaining part of the angle has already been disposed of.

It is convenient to consider the two cases, k even and k odd, separately. We shall first discuss the case k even. Considering the integral J, taking $\arg y$ to be fixed and equal to $\pi/(4k)$, and using Cauchy's theorem, we can write

$$(5.1) \qquad J = \int_Q \exp\left[-y^k\{(1+w)^k - k(1+w) + (k-1)\}\right] dw,$$

where Q is the infinite straight line through the origin of the w-plane making an angle $\pi/(2k)$ with the real axis in the sense shown in Fig. 1. The change of contour is legitimate, since

$$\left|\exp\left[-y^k\{(1+w)^k - k(1+w) + (k-1)\}\right]\right|$$

$$\leqslant \exp\left[-|y|^k R^k \cos\tfrac{1}{4}\pi + O(|y|^k R^{k-1})\right]$$

$$\{w = \pm Re^{-i\phi},\ 0 \leqslant \phi \leqslant \pi/(2k)\}.$$

The right-hand side of (5.1) is a regular function of y in $0 < \arg y < \pi/k$; hence, by the principle of analytic continuation, the formula (5.1) will hold in this angle, and in particular in

$$\frac{\pi}{2k} - \frac{\pi}{8k^2} \leqslant \arg y \leqslant \frac{\pi}{2k} + \frac{\pi}{8k^2}.$$

Using polar coordinates in place of the complex variable v, namely

$$v = re^{i\theta}, \quad \text{where} \quad v = 1 + w = 1 + \rho e^{-i\pi/(2k)} \quad (\rho \text{ real}),$$

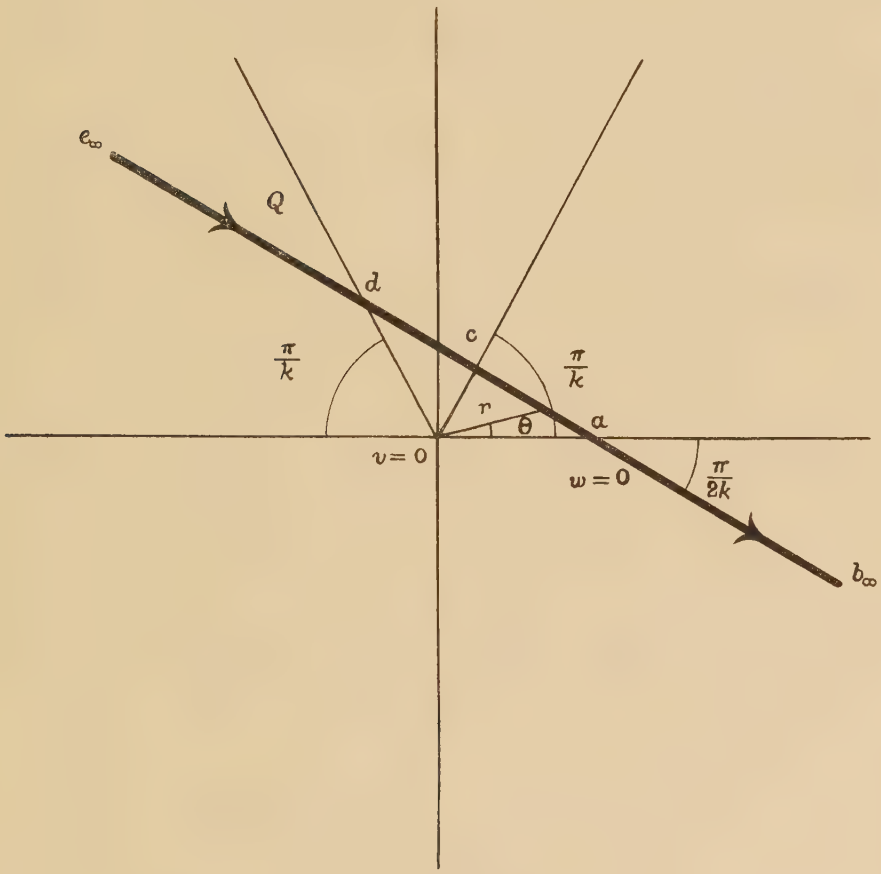

Fig. 1. *k even.*

and putting

$$\arg y = \frac{\pi}{2k} + \frac{\alpha}{k} \quad \text{where} \quad -\frac{\pi}{8k} \leqslant \alpha \leqslant \frac{\pi}{8k},$$

let

$$y^k\{v^k - kv + (k-1)\} \equiv Z \equiv X + iY.$$

Then $$\frac{dZ}{d\rho} = \frac{dZ}{dv}\frac{dv}{d\rho} = e^{-i\pi/(2k)} y^k \{kv^{k-1} - k\},$$

or $$\frac{dX}{d\rho} = k|y|^k \left[r^{k-1} \sin\left\{\frac{\pi}{2k} - \alpha - (k-1)\theta\right\} - \sin\left(\frac{\pi}{2k} - \alpha\right)\right],$$

and $$X = |y|^k \{kr \sin(\theta+\alpha) - (k-1)\sin\alpha - r^k \sin(k\theta+\alpha)\}.$$

Now on the part ab_∞ of the contour Q, $-\pi/k < \theta < 0$, $dX/d\rho > 0$*; on ac, $0 < \theta \leqslant \pi/k$, $dX/d\rho < 0$†; on de_∞, $\pi - \pi/k \leqslant \theta < \pi - \pi/(2k)$, $dX/d\rho < 0$‡; at the point d, $\theta = \pi - \pi/k$, $X > 0$§ for all $k \geqslant 4$; also, at the point a, $\theta = 0$, $X = 0$. Hence $X > 0$ over the parts cb_∞ and de_∞ except at the point a for all $-\pi/(8k) \leqslant \alpha \leqslant \pi/(8k)$.

It remains now to consider the part cd, that is to say the range

$$\frac{\pi}{k} \leqslant \theta \leqslant \pi - \frac{\pi}{k}.$$

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* Since $r > 1$, we have

$$\frac{dX}{d\rho} > k|y|^k \left[\sin\left\{\frac{\pi}{2k} - \alpha - (k-1)\theta\right\} - \sin\left(\frac{\pi}{2k} - \alpha\right)\right]$$

$$= 2k|y|^k \sin\left\{-\frac{(k-1)\theta}{2}\right\} \cos\left\{\frac{\pi}{2k} - \alpha - \frac{(k-1)\theta}{2}\right\} > 0.$$

† Since $r < 1$, we have

$$\frac{dX}{d\rho} < k|y|^k \left[\sin\left\{\frac{\pi}{2k} - \alpha - (k-1)\theta\right\} - \sin\left(\frac{\pi}{2k} - \alpha\right)\right] \quad \left(\text{if } 0 < (k-1)\theta < \frac{\pi}{2k} - \alpha\right)$$

$$= 2k|y|^k \sin\left\{-\frac{(k-1)\theta}{2}\right\} \cos\left\{\frac{\pi}{2k} - \alpha - \frac{(k-1)\theta}{2}\right\} < 0;$$

for the remaining range

$$\frac{\pi}{2k} - \alpha \leqslant (k-1)\theta \leqslant \frac{(k-1)\pi}{k}, \quad \sin\left\{\frac{\pi}{2k} - \alpha - (k-1)\theta\right\} \leqslant 0;$$

hence $dX/d\rho < 0$.

‡ Since $r > 1$, we have

$$\frac{dX}{d\rho} < k|y|^k \left[\sin\left\{\frac{\pi}{2k} - \alpha - (k-1)\theta\right\} - \sin\left(\frac{\pi}{2k} - \alpha\right)\right]$$

$$= 2k|y|^k \sin\left\{-\frac{(k-1)\theta}{2}\right\} \cos\left\{\frac{\pi}{2k} - \alpha - \frac{(k-1)\theta}{2}\right\} < 0.$$

§ Since $r = 1$, we have

$$X = |y|^k \left[k \sin\left(\pi - \frac{\pi}{k} + \alpha\right) - (k-1)\sin\alpha - \sin\{(k-1)\pi + \alpha\}\right]$$

$$= |y|^k \left[k \sin\left(\frac{\pi}{k} - \alpha\right) - (k-1)\sin\alpha + \sin\alpha\right] \geqslant k|y|^k \left[\sin\frac{7\pi}{8k} - \sin\frac{\pi}{k}\right] > 0.$$

If $\frac{5}{8} \leqslant r \leqslant 1$,

$$X \geqslant |y|^k \left\{ \tfrac{5}{8} k \sin\left(\frac{\pi}{k} - \frac{\pi}{8k}\right) - k \sin \frac{\pi}{8k} - 1 \right\} > 0^*$$

for all $k \geqslant 4$. If $r < \frac{5}{8}$, we have, on the contour Q,

$$r = \frac{\sin \pi/(2k)}{\sin\{\theta + \pi/(2k)\}}; \quad \text{also} \quad \frac{\sin(\theta+\alpha)}{\sin\{\theta+\pi/(2k)\}} \geqslant \tfrac{3}{8}\dagger;$$

therefore

$$X \geqslant |y|^k \left\{ k \sin \frac{\pi}{2k} \frac{\sin(\theta+\alpha)}{\sin\{\theta+\pi/(2k)\}} - k \sin \alpha - r^k \right\}$$

$$\geqslant |y|^k \left\{ k \sin \frac{\pi}{2k} \tfrac{3}{8} - k \sin \frac{\pi}{8k} - (\tfrac{5}{8})^k \right\} > 0 \quad (k \geqslant 4).$$

Thus $X > 0$ over the contour Q for all $\rho \neq 0$ and all $-\pi/(8k) \leqslant \alpha \leqslant \pi/(8k)$; but $X/(\rho^2 |y|^k)$, if defined to be $\frac{1}{2}k(k-1)\sin(\pi/k - \alpha) > 0$ when $\rho = 0$, is a positive and continuous function of the two variables ρ and α and tends to infinity uniformly in the above range of α as $\rho \to \pm\infty$; it therefore has a positive lower bound, B say, for all ρ and all α in the range.

Next consider the case k odd ($k \geqslant 5$). Considering the integral J, assuming y real and positive, and putting $yw = t$, we get

$$J = \frac{1}{y} \int_{N_1} \exp\left[-\{(y+t)^k - ky^{k-1}(y+t) + (k-1)y^k\}\right] dt,$$

where N_1 is obtained by displacing the contour M parallel to itself a distance $-y$. By an application of Cauchy's theorem we get

$$J = \frac{1}{y} \int_G \exp\left[-\{(y+t)^k - ky^{k-1}(y+t) + (k-1)y^k\}\right] dt, \tag{5.2}$$

where G is the infinite straight line through the origin of the t-plane making an angle $-\pi/(2k)$ with the positive real axis. The change of contour is

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* Since $\sin\theta/\theta$ is a decreasing function in $0 \leqslant \theta \leqslant \pi$, so that

$$X \geqslant |y|^k \left\{ \tfrac{5}{8} \cdot 4 \sin \tfrac{7}{32}\pi - \tfrac{1}{8}\pi - 1 \right\} > 0.$$

† $$\frac{\sin(\theta+\alpha)}{\sin\{\theta+\pi/(2k)\}} \geqslant \tfrac{3}{8} \quad \text{if} \quad 8\sin(\theta+\alpha) \geqslant 3\sin\left(\theta + \frac{\pi}{2k}\right),$$

i.e. if $$5\sin(\theta+\alpha) \geqslant 3\left\{\sin\left(\theta+\frac{\pi}{2k}\right) - \sin(\theta+\alpha)\right\},$$

i.e. if $$5\sin(\theta+\alpha) \geqslant 6\cos\left(\theta + \tfrac{1}{2}\alpha + \frac{\pi}{4k}\right)\sin\left(\frac{\pi}{4k} - \tfrac{1}{2}\alpha\right).$$

But

$$\sin(\theta+\alpha) \geqslant \sin\left(\frac{\pi}{k} - \frac{\pi}{8k}\right) > \sin\frac{5\pi}{8k} \text{ and } \cos\left(\theta + \tfrac{1}{2}\alpha + \frac{\pi}{4k}\right)\sin\left(\frac{\pi}{4k} - \tfrac{1}{2}\alpha\right) \leqslant \sin\left(\frac{\pi}{4k} + \frac{\pi}{16k}\right) = \sin\frac{5\pi}{16k}.$$

Thus the inequality is satisfied if $\sin\frac{5\pi}{8k} \geqslant \frac{6}{5}\sin\frac{5\pi}{16k}$, *i.e.* if $2\cos\frac{5\pi}{16k} \geqslant \frac{6}{5}$, which is the case.

legitimate because, when

$$t = Re^{-i\phi}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2k},$$

we have

$$\begin{aligned}
&|\exp[-\{(y+t)^k - ky^{k-1}(y+t) + (k-1)y^k\}]| \\
&\quad = |\exp[-t^k - kyt^{k-1} + O(|t|^{k-2})]| \\
&\quad = \exp[-R^k \cos k\phi - ky\, R^{k-1} \cos(k-1)\phi + O(R^{k-2})] \\
&\quad \leqslant \exp\left[-ky \sin\frac{\pi}{2k}\, R^{k-1} + O(R^{k-2})\right],
\end{aligned}$$

and, when $t = -Re^{-i\phi}, \quad \frac{\pi}{2k} \leqslant \phi < \frac{\pi}{k}$,

$$|\exp[-\{(y+t)^k - ky^{k-1}(y+t) + (k-1)y^k\}]|$$

$$\leqslant \begin{cases} \exp\left[-k \sin\frac{\pi}{4k}\, yR^{k-1} + O(R^{k-2})\right] & \left(\frac{\pi}{2k} \leqslant \phi \leqslant \frac{(2k-1)\pi}{4k(k-1)}\right) \\ \exp\left[-\sin\frac{\pi}{4(k-1)}\, R^k + kyR^{k-1} + O(R^{k-2})\right] & \left(\frac{(2k-1)\pi}{4k(k-1)} \leqslant \phi < \frac{\pi}{k}\right). \end{cases}$$

Now, since t^k is purely imaginary on the contour G, we see, by considering the term kyt^{k-1}, that the integral (5.2) is convergent, and represents a regular function of y, in $-\pi/(2k) < \arg y < \pi - \pi/(2k)$; hence, by analytic continuation, the formula (5.2) remains valid in this angle and in particular in the angle

$$\frac{\pi}{2k} - \frac{\pi}{8k^2} \leqslant \arg y \leqslant \frac{\pi}{2k} + \frac{\pi}{8k^2}.$$

If we now transform back by putting $t = yw$, the w-contour will now be derived from the t-contour by a rotation through an angle of magnitude $-\arg y$; denote this new contour by H. We thus have

$$J = \int_H \exp[-y^k\{(1+w)^k - k(1+w) + k - 1\}]\, dw. \tag{5.3}$$

On the part fg_∞ of the contour H, Fig. 2, writing

$$w = \rho \exp\{-i[\pi/(2k) + \arg y]\},$$

we have

$$\Re[y^k\{(1+w)^k - k(1+w) + (k-1)\}] = \rho^2 |y|^k \sum_{s=0}^{k-3} c_s \rho^s \geqslant B_1 \rho^2 |y|^k \quad (\rho \geqslant 0),$$

where

$$c_s = \binom{k}{s+2} \cos\left\{(k-s-2)\arg y - \frac{(s+2)\pi}{2k}\right\} > 0,$$

in the range $\frac{\pi}{2k} - \frac{\pi}{8k^2} \leqslant \arg y \leqslant \frac{\pi}{2k} + \frac{\pi}{8k^2}$.

For the other part of the contour ($\rho < 0$), we have, with the same notation as in the even case,

$$X = |y|^k \{-r^k \sin(k\theta+\alpha)+kr\sin(\theta+\alpha)-(k-1)\sin\alpha\}$$

and

$$\frac{dX}{d\rho} = k|y|^k\left[r^{k-1}\sin\left\{\frac{\pi}{k}-\alpha+\frac{\alpha}{k}-(k-1)\theta\right\}-\sin\left(\frac{\pi}{k}-\alpha+\frac{\alpha}{k}\right)\right].$$

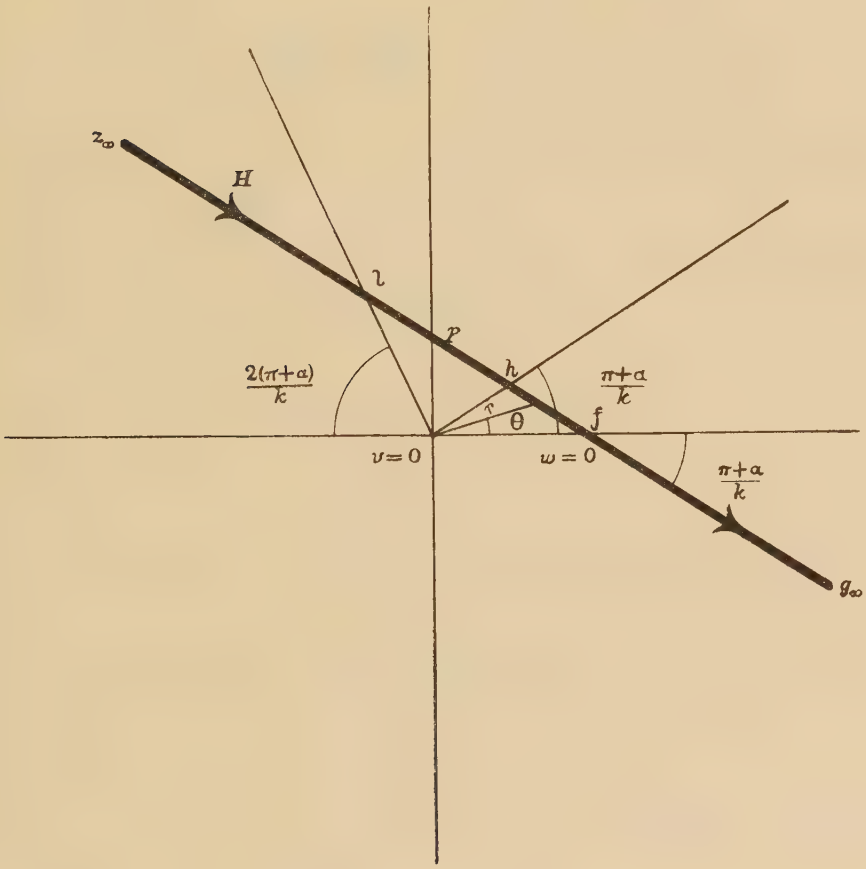

Fig. 2. k odd.

Now, on the part fh, $0 < \theta \leqslant \pi/k+\alpha/k$, $dX/d\rho < 0$; on the part lz_∞, $\pi-(2\pi)/k-(2\alpha)/k \leqslant \theta \leqslant \pi-\pi/k-\alpha/k$, $dX/d\rho < 0$; at the point l,

$$\theta = \pi-\frac{2\pi}{k}-\frac{2\alpha}{k}, \quad X > 0;$$

also at the point f, $\theta = 0$, $X = 0$. Hence $X > 0$ over the parts fh and lz_∞ (except at the point f) for all $-\pi/(8k) \leqslant \alpha \leqslant \pi/(8k)$.

It remains now to consider the part hl, that is to say, the range $\pi/k+\alpha/k \leqslant \theta \leqslant \pi-(2\pi)/k-(2\alpha)/k$. On the part pl, where

$$\tfrac{1}{2}\pi \leqslant \theta \leqslant \pi-\frac{2\pi}{k}-\frac{2\alpha}{k},$$

we have $r=\dfrac{\sin\{(\pi+\alpha)/k\}}{\sin\{\theta+(\pi+\alpha)/k\}}$; hence

$$X \geqslant |y|^k\left\{-1+k\sin\left(\frac{\pi+\alpha}{k}\right)\frac{\sin(\theta+\alpha)}{\sin\{\theta+(\pi+\alpha)/k\}}-(k-1)\sin\alpha\right\}.$$

But $\dfrac{\sin(\theta+\alpha)}{\sin\{\theta+(\pi+\alpha)/k\}} \geqslant 1$*. Therefore

$$X \geqslant |y|^k\left\{-1+k\sin\frac{\pi}{k}-(k-1)\sin\frac{\pi}{8k}\right\} \quad \left(0 \leqslant \alpha \leqslant \frac{\pi}{8k}\right)$$

$$> |y|^k\left\{-1+k\sin\frac{\pi}{k}-k\sin\frac{\pi}{8k}\right\} > 0,$$

and

$$X \geqslant |y|^k\left\{-1+k\sin\left(\frac{\pi}{k}-\frac{\pi}{8k^2}\right)\right\} \quad \left(-\frac{\pi}{8k} \leqslant \alpha < 0\right)$$

$$> |y|^k\left\{-1+k\sin\tfrac{6}{8}\frac{\pi}{k}\right\} > 0.$$

Finally, consider the part ph where $(\pi+\alpha)/k \leqslant \theta \leqslant \tfrac{1}{2}\pi$, remembering that $k \geqslant 5$. If $\tfrac{3}{4} \leqslant r \leqslant 1$, we have

$$\begin{cases} X \geqslant |y|^k\left\{-1+\tfrac{3}{4}k\sin\left(\dfrac{\pi}{k}-\dfrac{\pi}{8k^2}-\dfrac{\pi}{8k}\right)\right\} > |y|^k\left\{-1+\tfrac{3}{4}k\sin\dfrac{3\pi}{4k}\right\} > 0 \\ \qquad\qquad \left(-\dfrac{\pi}{8k} \leqslant \alpha \leqslant 0\right), \\ X \geqslant |y|^k\left\{-1+\tfrac{3}{4}k\sin\dfrac{\pi}{k}-(k-1)\sin\dfrac{\pi}{8k}\right\} \\ \qquad > |y|^k\left\{-1+\tfrac{3}{4}k\sin\dfrac{\pi}{k}-k\sin\dfrac{\pi}{8k}\right\} > 0 \quad \left(0 \leqslant \alpha \leqslant \dfrac{\pi}{8k}\right). \end{cases}$$

* This inequality is true if

$$\sin(\theta+\alpha) \geqslant \sin\left(\theta+\frac{\pi+\alpha}{k}\right), \text{ i.e. if } 0 \geqslant \sin\left(\theta+\frac{\pi+\alpha}{k}\right)-\sin(\theta+\alpha),$$

i.e. if

$$0 \geqslant 2\cos\left(\theta+\frac{\pi}{2k}+\frac{\alpha}{2k}+\tfrac{1}{2}\alpha\right)\sin\left(\frac{\pi+\alpha}{2k}-\tfrac{1}{2}\alpha\right),$$

which is the case.

If $r < \frac{3}{4}$, we have

$$X \geqslant |y|^k \left\{ -r^k + k \sin\left(\frac{\pi+\alpha}{k}\right) \frac{\sin(\theta+\alpha)}{\sin\{\theta+(\pi+\alpha)/k\}} - (k-1)\sin\alpha \right\}.$$

Now* $\dfrac{\sin(\theta+\alpha)}{\sin\{\theta+(\pi+\alpha)/k\}} \geqslant \frac{3}{8}$; therefore

$$X \geqslant |y|^k \left\{ -(\tfrac{3}{4})^k + \tfrac{3}{8}k \sin\left(\frac{\pi}{k} - \frac{\pi}{8k^2}\right) - (k-1)\sin\frac{\pi}{8k} \right\} > 0.$$

Thus $X > 0$ over the contour H for all $\rho \neq 0$ and all $\pi/(8k) \leqslant \alpha \leqslant \pi/(8k)$. Using the same argument as in the even case, we see that $X \geqslant B_2\rho^2|y|^k$ at all points of the contour H and for all $-\pi/(8k) \leqslant \alpha \leqslant \pi/(8k)$. (All the B's are positive numbers depending only on k.)

Having proved that $\Re[y^k\{(1+w)^k - k(1+w) + k - 1\}]$ is greater than or equal to $B\rho^2|y|^k$ on the contour Q, and is greater than or equal to $B_2\rho^2|y|^k$ on the contour H, we can now easily show, by the method used in § 4, that the formula (4.6) holds uniformly in

$$\frac{\pi}{2k} - \frac{\pi}{8k^2} \leqslant \arg y \leqslant \frac{\pi}{2k} + \frac{\pi}{8k^2}.$$

The integrals corresponding to J_1 and J_2 in § 4 are convergent, since on either the contour Q, or H, we have

$$\Re\{\tfrac{1}{2}k(k-1)y^k w^2\} \geqslant \tfrac{1}{2}k(k-1)\sin\frac{7\pi}{8k}|y|^k\rho^2 > B_3|y|^k\rho^2.$$

Combining the result of this section with that of (4.5), we obtain the asymptotic expansion (4.5) holding uniformly in

$$0 \leqslant \arg y \leqslant \frac{\pi}{2k} + \frac{\pi}{8k^2},$$

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* The inequality holds if

$$8\sin(\theta+\alpha) \geqslant 3\sin\left(\theta + \frac{\pi+\alpha}{k}\right),$$

i.e. if

$$5\sin(\theta+\alpha) \geqslant 6\cos\left(\theta + \frac{\pi+\alpha}{2k} + \tfrac{1}{2}\alpha\right)\sin\left(\frac{\pi+\alpha}{2k} - \tfrac{1}{2}\alpha\right).$$

Now

$$\sin(\theta+\alpha) \geqslant \sin\left(\frac{\pi}{k} - \frac{\pi}{8k^2} - \frac{\pi}{8k}\right) > \sin\frac{(7k-1)\pi}{8k^2}.$$

Thus the inequality holds if

$$\sin\frac{(7k-1)\pi}{8k^2} \geqslant \tfrac{8}{5}\sin\frac{(9k-1)\pi}{16k^2},$$

which is the case.

or, written in terms of x,

$$(5.4)\quad I_k(x) = \exp\left[(k-1)\left(\frac{x}{k}\right)^{k/(k-1)}\right]\left\{\sum_{s=0}^{n-1} C_s\left(\frac{x}{k}\right)^{-[k(s+\frac{1}{2})-1]/(k-1)} + O(|x|^{-[k(n+\frac{1}{2})-1]/(k-1)})\right\}$$

uniformly in

$$0 \leqslant \arg x \leqslant \tfrac{1}{2}\pi - \frac{\pi}{2k} + \frac{(k-1)\pi}{8k^2} \quad \text{as} \quad |x| \to \infty.$$

6. The preceding discussion does not, as it stands, cover the case $k = 3$. But this case is essentially simpler than the general case. Starting from (5.2) we obtain at once (without the use of elaborate inequalities) the desired asymptotic expansion (5.4), and indeed in the wider range $-\frac{1}{3}\pi+\delta \leqslant \arg x \leqslant \frac{5}{3}\pi-\delta$. In this case there is a simple general formula for the coefficients C_s, namely

$$C_s = \frac{1}{(2s)!}\,\frac{\Gamma(3s+\frac{1}{2})}{3^{3s+\frac{1}{2}}}.$$

We may note that the function $I_3(x)$ is connected with Airy's integral

$$A(x) = \int_0^\infty \cos(t^3+xt)\,dt,$$

and so with Bessel functions. For it may be shown by an application of Cauchy's theorem that, when $\arg x = \frac{2}{3}\pi$, the contour M in the formula (4.1) for $I_3(x)$ may be replaced by the infinite line making an angle $-\frac{1}{6}\pi$ with the positive real axis. Making the substitution $u = e^{-\frac{1}{6}\pi i}t$ in the transformed integral, we obtain

$$I_3(x) = \int_{-\infty}^{\infty} \exp(it^3+xe^{-\frac{1}{6}\pi i}t)\,e^{-\frac{1}{6}\pi i}\,dt$$

$$= 2e^{-\frac{1}{6}\pi i}A(xe^{-\frac{2}{3}\pi i}),$$

whence* $$I_3(x) = -\tfrac{2}{3}ix^{\frac{1}{2}}K_{\frac{1}{3}}\left(-\frac{2x^{\frac{3}{2}}}{3^{\frac{3}{2}}}\right),$$

and this formula is valid, by the theory of analytic continuation, for general values of x. From the known asymptotic expansion† of the

* See Watson's *Theory of Bessel functions*, 190 (2). Our function $I_k(x)$ is, of course, not to be confused with the function so denoted in the theory of Bessel functions.

† Watson, *loc. cit.*, 201 (1).

function $K_{\frac{1}{3}}(z)$ we can now infer at once the expansion (5.4) for $I_3(x)$, again in the angle $-\frac{1}{3}\pi+\delta \leqslant \arg x \leqslant \frac{5}{3}\pi-\delta$.

7. Returning to the general case, we now show how the algebraic asymptotic expansion of $F_k(x)$ and the exponential expansion of $I_k(x)$ together lead to a mixed asymptotic expansion for $F_k(x)$. We have

$$I_k(x) = F_k(x)+\omega F_k(-\omega x), \tag{7.1}$$

where $\omega = 1$ (k even), $\omega = e^{-\pi i/k}$ (k odd).

Now, using the results of § 3 and noting that $\omega^k = (-1)^k$, we obtain

$$\omega F_k(-\omega x) = -\sum_{p=0}^{n-1} \frac{(-)^p}{p!}\,\frac{\Gamma(pk+1)}{(-x)^{pk+1}} + O\left(\frac{1}{|x|^{nk+1}}\right) \tag{7.2}$$

uniformly in the angle defined by

$$\left|\arg x - \frac{\epsilon\pi}{k}\right| \leqslant \tfrac{1}{2}\pi+\frac{\pi}{2k}-\delta,$$

where $\epsilon = 0$ or 1 according as k is even or odd. In either case (7.2) holds in

$$0 \leqslant \arg x \leqslant \tfrac{1}{2}\pi+\frac{\pi}{2k}-\delta,$$

and so certainly in the angle specified in (5.4). Hence, combining (7.1), (7.2), and (5.4), and noting that $F_k(x)$ takes conjugate values for conjugate values of x, we obtain

$$F_k(x) = \exp\left\{(k-1)\left(\frac{x}{k}\right)^{k/(k-1)}\left\{\sum_{s=0}^{n-1} C_s\left(\frac{x}{k}\right)^{-[k(s+\frac{1}{2})-1]/(k-1)} + O(|x|^{-[k(n+\frac{1}{2})-1]/(k-1)}\right\}\right.$$
$$\left. + \sum_{p=0}^{n-1} \frac{(-)^p}{p!}\,\frac{\Gamma(pk+1)}{(-x)^{pk+1}} + O\left(\frac{1}{|x|^{nk+1}}\right)\right. \tag{7.3}$$

uniformly in

$$|\arg x| \leqslant \tfrac{1}{2}\pi-\frac{\pi}{2k}+\frac{(k-1)\pi}{8k^2} \quad as \quad |x|\to\infty.$$

The formula (7.3) can be extended to hold uniformly, for example, in $|\arg x| \leqslant \frac{1}{2}\pi$. This is so because, first, the algebraic asymptotic

expansion (3.2) holds uniformly in

$$\tfrac{1}{2}\pi - \frac{\pi}{2k} + \frac{(k-1)\pi}{8k^2} \leqslant \arg x \leqslant \tfrac{1}{2}\pi,$$

and, secondly, $$\cos\left(\frac{k}{k-1}\arg x\right) \leqslant -\sin\frac{\pi}{8k}$$

in this angle, so that the exponential part of the asymptotic expansion (7.3) is negligible in comparison with the algebraic part.

We observe that $$C_0 = \sqrt{\left(\frac{2\pi}{k(k-1)}\right)};$$

hence, if we take $n = 1$, we obtain the formula quoted in § 1.

The discussion of the function $F_k(x)$, as regards its asymptotic behaviour in the whole plane, is now complete.

I should like to express my warmest thanks to Mr. A. E. Ingham, under whose supervision and with whose generous help this paper was written.

ON THE RELATIONS BETWEEN THE ROTATION-GROUPS OF THE REGULAR POLYTOPES AND PERMUTATION-GROUPS

By D. M. Y. SOMMERVILLE.

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1. If N_p denotes as usual the number of p-dimensional boundaries of a regular polytope of n dimensions, and $N_{p,q}$ the number of p-dimensional boundaries which lie in or pass through (according as p is less than or greater than q) each q-dimensional boundary, the order of the group of rotations of the polytope is*

$$(n-2)!\, N_{0,n-2} N_{n-2}.$$

2.1. In all dimensions there are two groups:

(1) the *simplex* group (triangle, tetrahedron, 5-cell, etc.) whose order is $\frac{1}{2}(n+1)!$, and

(2) the *orthotope* group (square, cube, 8-cell, etc., and their reciprocals, square, octahedron, 16-cell, etc.) whose order is $2^{n-1} . n!$.

2.2. In three dimensions there is also the icosahedron-dodecahedron group of order 60.

In four dimensions we have the group of the 24-cell, of order 576; and the group of the 600-cell and 120-cell, of order 7200.

3.1. It is well known that the simplex group is simply isomorphic with the group of the even permutations of $n+1$ things $G^{n+1}_{\frac{1}{2}(n+1)!}$, and is

* P. H. Schoute, *Mehrdimensionale Geometrie*, 2 (1905), 255. Schoute published this result in *Rep. Brit. Assoc.*, 64 (Oxford, 1894), 562–563. An independent investigation of the orders of the groups, without knowledge of previous work, was given by E. L. Ince, *Proc. London Math. Soc.* (2), 17 (1918), xxxv–xxxviii.

therefore hemihedrically or (1, 2)-isomorphic with $G^{n+1}_{(n+1)!}$, the symmetrical permutation group*.

If the $n+1$ vertices are denoted by $A, B, C, \ldots$, any permutation of these $n+1$ letters leaves the connections of the vertices unaltered, but an odd permutation produces an inversion or reflection. Rotations correspond to even permutations. The complete symmetrical permutation-group is equivalent to the "extended" group, which includes reflections in planes, etc., of symmetry.

3.2. For the sake of completeness and convenience of reference we give the detailed operations of the tetrahedron group and the group of the 5-cell in space of four dimensions arranged in conjugate sets. (For $n > 3$ the simplex group in space of n dimensions is simple.)

Tetrahedron group, G^4_{12}.

1	Identity.	
4	(ABC), (DCB), ...	Rotations $\frac{1}{3}$ and $\frac{2}{3}$ respectively about an axis through a vertex perpendicular to the opposite face.
4	(CBA), (BCD), ...	
3	$(BC)(AD)$.	Rotations $\frac{1}{2}$ about an axis joining the mid-points of two opposite edges.
12		

(The tetrahedron group has as a self-conjugate subgroup the quadratic group $\{1, (BC)(AD), (CA)(BD), (AB)(CD)\}$ or $G_4{}^4$, and is (4, 1)-isomorphic with the cyclic group $G_3{}^3$.)

Simplex group in S_4, G^5_{60}.

		Rotations
1	Identity	1
20	(ABC)	$\frac{1}{3}$
15	$(BC)(AD)$	$\frac{1}{2}$
12	$(ABCDE)$	$\frac{1}{5}, \frac{2}{5}$
12	$(ABCED)$	$\frac{2}{5}, \frac{1}{5}$
60		

* P. H. Schoute, *Mehrdimensionale Geometrie*, 2 (1905), 256.

To obtain the other operations conjugate to $(ABCDE)$ permute any three consecutive letters cyclically.

In space of four dimensions the general motion of a body having one point fixed leaves two completely orthogonal planes both invariant, but is of two types*:

(1) *simple rotations* in which all the points of one plane are fixed,

(2) *double rotations* in which the points of each of the two planes rotate in their planes about the common point.

The symbols used above for the rotations denote the fractions of the complete rotation through four right angles.

The groups of the simplex and the orthotope in space of four dimensions were described by S. L. van Oss†. I have not, however, had access to his paper.

4.1. *The orthotope series.*

In space of any dimensions the orthotope and its reciprocal have the same rotation-group. It will be more convenient to use the reciprocal figure, which is formed of $2n$ vertices lying in pairs on n mutually perpendicular axes through the centre O, and equidistant from O.

Denote the vertices, in opposite pairs, by A_p, A_p' $(p = 1, 2, \ldots, n)$.

The n lines $A_p A_p'$ are diagonals; all other joins of pairs of vertices are edges—in number $2^2 . {}_nC_2 = 2n(n-1)$.

The number of triangular faces is $2^3 . {}_nC_3$, and they consist of all sets of three vertices such as A_p, A_q, A_r, in which the subscripts are all different, and the letters may or may not be accented.

In general the boundaries of $r-1$ dimensions are $2^r . {}_nC_r$ simplexes, represented by any set of r letters with subscripts all different, and with or without accents.

4.2. Any rotation or reflection corresponds to a permutation of the n subscripts together with changes of accents; rotations correspond to

* See F. N. Cole, "On rotations in space of four dimensions", *American J. o Math.*, 12 (1889), 191-210.

† S. L. van Oss, "Die Bewegungsgruppen der regelmässigen Gebilde von vier Dimensionen", *Diss. Giessen* (1894).

even (or odd) permutations with an even (or odd, respectively) number of changes of accent. This verifies that the order of the extended group is $2^n.n!$, and the order of the rotation-group $2^{n-1}.n!$. A rotation or reflection is also represented by a permutation of the $2n$ symbols A_p, A_p', subject to the condition that, if A_p is changed into A_q, A_p' is simultaneously changed into A_q'. Hence, when a rotation or reflection is represented by a product of independent cycles of permutations of the $2n$ symbols A_p, A_p', every cycle must either (1) consist of a succession of symbols followed by the same succession with all the accents changed (double cycle), or (2) be accompanied by another cycle which differs from it only in having all the accents changed (single cycles).

To every permutation of the n subscripts correspond 2^n rotations or reflections. This suggests the multiple isomorphism between the permutation-group and the rotation-group which we proceed to establish. This isomorphism was noticed in the case of four dimensions by Maschke*, but without explicit representation. He represents the operations analytically by linear substitutions.

4.3. Denote the symmetrical permutation-group of n things by G_0, the rotation-group of the polytope by G, and the extended group by G'.

The transposition operations $(A_p A_p')$ and their products form a subgroup Γ' of the extended group G', of order

$$1+{}_nC_1+{}_nC_2+\ldots+{}_nC_n=2^n;$$

and the even transpositions form a subgroup Γ of G, of order 2^{n-1}. This group, whose operations, other than identity, are all of order 2, is an extension to n dimensions of the "quadratic group".

We shall show that Γ is a self-conjugate subgroup of G.

4.4. Consider any monocyclic operation of G_0, say

$$S_0 \equiv (a \ldots lm \ldots d),$$

and let P stand for either A_p or A_p' and P' for A_p' or A_p respectively.

* H. Maschke, "The representation of finite groups, especially of the rotation-groups of the regular bodies of three- and four-dimensional space, by Cayley's colour diagrams", *American J. of Math.*, 18 (1896), 156–194.

Then we have a corresponding operation of G', viz.

$$S = (A \dots LM \dots D)(A' \dots L'M' \dots D').$$

Multiplying this by (AA') we get

$$S(AA') = (A \dots LM \dots DA' \dots L'M' \dots D') = (DD')S = S', \text{ say}; \quad (4.41)$$

and

$$S'(AA') = S = (DD')S'.$$

Multiplying again by (MM'), we get

$$S(AA')(MM') = (A \dots LM' \dots D')(A' \dots L'M \dots D) = (LL')(DD')S. \quad (4.42)$$

Further, the cycles of any operation contain no common symbols, and (AA') is commutative with any cycle which does not contain the symbol A.

Hence, if P is *any* operation of G', and T any operation of Γ', there is another operation T' of Γ' such that

$$TP = PT'.$$

Therefore

$$P^{-1}TP = T',$$

and the group Γ' is self-conjugate in G'; similarly Γ is self-conjugate in G.

4.5. From (4.41) and (4.42) it follows that, if P is any operation of G' and T any operation of Γ', PT consists of cycles containing the same succession of symbols as those of P, but with the following changes:

(1) if the cycle contains an even number of the subscripts involved in T, it is only changed as regards accents;

(2) if the cycle contains an odd number of the subscripts involved in T, it becomes double if it was single, and *vice versa*.

4.6. If S_{01} and S_{02} are distinct operations of G_0, S_1 and S_2 are two corresponding operations of G', and T_1, T_2 are distinct operations of Γ', then S_1T_1 and S_1T_2 are distinct, and S_1T_1 and S_2T_2 are distinct operations of G'.

If S_0 is an even permutation of G_0, any corresponding operation S belongs to the group G, and, when S is multiplied by all the operations of Γ, it gives rise to 2^{n-1} different operations all belonging to the rotation-group.

If S_0 is an odd permutation of G_0, a corresponding operation S belongs to G', but not to G, and, when multiplied by all the odd transpositions of the group Γ', gives rise to 2^{n-1} different operations belonging to the rotation-group.

Let S_{01}, S_{02} be any two operations of G_0, and let $S_{01}S_{02} = S_{03}$; also let S_1, S_2, and S_3 be the corresponding operations of G' which involve only the unaccented symbols, then $S_1S_2 = S_3$. If S_1' and S_2' are any corresponding operations of G, so that $S_1' = S_1T_1$ and $S_2' = S_2T_2 = T_3S_2$, then $S_1'S_2' = S_1T_1T_3S_2 = S_1T_4S_2 = S_1S_2T_5 = S_3T_5 = S_3'$. This establishes the multiple isomorphism of the rotation-group of the orthotope to the symmetric permutation-group of n things. *The operations which correspond to the identical operation are the 2^{n-1} operations of the group Γ; to each even permutation S_0 of G_0 correspond the 2^{n-1} operations ST, where T is an operation of Γ, and to each odd permutation S_0 and G_0 correspond the 2^{n-1} operations ST, where T is an odd transposition of Γ'.*

4.7. It is useful to give in full the rotation-groups of the octahedron and cube in three dimensions, and the orthotope and 16-cell in four dimensions.

Octahedral group $\equiv G_{24}^4$.

		Rotations
1	Identity	1
6	$(BCB'C')$	$\frac{1}{4}$ about a diagonal
3	$(BB')(CC')$	$\frac{1}{2}$ about a diagonal
8	$(ABC)(A'B'C'),$ $(A'BC)(AB'C')$	$\frac{1}{3}$ about axis perpendicular to opposite faces
6	$(BC)(B'C')(AA'),$ $(BC')(B'C)(AA')$	$\frac{1}{2}$ about axis perpendicular to opposite edges.
24		

[The octahedral group is also simply isomorphic with G_{24}^4, as well as being (4, 1)-isomorphic with $G_6{}^3$, and also (12, 1)-isomorphic with the cyclic group $G_2{}^2$, since it has $G_4{}^4$ and G_{12}^4 as self-conjugate subgroups.]

Orthotope group in S_4.

The even permutations of the four letters A, B, C, D are duplicated with accents and multiplied by the even transpositions

$$1;\quad \begin{matrix}(BB')(CC'), & (CC')(AA'), & (AA')(BB') \\ (AA')(DD'), & (BB')(DD'), & (CC')(DD');\end{matrix}\quad (AA')(BB')(CC')(DD');$$

and the duplicated odd permutations are multiplied by the odd transpositions

$$(AA'),\quad (BB'),\quad (CC'),\quad (DD');$$

$$(BB')(CC')(DD'),\quad (CC')(AA')(DD'),\quad (AA')(BB')(DD'),$$

$$(AA')(BB')(CC').$$

In the following table one only of each set of conjugate operations is given.

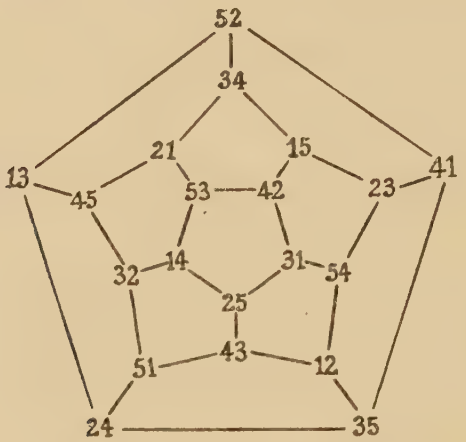
Permutation Group G_{24}^4	Number of Operations	Orthotope Group	Number of Operations	Rotations
Identity	1	1	1	1
(even)		$(BB')(CC')$	6	$\frac{1}{2}$
		$(AA')(BB')(CC')(DD')$	1	$\frac{1}{2}, \frac{1}{2}$
(BC)	6	$(BC)(B'C')(AA')$	24	$\frac{1}{2}$
(odd)		$(BCB'C')$	12	$\frac{1}{4}$
		$(BCB'C')(AA')(DD')$	12	$\frac{1}{4}, \frac{1}{2}$
$(BC)(AD)$	3	$(BC)(AD)(B'C')(A'D')$	12	$\frac{1}{2}$
(even)		$(BCB'C')(ADA'D')$	6	$\frac{1}{4}, \frac{1}{4}$
		$(BCB'C'(AD'A'D)$	6	$\frac{1}{4}, \frac{1}{4}$
(ABC)	8	$(ABC)(A'B'C')$	32	$\frac{1}{3}$
(even)		$(ABCA'B'C')(DD')$	32	$\frac{1}{6}, \frac{1}{2}$
$(ABCD)$	6	$(ABCDA'B'C'D')$	24	$\frac{1}{8}, \frac{3}{8}$
(odd)		$(ABCD'A'B'C'D)$	24	$\frac{3}{8}, \frac{1}{8}$
	24		192	

[The orthotope group in S_4 is also (32, 1)-isomorphic with G_6^3, as well as being (8 . 1)-isomorphic with G_{24}^4.]

5. *The icosahedron-dodecahedron group in three dimensions.*

The order of this group is 60 and it is *simply isomorphic with the alternating group of permutations of five things.* This isomorphism may be exhibited by a special notation for the vertices of the dodecahedron. The

twenty vertices of a dodecahedron may be grouped in sets of eight in five different ways, forming vertices of five cubes, two cubes at each vertex. Mark each vertex of the first cube with the number 1, each vertex of the second cube with the number 2, and so on. Each vertex will then have two numbers attached to it, and opposite vertices will have the same pair



of numbers. By ordering the numbers of each pair, each vertex will be represented by a distinct *ordered pair*, opposite vertices having the two numbers in opposite order. When the order for one vertex is fixed we can fix the order for all the other vertices so that no two vertices of the same pentagon have the same number standing first.

The sixty operations of the group are as follows:

Permutations	Number	Rotations
Identity	1	1
(123)	20	$\frac{1}{3}$ about axis through vertices 45, 54
(12)(34)	15	$\frac{1}{2}$ about axis through midpoint of edges (12, 43) and (21, 34)
(12345)	12	$\frac{1}{5}$ about axis through centres of pentagons (14, 25, 31, 42, 53) and (41, 52, 13, 24, 35)
(12354)	12	$\frac{2}{5}$ about axis through centres of pentagons (14, 53, 21, 45, 32) and (41, 35, 12, 54, 23)
	60	

6.1. *The group of the 24-cell in four dimensions.*

The twenty-four vertices of the 24-cell consist of the sixteen vertices of an orthotope or 8-cell, together with the centres of the eight cubes of

the orthotope. Further, the sixteen vertices of the orthotope consist of the vertices of two 16-cells, and the cell-centres are the vertices of a third 16-cell.

Take one cube of the 8-cell; its eight vertices consist of the vertices of two tetrahedra. Denote one set by A_1', B_1', C_1', D_1, and the opposite vertices of the cube by A_2, B_2, C_2, D_2. Then denote the opposite vertices of the 8-cell (which are also opposite vertices of the 24-cell) by the same symbols with the accents changed. We have then the four cubes:

$$(A_3) \quad A_1 B_1' C_1' D_1' \quad A_2 B_2' C_2' D_2,$$

$$(B_3) \quad A_1' B_1 C_1' D_1' \quad B_2 A_2' D_2 C_2',$$

$$(C_3) \quad A_1' B_1' C_1 D_1' \quad C_2 D_2 A_2' B_2',$$

$$(D_3) \quad A_1 B_1 C_1 D_1' \quad D_2' C_2' B_2' A_2',$$

and by changing all the accents we obtain the four opposite cubes. Opposite vertices of each cube are equidistant from the ends in each row of symbols.

We now take the centres of the eight cubes A_3, ..., A_3', ..., as indicated, to complete the twenty-four vertices of the 24-cell, forming the vertices of three 16-cells A_1, ..., A_2, ..., A_3, ... Through each vertex of the 24-cell there are eight edges, and the eight vertices to which a vertex is joined are the vertices of a cube. Thus the above scheme indicates the vertices to which A_3, B_3, C_3, and D_3 are joined. The remaining connections are represented by the following scheme:

$$A_1 \text{ joined to } A_2 B_2' C_2' D_2' \quad A_3 B_3' C_3' D_3,$$

$$B_1 \quad ,, \quad ,, \quad A_2' B_2 C_2' D_2' \quad B_3 A_3' D_3 C_3',$$

$$C_1 \quad ,, \quad ,, \quad A_2' B_2' C_2 D_2' \quad C_3 D_3 A_3' B_3',$$

$$D_1 \quad ,, \quad ,, \quad A_2 B_2 C_2 D_2' \quad D_3' C_3' B_3' A_3';$$

$$A_2 \quad ,, \quad ,, \quad A_3 B_3' C_3' D_3' \quad A_1 B_1' C_1' D_1,$$

$$B_2 \quad ,, \quad ,, \quad A_3' B_3 C_3' D_3' \quad B_1 A_1' D_1 C_1',$$

$$C_2 \quad ,, \quad ,, \quad A_3' B_3' C_3 D_3' \quad C_1 D_1 A_1' B_1',$$

$$D_2 \quad ,, \quad ,, \quad A_3 B_3 C_3 D_3' \quad D_1' C_1' B_1' A_1'.$$

6.2. Each vertex is thus the centre of a cube, and the separation of the vertices into three sets in this way is unique, for, starting with any vertex, we obtain round it the eight vertices of a cube, and the eight opposite

vertices complete an 8-cell; the eight vertices of the cube can be separated in just one way into the vertices of two tetrahedra, and therefore the vertices of the 8-cell are separated in just one way into the vertices of the two 16-cells; the centres of the eight cubes (including the original vertex) form the vertices of the third 16-cell.

6.3. Hence every operation which transforms the 24-cell into itself must keep the three 16-cells intact and permute them among themselves. This suggests that *the group of the 24-cell is multiply isomorphic with the permutation-group of three things.*

All the operations which transform each 16-cell into itself evidently form a subgroup Γ. Let T be any operation of this subgroup, and let S be any other operation of the group. Then ST permutes the three 16-cells in the same way as S; therefore STS^{-1} transforms each 16-cell into itself and thus belongs to Γ. Γ is therefore self-conjugate. Making Γ correspond to the identical operation, to any operation of the permutation-group of three symbols corresponds a set of operations permuting the three 16-cells in the same way; if S is any one such operation, the whole set is obtained by multiplying S by the operations of Γ. Hence the isomorphism is established. Since the order of the permutation-group is 6, and the order of the whole group is 576, the order of Γ is 96, and the isomorphism is 96-ply.

6.4. The operations of Γ are obtained from those operations of the group of the 16-cell which are derived from the even permutations of four things (see §4.7). The cubes

$$A_1B_1'C_1'D_1' \quad A_2B_2'C_2'D_2 \quad \text{and} \quad A_1B_1'C_1'D_1 \quad A_3B_3'C_3'D_3'$$

are changed by the operations on the 16-cell $A_1 \ldots$ as follows:

Operation of 16-cell group	$A_1B_1'C_1'D_1'$ $A_2B_2'C_2'D_2$ changed into	$A_1B_1'C_1'D_1$ $A_3B_3'C_3'D_3'$ changed into
$(AA')(BB')(CC')(DD')$	$A_1'B_1C_1D_1$ $A_2'B_2C_2D_2'$	$A_1'B_1C_1D_1'$ $A_3'B_3C_3D_3$
$(BB')CC')$	$A_1B_1C_1D_1'$ $D_2'C_2'B_2'A_2'$	$A_1B_1C_1D_1$ $D_3C_3'B_3'A_3'$
$(BC)(AD)\ldots$	$D_1C_1'B_1'A_1'$ $A_2B_2C_2D_2$	$D_1C_1'B_1'A_1$ $D_3'C_3'B_3'A_3$
$(BCB'C')(ADA'D')$	$D_1C_1'B_1A_1$ $B_2A_2D_2'C_2'$	$D_1C_1'B_1A_1'$ $C_3'D_3'A_3'B_3$
$(ABC)\ldots$	$B_1C_1'A_1'D_1'$ $B_2C_2'A_2'D_2$	$B_1C_1'A_1'D_1$ $B_3C_3'A_3'D_3'$
$(ABCA'B'C')(DD')$	$B_1C_1'A_1D_1$ $C_2'B_2D_2'A_2$	$B_1C_1'A_1D_1'$ $C_3'B_3D_3A_3$

Hence we have the operations of the self-conjugate subgroup Γ (for simplicity, repetitions of cycles with the accents changed are indicated by ...).

Self-conjugate subgroup Γ of the 24-cell group	No. of operations	Rotations
Identity	1	1
$(A_1A_1')(B_1B_1')(C_1C_1')(D_1D_1')(A_2A_2')(B_2B_2')(C_2C_2')(D_2D_2')(A_3A_3')(B_3B_3')(C_3C_3')(D_3D_3')$	1	$\frac{1}{2}, \frac{1}{2}$
$(B_1B_1')(C_1C_1')(A_2D_2')(B_2C_2)(A_3D_3)(B_3C_3)\ldots$ $(A_1A_1')(D_1D_1')(A_2D_2)(B_2C_2')(A_3D_3')(B_3C_3')\ldots$ (permute ABC cyclically)	6	$\frac{1}{2}$
(Two other sets of six conjugate operations by permuting 123 cyclically)	12	
$(B_1C_1\ldots)(A_1D_1\ldots)(A_2B_2\ldots)(C_2D_2\ldots)(C_3A_3\ldots)(B_3D_3\ldots)$	6	
$(B_1C_1\ldots)(D_1A_1\ldots)(C_2A_2\ldots)(D_2B_2\ldots)(A_3B_3\ldots)(D_3C_3\ldots)$	6	$\frac{1}{4}, \frac{1}{4}$
(Permute ABC cyclically and take cubes)		
$(A_1B_1C_1)(A_2B_2C_2)(A_3B_3C_3)\ldots$ $(A_1'B_1C_1)(D_2'C_2B_2)(D_3C_3B_3)\ldots$ $(D_1C_1'B_1)(D_2A_2'C_2)(D_3B_3'A_3)\ldots$ (permute ABC and also 123 cyclically)	16	
$(D_1C_1B_1')(D_2B_2A_2')(D_3A_3C_3')\ldots$		$\frac{1}{3}$
(Another set of sixteen conjugate operations by taking their squares)	16	
$(A_1C_1'B_1\ldots)(D_1D_1')(A_2C_2'B_2\ldots)(D_2D_2')(A_3C_3'B_3\ldots)(D_3D_3')$ $(A_1C_1B_1'\ldots)(D_1D_1')(D_2B_2C_2'\ldots)(A_2A_2')(D_3B_3'C_3\ldots)(A_3A_3')$ (permute ABC and also 123 cyclically) $(D_1'C_1B_1\ldots)(A_1A_1')(D_2'B_2A_2\ldots)(C_2C_2')(D_3'A_3C_3\ldots)(B_3B_3')$ $(D_1C_1B_1\ldots)(A_1A_1')(D_2A_2C_2\ldots)(B_2B_2')(D_3B_3A_3\ldots)(C_3C_3')$	16	$\frac{1}{6}, \frac{1}{2}$
(Another set of sixteen conjugate operations by taking their 5th powers)	16	
	96	

6.5. The operations of the 16-cell group which are derived from the odd permutations give rise to operations which interchange two of the 16-cells.

Operation of 16-cell group	$A_1B_1'C_1'D_1'$ $A_2B_2'C_2'D_2$ changed to	$A_1B_1'C_1'D_1$ $A_3B_3'C_3'D_3'$ changed to
$(BC)(B'C')(AA')$	$A_1'C_1'B_1'D_1'$ $D_3'B_3C_3A_3$	$A_1'C_1'B_1'D_1$ $D_2B_2C_2A_2$
$(BCB'C')$	$A_1C_1'B_1D_1'$ $C_3'A_3D_3B_3$	$A_1C_1'B_1D_1$ $C_2'A_2D_2'B_2$
$(BCB'C')(AA')(DD')$	$A_1'C_1'B_1D_1$ $B_3D_3'A_3'C_3'$	$A_1'C_1'B_1D_1'$ $B_2D_2A_2'C_2'$
$(ABCDA'B'C'D')$	$B_1C_1'D_1'A_1$ $B_3C_3'D_3A_3$	$B_1C_1'D_1'A_1'$ $C_2'B_2A_2'D_2$
$(ABCD'A'B'C'D)$	$B_1C_1'D_1A_1'$ $C_3'B_3A_3'D_3'$	$B_1C_1'D_1A_1$ $B_2C_2'D_2'A_2$

Hence we have the operations which interchange the two 16-cells (2) and (3), not necessarily in conjugate sets:

	Number of operations	Rotations
$(A_1A_1')(B_1C_1)(A_2D_3')(A_3D_2)(B_2B_3')(C_2C_3')\ldots$	24	$\frac{1}{2}$
$(B_1C_1B_1'C_1')(A_2C_3'D_2'B_3')(A_3C_2'D_3B_2')\ldots$	12	$\frac{1}{4}$
$(B_1C_1B_1'C_1')(A_1A_1')(D_1D_1')(A_2B_3D_2'C_3)(A_3B_2D_3C_2)\ldots$	12	$\frac{1}{4}, \frac{1}{2}$
$(A_1B_1C_1D_1A_1'B_1'C_1'D_1')(A_2B_3B_2'C_3'A_2'B_3'B_2C_3)(A_3C_2'D_3D_2'A_3'C_2D_3'D_2)$	24	$\frac{1}{8}, \frac{3}{8}$
$(A_1B_1C_1D_1'\ldots)(A_2C_3'D_2'D_3\ldots)(A_3B_2B_3'C_2'\ldots)$	24	$\frac{3}{8}, \frac{1}{8}$
	96	

Two other similar sets of ninety-six each interchange the 16-cells (3) and (1), and (1) and (2).

6.6. Then the operation $(A_1A_2A_3)(B_1B_2B_3)(C_1C_2C_2)(D_1D_2D_3)\ldots$ also belongs to the group and permutes the three 16-cells according to the permutation (123) of the subscripts. Multiplying the operations of Γ by this operation and its square, we obtain two further sets of ninety-six operations, each containing twenty-four rotations $(\frac{1}{3}, \frac{1}{3})$, twenty-four rotations $(\frac{1}{6}, \frac{1}{6})$, and forty-eight rotations $(\frac{1}{12}, \frac{5}{12})$.

7.1. *The group of the* 120*-cell and* 600*-cell in four dimensions.*

This group is of order 7200 and its operations have been enumerated by S. L. van Oss* as follows:

Rotation-group of the 600-cell.

Symbol of operation	Rotations	Number in set	Symbol of operation	Rotations	Number in set
A	$\frac{1}{2}$	450	L	$\frac{1}{30}, \frac{11}{30}$	480
B	$\frac{1}{3}$	400	L'	$\frac{7}{30}, \frac{13}{30}$	480
C	$\frac{1}{5}$	144	M	$\frac{1}{15}, \frac{4}{15}$	480
C'	$\frac{2}{5}$	144	M'	$\frac{2}{15}, \frac{7}{15}$	480
D	$\frac{1}{6}, \frac{1}{2}$	400	N	$\frac{1}{4}, \frac{1}{4}$	60
E	$\frac{1}{10}, \frac{1}{2}$	144	P	$\frac{1}{6}, \frac{1}{6}$	40
E'	$\frac{3}{10}, \frac{1}{2}$	144	Q	$\frac{1}{3}, \frac{1}{3}$	40
F	$\frac{1}{5}, \frac{2}{5}$	288	R	$\frac{1}{10}, \frac{1}{10}$	24
G	$\frac{1}{10}, \frac{3}{10}$	288	R'	$\frac{3}{10}, \frac{3}{10}$	24
H	$\frac{1}{12}, \frac{5}{12}$	1200	S	$\frac{1}{5}, \frac{1}{5}$	24
K	$\frac{1}{20}, \frac{9}{20}$	720	S'	$\frac{2}{5}, \frac{2}{5}$	24
K'	$\frac{3}{20}, \frac{7}{20}$	720	T	$\frac{1}{2}, \frac{1}{2}$	1
			1	1	1
					7200

7.2. A multiple isomorphism of the group G_{7200} with the icosahedral group is implicit in a representation given by Goursat†. According to

* S. L. van Oss, "Das regelmässige Sechshundertzell und seine selbstdeckenden Bewegungen", *Amsterdam, Verh. K. Akad. Wet.* (§1), 7, No. 1 (1899). Reference may be made also to P. H. Schoute, "Regelmässige Schnitte und Projectionen des Hundertzwanzigzelles und Sechshundertzelles", *Amsterdam, Verh. K. Akad. Wet.* (§ 1), 2, No. 7 (1894), in which detailed tables of the coordinates of the vertices of the 600-cell and the 120-cell are given, referred to the several sets of principal axes. The following misprint in "Tabelle I" should be noted: under D, w_2, vertex 27, for "2" read "-2". The author's *Introduction to the geometry of n dimensions* (London, 1929) gives a compact table of the coordinates of the vertices of the 600-cell referred to a set of rectangular axes which pass through vertices.

† E. Goursat, "Sur les substitutions orthogonales et les divisions régulières de l'espace", *Annales Sci. de l'Ecole Normale Sup.* (3), 6 (1889), 9–102.

the method of Goursat the groups of the regular polytopes in S_4 are represented by means of linear substitutions. The general rotation, or displacement with one point fixed, in space of four dimensions is represented by the general linear transformation which leaves the hypersphere $x^2+y^2+z^2+w^2=1$ unaltered, and whose determinant is positive; transformations with a negative determinant correspond to reflections. These transformations can also be interpreted as collineations which leave unaltered the sphere $x^2+y^2+z^2+w^2=0$ in three dimensions, and transform the generating lines of each system into generating lines of the same system. If the two systems of generating lines are defined by parameters ξ and η, the transformations in question are represented by pairs of linear substitutions on ξ and η separately, of the form

$$\xi' = \frac{a\xi+b}{a'-b'\xi}, \quad \eta' = \frac{c\eta+d}{c'-d'\eta},$$

where a, a', ..., represent conjugate complex numbers. Thus any group of collineations keeping the sphere $\Sigma x^2=0$ unaltered is compounded of two groups which we may denote by $\{\xi\}$ and $\{\eta\}$. The problem of determining all the finite groups compounded of two groups in this way is solved completely in Goursat's memoir.

7.3. Now to every substitution which leaves $\Sigma x^2=0$ unaltered correspond two which leave $\Sigma x^2=1$ unaltered, for the substitution which changes the sign of each of the four variables is only the identical operation in the former group, but in the latter represents a double rotation T interchanging pairs of opposite points. (In space of odd dimensions this operation is a reflection.) Hence to every group of collineations in S_3 of this form corresponds a group of rotations in S_4 which, if it contains the operation T, is (2, 1)-isomorphic with it.

The group of the 5-cell in S_4 does not contain the operation T, since the 5-cell does not possess opposite vertices. Its group is just that given by Goursat as No. XXXII, and is compounded of two icosahedral groups in which there is a certain (1, 1) correspondence between the operations; it is, of course, simply isomorphic with the icosahedral group.

The groups of the 16-cell and 24-cell both contain T, and are (2, 1)-isomorphic with Goursat's groups XXVII and XXVIII respectively.

The group of the 600-cell, G_{7200}, is (2, 1)-isomorphic with Goursat's group XXX (or G_{3600}), which is the direct product of two icosahedral groups, each being therefore a self-conjugate subgroup of G_{3600}. Hence G_{7200} contains two self-conjugate subgroups of order 120 which are

(2, 1)-isomorphic with the icosahedral group. We have still to determine these two subgroups.

7.4. The operation T is permutable with every operation of G_{7200}. For suppose that the operation S changes the vertex A into B, and therefore the opposite vertex A' into the opposite vertex B'; then

$$STS^{-1} = (AB \dots A'B' \dots)(AA')(BB') \dots (BA \dots B'A' \dots)$$
$$= (AA') \dots = T.$$

Also $T^2 = 1$, hence $\{T, 1\}$ is a self-conjugate subgroup of G_{7200} of order 2. The (2, 1)-isomorphism of G_{7200} with G_{3600} is thus indicated, the two operations 1 and T corresponding to 1 in G_{3600}.

7.5. Let P be any operation of G_{3600} of order n, P' and TP' the corresponding operations of G_{7200}. Then $(P')^n = 1$ or T. If n is odd, $(TP')^n = T(P')^n$; hence one of the two operations P' and TP' is of order n and the other is of order $2n$. If n is even, $(TP')^n = (P')^n$; hence P' and TP' are either both of order n or both of order $2n$.

Now, since G_{3600} is the direct product of two permutable icosahedral groups, the number of operations of different orders is as follows:

(G_{3600}) Order of operation	1	2	3	5	6	10	15
Number	1	255	440	624	600	720	960

The operations of order 3, 5, 15 give rise to all the operations of G_{7200} of orders 3, 5, 15 and 6, 10, 30. The 600 operations of order 6 give the 1,200 operations of G_{7200} of order 12, and the 720 operations of order 10 give the 1,440 operations of order 20. There remain the 451 operations of G_{7200} of order 2 and the sixty of order 4. One of these operations of order 2 is T, the remaining 450 are derived from the 225 operations PQ, where P and Q are operations of order 2 in the two groups $\{\xi\}$ and $\{\eta\}$ respectively. The sixty operations of order 4 come from the thirty operations P, Q.

Hence the self-conjugate subgroups of G_{7200} of order 120 each consist of 1, T, and thirty operations of order 4, twenty of order 3, twenty of order 6, twenty-four of order 5, and twenty-four of order 10. The operations labelled N, P, Q, R, R', S, and S' in §7.1 each consist of two conjugate sets; one each of these, together with 1 and T, form the two self-conjugate subgroups*.

* The Referee remarks that the two self-conjugate subgroups are actually the "extended" icosahedral group, in Klein's sense of the term "extended", which differs from the sense in which the term is used in this paper.

7.6. There are two well-known groups of order 120 which have to be distinguished from this group; these are the symmetric permutation group on five symbols, G^5_{120}, and the extended icosahedral group consisting of rotations and reflections. For comparison, the number of operations of various orders of the three groups are tabulated as follows:

Order of operation	1	2	3	4	5	6	10
Symmetric group G^5_{120}	1	25	20	30	24	20	—
Extended icosahedral group	1	31	20	—	24	20	24
Self-conjugate subgroup of G_{7200}	1	1	20	30	24	20	24

The final result with regard to the 600-cell group may be summarised. G_{7200} has a self-conjugate subgroup $\{1, T\}$ of order 2, where T is the operation which interchanges opposite vertices. It is thus (2, 1)-isomorphic with G_{3600}, the direct product of two icosahedral groups. It has also two self-conjugate subgroups of order 120, and is therefore in two ways (120, 1)-isomorphic with the icosahedral group.

I am indebted to the Referee for the correction of several errors, and especially for the indication of the results relating to the 600-cell group, where I fell into error owing to being unable at first to obtain access to Goursat's memoir.

A CLASS OF FOURIER KERNELS

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Introduction and formalities.

1.1. We may say that $K(x)$ is a *Fourier kernel* if it gives rise to a "Fourier formula"

$$F(x) = \int_0^\infty K(xu)\,du \int_0^\infty K(uy)\,F(y)\,dy \tag{1.1.1}$$

for an arbitrary function $F(x)$, or (what is, from a formal point of view, the same thing) to a "reciprocity"

$$G(x) = \int_0^\infty K(xy)\,F(y)\,dy. \tag{1.1.2}$$

$$F(x) = \int_0^\infty K(xy)\,G(y)\,dy, \tag{1.1.3}$$

The classical example is $K(x) = \sqrt{(2/\pi)}\cos x$.

Suppose that

$$k(s) = \int_0^\infty x^{s-1}K(x)\,dx \tag{1.1.4}$$

(and similarly with other letters). Then, starting from (1.1.2), and integrating formally, we obtain

$$g(s) = \int_0^\infty x^{s-1}dx \int_0^\infty K(xy)\,F(y)\,dy = \int_0^\infty F(y)\,dy \int_0^\infty x^{s-1}K(yx)\,dx$$

$$= \int_0^\infty y^{-s}F(y)\,dy \int_0^\infty w^{s-1}K(w)\,dw = f(1-s)\,k(s).$$

Similarly (1.1.3) gives

$$f(s) = g(1-s)\,k(s);$$

† We are indebted to Mr. J. Cossar for reading the proofs and correcting a number of mistakes.

and, if we change s into $1-s$ in one of these equations, and multiply, we deduce that

$$k(s)\,k(1-s)=1. \tag{1.1.5}$$

Solving (1.1.4) by Mellin's inversion formula, we obtain

$$K(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\,k(s)\,ds. \tag{1.1.6}$$

We may therefore expect that any Fourier kernel must be expressible in this form, with a $k(s)$ subject to (1.1.5).

1.2. This condition may also be expected to be in some sense sufficient.

A characteristic property of a Fourier kernel $K(x)$ is that, if

$$K_1(x)=\int_0^x K(u)\,du, \tag{1.2.1}$$

then

$$\int_0^\infty K(yu)\,\frac{K_1(xu)}{u}\,du=\begin{cases}1 & (0<y<x)\\ 0 & (y>x).\end{cases} \tag{1.2.2}$$

In fact, (1.2.2) leads to

$$\int_0^x F(y)\,dy=\int_0^\infty F(y)\,dy\int_0^\infty K(yu)\,\frac{K_1(xu)}{u}\,du=\int_0^\infty\frac{K_1(xu)}{u}\,du\int_0^\infty K(uy)\,F(y)\,dy,$$

from which (1.1.1) follows by formal differentiation.

On the other hand, (1.1.6) gives

$$K_1(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{1-s}}{1-s}\,k(s)\,ds. \tag{1.2.3}$$

Taking $c=\frac{1}{2}$ in (1.1.6) and (1.2.3), we obtain

$$\begin{aligned}\int_0^\infty K(yu)\,\frac{K_1(xu)}{u}\,du&=\lim_{\lambda\to\infty}\int_{1/\lambda}^{\lambda}K(yu)\,\frac{K_1(xu)}{u}\,du\\
&=\lim_{\lambda\to\infty}\frac{1}{4\pi^2}\int_{1/\lambda}^{\lambda}\frac{du}{u}\int_{-\infty}^{\infty}(yu)^{-\frac{1}{2}-it}\,k(\tfrac{1}{2}+it)\,dt\int_{-\infty}^{\infty}\frac{(xu)^{\frac{1}{2}-it'}}{\frac{1}{2}-it'}\,k(\tfrac{1}{2}+it')\,dt'\\
&=\lim_{\lambda\to\infty}\frac{1}{4\pi^2}\int_{-\infty}^{\infty}y^{-\frac{1}{2}-it}\,k(\tfrac{1}{2}+it)\,dt\int_{-\infty}^{\infty}\frac{x^{\frac{1}{2}-it'}}{\frac{1}{2}-it'}\,k(\tfrac{1}{2}+it')\,dt'\int_{1/\lambda}^{\lambda}u^{-1-it-it'}\,du\\
&=\lim_{\lambda\to\infty}\frac{1}{2\pi^2}\int_{-\infty}^{\infty}y^{-\frac{1}{2}-it}\,k(\tfrac{1}{2}+it)\,dt\int_{-\infty}^{\infty}\frac{x^{\frac{1}{2}-it'}}{\frac{1}{2}-it'}\,k(\tfrac{1}{2}+it')\,\frac{\sin\mu(t'+t)}{t'+t}\,dt',\end{aligned}$$

where

$$\mu = \log \lambda. \tag{1.2.4}$$

If we pass to the limit formally in accordance with the ordinary rules, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}+it} k(\tfrac{1}{2}+it)\,k(\tfrac{1}{2}-it)\,\frac{dt}{\frac{1}{2}+it} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}+it}\frac{dt}{\frac{1}{2}+it},$$

by (1.1.5), and the value of this integral is 1 if $y < x$ and 0 if $y > x$.

1.3. We have stated our analysis in terms of the symmetrical formulae arising from a Fourier kernel. We may, however, apply similar analysis to the unsymmetrical equations

$$F(x) = \int_0^\infty K(xu)\,du \int_0^\infty H(uy)\,F(y)\,dy, \tag{1.3.1}$$

$$G(x) = \int_0^\infty H(xy)\,F(y)\,dy, \tag{1.3.2}$$

$$F(x) = \int_0^\infty K(xy)\,G(y)\,dy. \tag{1.3.3}$$

The equation (1.1.5) is then replaced by

$$k(s)\,h(1-s) = 1, \tag{1.3.4}$$

and the "solving kernel" of the equation (1.3.2) is defined by

$$H(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\,h(s)\,ds. \tag{1.3.5}$$

1.4. If $K(x)$ is a Fourier kernel, and

$$F(x) = \lambda\int_0^\infty K(xy)\,F(y)\,dy, \tag{1.4.1}$$

then it follows from the Fourier formulae that $\lambda^2 = 1$. Thus a Fourier kernel has two *Eigenwerte* only, viz. ± 1. With each of these values is associated a mass of *Eigenfunktionen* (self-reciprocal functions) of very diverse types.

If $K(x)$ is not a Fourier kernel, and we operate on (1.4.1) as in § 1.1, we obtain

$$\lambda^2\,k(s)\,k(1-s) = 1,$$

which is in general false. A typical case is that in which

$$K(x) = e^{-x}, \quad k(s) = \Gamma(s).$$

We considered this case in our paper **11**. The inference from our present analysis is that there can be no solutions for which

$$\int_0^\infty f(x)\, x^{s-1}\, dx$$

exists. Actually, every real λ is an *Eigenwert*, and with each is associated (substantially) just one function.

1.5. Before proceeding further we illustrate our formalities by examples.

(1) If

$$K(x) = x^{\frac{1}{2}} J_\nu(x), \tag{1.5.1}$$

then†

$$k(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})}, \tag{1.5.2}$$

a solution of (1.1.5). The cases $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$ give

$$K(x) = \sqrt{\left(\frac{2}{\pi}\right)} \cos x, \quad k(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2}-\frac{1}{2}s)};$$

$$K(x) = \sqrt{\left(\frac{2}{\pi}\right)} \sin x, \quad k(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)}.$$

These functions give the Hankel and Fourier formulae.

(2) If $$K(x) = x^{\frac{1}{2}} Y_\nu(x),$$

then‡

$$k(s) = -2^{s-\frac{1}{2}} \pi^{-1} \Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\nu+\tfrac{1}{4})\, \Gamma(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4}) \cos(\tfrac{1}{2}s-\tfrac{1}{2}\nu+\tfrac{1}{4})\pi.$$

This is not a solution of (1.1.5), and $K(x)$ is not a Fourier kernel. Here (1.3.4) gives

$$h(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})} \tan \tfrac{1}{2}(s+\nu+\tfrac{1}{2})\pi.$$

If we substitute this value of $h(s)$ in (1.3.5), with $0 < c < 1$, and evaluate the integral by calculation of the residues at poles on the left, we find that

$$H(x) = x^{\frac{1}{2}} \mathbf{H}_\nu(x),$$

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† Watson (**18**), §13.24 (1).
‡ Watson (**18**), §13.24 (5).

where $\mathbf{H}_\nu(x)$ is Struve's function†. This transformation was found by Titchmarsh‡.

(3) Fox§ has discovered and discussed a number of very general transformations in which $K(x)$ and $H(x)$ are linear combinations of generalized hypergeometric functions. From our present point of view, these originate as follows.

Suppose that $\alpha_1 > 0$, that ρ_1 and ρ_2 are any real numbers other than negative integers, and that

$$\phi = \alpha_1 - \rho_1 - \rho_2 + \tfrac{1}{2},$$

and let $$k(s) = 2^{s-1} \frac{\Gamma(\alpha_1 + \frac{1}{2}\phi - \frac{1}{2}s)\,\Gamma(\frac{1}{2}s - \frac{1}{2}\phi)}{\Gamma(\rho_1 + \frac{1}{2}\phi - \frac{1}{2}s)\,\Gamma(\rho_2 + \frac{1}{2}\phi - \frac{1}{2}s)}.$$

Then $K(x)$ may be calculated from (1.1.6), with $0 < c < \alpha_1$; we find

$$K(x) = (\tfrac{1}{2}x)^{-\phi} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)}{\Gamma(\rho_1+n)\,\Gamma(\rho_2+n)} \frac{(-\frac{1}{4}x^2)^n}{n!} = (\tfrac{1}{2}x)^{-\phi}\, {}_1F_2(\alpha_1, \rho_1, \rho_2; -\tfrac{1}{4}x^2),$$

in the notation of Barnes. If now we calculate $h(s)$ from (1.3.4), and $H(x)$ from (1.3.5), by summation of residues, we find

$$H(x) = H_1(x) + H_2(x),$$

where

$$H_1(x) = \frac{\sin(\alpha_1 - \rho_1)\pi}{\sin(\rho_2 - \rho_1)\pi} (\tfrac{1}{2}x)^{2\rho_1+\phi-1}\, {}_1F_2(1-\alpha_1+\rho_1,\ 1-\rho_2+\rho_1,\ \rho_1;\ -\tfrac{1}{4}x^2),$$

and $H_2(x)$ is derived from $H_1(x)$ by interchange of ρ_1 and ρ_2. The formulae thus obtained are those of Fox's Theorem 1, in the special case $p = 1$. In the general case $k(s)$ is a more complex product of gamma-functions of the same type.

The case $\alpha_1 = 1$, $\rho_1 = \frac{3}{2}$, $\rho_2 = \nu + \frac{3}{2}$ gives example (2) above. The case $\alpha_1 = 1$, $\rho_1 = a+1$, $\rho_2 = \nu + a + 1$ gives a more general transformation found by Hardy‖ and discussed by Cooke¶. The case $\alpha_1 = \nu + \frac{3}{2}$, $\rho_1 = \nu + 1$, $\rho_2 = 2\nu + 1$ gives

$$K(x) = \tfrac{1}{2}\sqrt{\pi}\, \frac{d}{dx}\{x J_\nu^2(\tfrac{1}{2}x)\}, \quad H(x) = -\sqrt{\pi}\, J_\nu(\tfrac{1}{2}x)\, Y_\nu(\tfrac{1}{2}x);$$

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† Watson (**18**), §10.4, §13.24 (2).
‡ Titchmarsh (**15**).
§ Fox (**4**).
‖ Hardy (**7**).
¶ Cooke (**3**).

a transformation due to Bateman†. The case $a_1 = \nu+a+\frac{1}{2}$, $\rho_1 = \nu+a+1$, $\rho_2 = 2\nu+a+1$ gives a more general transformation due to Titchmarsh‡. Fuller details concerning these transformations will be found in § 5.2 of Fox's paper.

If we take

$$k(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(a_1+\frac{1}{2}s)\,\Gamma(a_2+\frac{1}{2}s)\,\Gamma(a_3-\frac{1}{2}s)}{\Gamma(b_1+\frac{1}{2}s)\,\Gamma(b_2-\frac{1}{2}s)\,\Gamma(b_3-\frac{1}{2}s)},$$

where

$$a_1+a_2+a_3+\tfrac{1}{2} = b_1+b_2+b_3,$$

we obtain examples of Fox's Theorem 2. For example, if

$$a_1 = \tfrac{1}{2}\mu+\tfrac{1}{2}\nu+\tfrac{1}{2}, \quad a_2 = \tfrac{1}{2}-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu, \quad a_3 = \tfrac{1}{2},$$

$$b_1 = 1, \quad b_2 = \tfrac{1}{2}\nu-\tfrac{1}{2}\mu+\tfrac{1}{2}, \quad b_3 = \tfrac{1}{2}\mu-\tfrac{1}{2}\nu+\tfrac{1}{2},$$

$K(x)$ and $H(x)$ are each combinations of two hypergeometric functions, and can be reduced to the forms§

$$K(x) = \frac{\sqrt{\pi}}{2\sqrt{2}\sin\frac{1}{2}(\mu+\nu)\pi}\, x\{J_{-\mu}(\tfrac{1}{2}x)\,J_{-\nu}(\tfrac{1}{2}x)-J_{\mu}(\tfrac{1}{2}x)\,J_{\nu}(\tfrac{1}{2}x)\},$$

$$H(x) = \frac{\sqrt{\pi}}{2\sqrt{2}\cos\frac{1}{2}(\mu-\nu)\pi}\,\frac{d}{dx}\{x\,[J_{\mu}(\tfrac{1}{2}x)\,J_{-\nu}(\tfrac{1}{2}x)+J_{-\mu}(\tfrac{1}{2}x)\,J_{\nu}(\tfrac{1}{2}x)]\}.$$

(4) If

$$K(x) = x^{\frac{1}{2}}\left\{Y_{\nu}(x)+\frac{2}{\pi}\cos a\pi K_{\nu}(x)\right\},$$

where $K_\nu(x)$ is the function of Basset and Macdonald, then|| $k(s)$ is the product of 2^{2s-1} by

$$\frac{\Gamma(\frac{1}{4}s+\frac{1}{4}\nu+\frac{1}{8})\,\Gamma(\frac{1}{4}s+\frac{1}{4}\nu+\frac{5}{8})\,\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{1}{8})\,\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{5}{8})}{\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{1}{8}+\frac{1}{2}a)\,\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s-\frac{1}{2}a)\,\Gamma(\frac{1}{4}s-\frac{1}{4}\nu+\frac{1}{8}-\frac{1}{2}a)\,\Gamma(\frac{7}{8}+\frac{1}{4}\nu-\frac{1}{4}s+\frac{1}{2}a)}.$$

In this case again $H(x)$ is the sum of two hypergeometric series. There are two interesting particular cases. If $\nu = 0$, $a = 1$, then

$$k(s) = -2^{2s-1}\left\{\frac{\Gamma(\frac{1}{4}s+\frac{1}{8})}{\Gamma(\frac{3}{8}-\frac{1}{4}s)}\right\}^2,$$

which satisfies (1.1.5), so that

$$K(x) = x^{\frac{1}{2}}\left\{Y_0(x)-\frac{2}{\pi}K_0(x)\right\}$$

† Bateman (**1**), 482, and (**2**), 75.

‡ Titchmarsh (**17**).

§ See Fox (**4**), § 5.3.

|| Watson (**18**), § 3.7, § 13.21 (8), § 13.24 (5).

is a Fourier kernel. If $\nu = 2$, $a = 0$, then

$$k(s) = 2^{2s-1}\left\{\frac{\Gamma(\frac{1}{4}s+\frac{9}{8})}{\Gamma(\frac{11}{8}-\frac{1}{4}s)}\right\}^2 \frac{\frac{1}{4}s-\frac{3}{8}}{\frac{1}{4}s+\frac{1}{8}},$$

and $k(s)\,k(1-s) = -1$, so that

$$K(x) = -H(x) = x^{\frac{1}{2}}\left\{Y_2(x)+\frac{2}{\pi}K_2(x)\right\}.$$

The formulae in this case are due to A. L. Dixon and Hardy†. Much more general formulae of a similar character have been found by Steen‡ and by Kuttner, whose results are not yet published.

(5) If $$k(s) = e^{-ai(s-\frac{1}{2})^2} \quad (a > 0),$$

then $$h(s) = e^{ai(s-\frac{1}{2})^2}.$$

Taking $c = \frac{1}{2}$ in (1.1.6), we find that

$$K(x) = \frac{1}{2\pi\sqrt{x}}\int_{-\infty}^{\infty} e^{ait^2}\cos(t\log x)\,dt = \frac{e^{\frac{1}{4}\pi i}}{2\sqrt{(\pi a x)}}\,e^{-i(\log x)^2/4a},$$

while $H(x)$ is the conjugate function. The Fourier formula thus obtained may be reduced by a change of variable to the exponential form of the ordinary Fourier formula. It is

$$F(x) = \frac{1}{4\pi a}\int_0^{\infty} e^{-i\{\log(xu)\}^2/4a}\frac{du}{\sqrt{(xu)}}\int_0^{\infty} e^{i\{\log(uy)\}^2/4a}\frac{F(y)}{\sqrt{(uy)}}\,dy;$$

and, if we put $a = \frac{1}{2}$ and

$$x = e^{\xi}, \quad u = e^{\zeta}, \quad y = e^{\eta}, \quad f(\xi) = e^{\frac{1}{2}i\xi^2+\frac{1}{2}\xi}F(e^{\xi}),$$

we obtain $$f(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi\zeta}d\zeta\int_{-\infty}^{\infty} e^{i\zeta\eta}f(\eta)\,d\eta.$$

This formula is not included in our standard form, since the limits are $-\infty$ and ∞ instead of 0 and ∞.

(6) If $$k(s) = e^{a(s-\frac{1}{2})^3} \quad (a > 0),$$

then $K(x)$ is a Fourier kernel. Taking $c = \frac{1}{2}$ in (1.1.6), we find

$$K(x) = \frac{1}{\pi\sqrt{x}}\int_0^{\infty}\cos(at^3+t\log x)\,dt.$$

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† Hardy (10); see also Hardy and Titchmarsh (12), 212.

‡ See Steen (14).

This is an "Airy" integral, and†

$$K(x) = \frac{1}{3\sqrt{(3ax)}} \left(\log \frac{1}{x}\right)^{\frac{1}{2}} \left[J_{\frac{1}{3}} \left\{ \frac{2}{3\sqrt{(3a)}} \left(\log \frac{1}{x}\right)^{\frac{3}{2}} \right\} \right.$$

$$\left. + J_{-\frac{1}{3}} \left\{ \frac{2}{3\sqrt{(3a)}} \left(\log \frac{1}{x}\right)^{\frac{3}{2}} \right\} \right] \quad (0 < x < 1),$$

$$K(x) = \frac{1}{3\pi\sqrt{(ax)}} (\log x)^{\frac{1}{2}} K_{\frac{1}{3}} \left\{ \frac{2}{3\sqrt{(3a)}} (\log x)^{\frac{3}{2}} \right\} \quad (x > 1).$$

(7) If

$$k(s) = 1,$$

then (1.1.5) is satisfied, but the integral (1.1.6) is not convergent. If, however, we regard (1.2.3), with $0 < c < 1$, as the definition of $K_1(x)$, we have

$$K_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{1-s}}{1-s}\, ds = \begin{cases} 1 & (x > 1) \\ 0 & (0 < x < 1). \end{cases}$$

And if we replace (1.1.2) and (1.1.3) by

$$F(x) = \frac{1}{x} \int_0^\infty G(y)\, dK_1(xy), \qquad G(x) = \frac{1}{x} \int_0^\infty F(y)\, dK_1(xy),$$

then our formulae become

$$F(x) = \frac{1}{x}\, G\left(\frac{1}{x}\right), \qquad G(x) = \frac{1}{x}\, F\left(\frac{1}{x}\right),$$

which are plainly consequences of one another.

(8) In all these examples $k(s)$ is analytic. Suppose, however, that $c = \frac{1}{2}$ and

$$k(\tfrac{1}{2} + it) = i \operatorname{sgn} t.$$

Then h, defined by (1.3.4), is $-k$. The integral (1.1.6) is not convergent, but it is formally

$$\frac{i}{2\pi x^{\frac{1}{2}}} \left(\int_0^\infty x^{-it}\, dt - \int_{-\infty}^0 x^{-it}\, dt \right) = -\frac{i}{2\pi x^{\frac{1}{2}}} \int_0^\infty (x^{it} - x^{-it})\, dt$$

$$= \frac{1}{\pi x^{\frac{1}{2}}} \int_0^\infty \sin(t \log x)\, dt = \frac{1}{\pi x^{\frac{1}{2}} \log x},$$

† Watson (18), §6.4 (1), (2).

the integral being summable $(C, 1)$†. Our formulae become

$$F(x) = \frac{1}{\pi}\int_0^\infty \frac{G(y)\,dy}{(xy)^{\frac{1}{2}}\log(xy)}, \qquad G(x) = -\frac{1}{\pi}\int_0^\infty \frac{F(y)\,dy}{(xy)^{\frac{1}{2}}\log(xy)}.$$

If we replace x and y by e^x and e^y, and interpret the integrals as principal values, we obtain formulae equivalent to those of the theory of "Hilbert transforms".

(9) If $k(s) = \cot\frac{1}{2}s\pi$, then (1.1.5) is satisfied. The integral (1.1.6) is of the same type as in (8). A formal application of the theorem of residues gives

$$K(x) = \frac{2}{\pi}\,\frac{1}{1-x^2},$$

and we again obtain formulae of the "Hilbert transform" type.

(10) We obtain formulae of a somewhat different type by taking

$$k(\tfrac{1}{2}+it) = e^{i/t}.$$

Then (1.1.5) is satisfied, and the formula for $K(x)$ is

$$K(x) = \frac{1}{2\pi x^{\frac{1}{2}}}\int_{-\infty}^\infty e^{i/t}x^{-it}\,dt = \frac{1}{\pi x^{\frac{1}{2}}}\int_0^\infty \cos\left(\frac{1}{t} - t\log x\right)dt.$$

The integral is summable $(C, 1)$ if $x \neq 1$, and has the value‡

$$-\left(x\log\frac{1}{x}\right)^{-\frac{1}{2}} J_1\left\{2\left(\log\frac{1}{x}\right)^{\frac{1}{2}}\right\} \quad (0<x<1), \qquad 0 \quad (x>1).$$

If $x = 1$, the integral for $K(x)$ diverges to infinity, and $K_1(x)$ has a discontinuity, as in example (7). The formula which results is therefore

$$G(x) = \frac{1}{x}F\left(\frac{1}{x}\right) - \int_0^{1/x} \frac{J_1\{2(-\log xy)^{\frac{1}{2}}\}}{(-xy\log xy)^{\frac{1}{2}}}\,F(y)\,dy.$$

If we put

$$x = e^{-\xi}, \quad y = e^{-\eta}, \quad e^{-\frac{1}{2}\xi}F(e^{-\xi}) = \phi(\xi), \quad e^{-\frac{1}{2}\xi}G(e^{-\xi}) = \psi(\xi),$$

we obtain

$$\psi(\xi) = \phi(-\xi) - \int_{-\xi}^\infty \frac{J_1\{2\sqrt{(\xi+\eta)}\}}{\sqrt{(\xi+\eta)}}\,\phi(\eta)\,d\eta.$$

† We could use $K_1(x)$ as in the last example, but this would involve a certain amount of calculation.

‡ See Watson (18), §6.23 (1), (2): we have to consider a limiting case.

The reciprocal formula is obtained by interchanging ϕ and ψ. The Fourier formula which results may be verified by using the integral†

$$\int_{-\lambda}^{\infty} \frac{J_1\{2\sqrt{(x+\lambda)}\}\,J_1\{2\sqrt{(x+\mu)}\}}{\sqrt{\{(x+\lambda)(x+\mu)\}}}\,dx = \frac{J_1\{2\sqrt{(\mu-\lambda)}\}}{\sqrt{(\mu-\lambda)}}.$$

1.6. There are two different directions in which we may look for a general theory of all these transformations.

(1) We may attempt to construct a theory of "transforms" similar to Plancherel's theory of Fourier transforms. Our functions will then, in the first instance at any rate, be functions of integrable square (of the Lebesgue class L^2); and our object will be to show that the formulae (1.1.2) and (1.1.3) are valid (at any rate in some generalized sense) for almost all x.

This is what has been done by Watson in the paper which follows. Watson has shown that, if only‡

$$k(\tfrac{1}{2}+it)\,k(\tfrac{1}{2}-it) = 1, \tag{1.6.1}$$

$$\int_{-\infty}^{\infty} \left| \frac{k(\tfrac{1}{2}+it)}{\tfrac{1}{2}+it} \right|^2 dt < \infty, \tag{1.6.2}$$

then there is a satisfactory generalization of Plancherel's theory.

(2) We may look for an analogue of the older theory of the Fourier integral. This is a "convergence theory", in the widened sense in which "convergence" includes summability by Cesàro's or similar means; and the functions to which it applies are not necessarily of integrable square. We wish to impose conditions on $k(s)$ such that, when $K(x)$ is defined by (1.1.6), then (1.1.1) shall be true in the classical or in some more general sense. It is essential (from our present point of view) that our conditions shall bear directly on $k(s)$ and on $k(s)$ alone.

This is what we have tried to do in the present paper. Our results have none of the generality of Watson's; we have found it necessary to subject $k(s)$ to very much more stringent conditions. We do not suggest that our theorems are at all so general as they should be; but we are

† Watson (**18**), § 13.47 (10).

‡ If $K(x)$ is real, $k(s)$ is real for real s, (1.6.1) is $|k(\frac{1}{2}+it)|^2 = 1$, and (1.6.2) is a consequence of (1.6.1).

disposed to think that our restrictions bear some relation to the facts, and that a convergence theory really requires conditions of a more drastic type.

We begin by supposing that $k(\frac{1}{2}+it)$, while not necessarily analytic, behaves much as

$$k_1(s) = \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(s)\cos\tfrac{1}{2}s\pi$$

behaves on the line $\sigma = \frac{1}{2}$. This function is the $k(s)$ which corresponds to the standard case of the Fourier cosine transformation. In other words we assume that $k(\frac{1}{2}+it)$ behaves, for large t, very much like

$$e^{i(t\log t - t)}.$$

On this hypothesis we prove Theorem 1. In the repeated integral here the inner integral is a Lebesgue integral, but the outer integral is defined by a summability convention of Rieszian type. It is a peculiarity of the theorem, which we cannot remove without further restrictions on $k(s)$, that this convention is required even when $F(y)$ is of bounded variation, when the ordinary Fourier formula is true in the classical sense. For this reason, though our formulae include those of Fox and other writers, Theorem 1 does not include the theorems which they prove.

We have proved a corresponding theorem (Theorem 2) in which $k(\frac{1}{2}+it)$ behaves like

$$e^{it^c} \quad (c > 1),$$

but the proof is long and we do not reproduce the details. There are doubtless similar theorems involving $e^{i\chi(t)}$, where χ is a general function subject to appropriate restrictions, but it is probable that the proof of any such theorem would be a very tedious task.

In § 6 we strengthen our hypotheses concerning $k(s)$, supposing now that $k(s)$ is analytic and behaves very much like $k_1(s)$ in a strip parallel to the imaginary axis. We are then able to prove a theorem (Theorem 3) in which the integral converges in the classical sense. This theorem includes (for example) the theorems of Fox.

In § 7 we prove a theorem concerning the special case considered in example (6) above. It would, no doubt, be possible to deal with the more general cases in which

$$k(s) = e^{-ai(s-\frac{1}{2})^{2p}}$$

or

$$k(s) = e^{a(s-\frac{1}{2})^{2p+1}},$$

and p is any positive integer, or in which $k(s)$ mimics the behaviour of these

functions sufficiently closely. The proofs would be long, and we confine ourselves to the simplest case†.

Our selection of cases for detailed discussion may seem somewhat arbitrary, but has been dictated by the desire (*a*) to keep the paper within reasonable bounds and (*b*) to include so far as possible the special transformations catalogued in § 1. Actually our analysis is sufficient to cover all the examples (1) to (7).

We conclude, in § 7, by proving a theorem which, unlike all those which precede, was suggested to us by Watson's work. The theorem covers only a special class of $F(x)$, and is of a different character in that we impose a condition directly on $K(x)$. If we do this, it is reasonable to expect results applicable to less specialized kernels.

Summability.

2.1. In the analysis which follows, $k(s) = k(\sigma+it)$ is defined for $\sigma = \frac{1}{2}$ only, and is a complex function of the real variable t. We retain the notation $k(\frac{1}{2}+it)$ for formal convenience.

THEOREM 1. *Suppose that* $k(\frac{1}{2}+it)$ *satisfies the following conditions*: (i) *it is bounded*; (ii)

$$k(\tfrac{1}{2}+it) = e^{i(t\log t-t)}\{\alpha+\beta(t)+\gamma(t)\}, \tag{2.1.1}$$

for $t>1$, *where* α *is constant,* β *is a constant multiple of a function which tends steadily to* 0, *and* γ *is* (*bounded and*) *integrable in* $(1, \infty)$; *and*

$$k(\tfrac{1}{2}+it) = e^{i(t\log|t|-t)}\{\alpha'+\beta'(t)+\gamma'(t)\}$$

for $t<-1$, α', β', *and* γ' *satisfying the same conditions as* α, β, *and* γ; (iii)

$$k(\tfrac{1}{2}+it)\,k(\tfrac{1}{2}-it) = 1. \tag{2.1.2}$$

Suppose also that $y^{-\frac{1}{2}}F(y)$ *is integrable in* $(0, 1)$, *and* $F(y)$ *in* $(1, \infty)$, *that* x *is positive, and that* $F(y)$ *is of bounded variation near* $y=x$. *Then*

$$\lim_{\lambda\to\infty}\int_{1/\lambda}^{\lambda}\left(1-\frac{|\log u|}{\log\lambda}\right)K(xu)\,du\int_0^\infty K(uy)\,F(y)\,dy = \tfrac{1}{2}\{F(x+0)+F(x-0)\}. \tag{2.1.3}$$

We say that a function $k(\frac{1}{2}+it)$ which satisfies (i), (ii), (iii) belongs to class **k**; the class includes all the examples of § 1 in which $k(s)$ is a

† Apart from the case $p=1$, which, after what was said under (5) above, requires no special discussion.

combination of gamma-functions. Consider, for example, Hankel's formulae, in which $k(s)$ is given by (1.5.2). Stirling's formula gives

$$k(s) = e^{\frac{1}{2}\pi i\nu}\, e^{i(t\log t - t)}\, t^{\sigma-\frac{1}{2}}\left\{1+\frac{b}{t}+O\left(\frac{1}{t^2}\right)\right\} \quad (t>1),$$

where b is a function of σ; and this is of the form (2.1.1) when $\sigma = \frac{1}{2}$.

We base the proof of Theorem 1 on a series of lemmas. Lemmas 1–4 are concerned with integrals of the form

$$\int e^{i\psi(t)}\,dt,$$

and are of a type familiar in the work of van der Corput and other writers. Lemmas 5–7 assert properties of the $K(x)$ corresponding to a $k(s)$ of the class **k**; and Lemma 8 is a case of a well-known theorem of Hobson.

Lemmas concerning integrals $\int e^{i\psi(t)}\,dt$.

3.1. LEMMA 1. *If $\psi'(t)$ is monotonic, and $\psi'(t) \geqslant r > 0$ or $\psi'(t) \leqslant -r < 0$, then*

$$\left|\int_a^b e^{i\psi(t)}\,dt\right| \leqslant \frac{4}{r}.$$

Suppose, for example, that $\psi'(t)$ increases and is not less than r. Then (with an obvious notation)

$$\int_a^b e^{i\psi(t)}\,dt = \int_\alpha^\beta \frac{e^{iu}}{\psi'(t)}\,du = \int_\alpha^\beta \frac{\cos u}{\psi'(t)}\,du + i\int_\alpha^\beta \frac{\sin u}{\psi'(t)}\,du$$

$$= \frac{1}{\psi'(a)}\int_\alpha^\gamma \cos u\,du + \frac{i}{\psi'(a)}\int_\alpha^{\gamma'} \sin u\,du,$$

and the modulus of this does not exceed $4/r$.

3.2. LEMMA 2. *If $\psi''(t) \geqslant r > 0$ or $\psi''(t) \leqslant -r < 0$, then*

$$\left|\int_a^b e^{i\psi(t)}\,dt\right| \leqslant \frac{8}{\sqrt{r}}.$$

Since $\psi'(t)$ is monotonic, it can vanish at most once, say for $t = t_0$. We prove the lemma on the hypothesis that $a < t_0 < b$, leaving to the reader the simple modifications required in other cases.

We have

$$(3.2.1) \qquad I = \int_a^b e^{i\psi(t)}\,dt = \int_a^{t_0-\rho} + \int_{t_0-\rho}^{t_0+\rho} + \int_{t_0+\rho}^b = I_1 + I_2 + I_3;$$

some parts of the integrals may be absent. Then $|I_2| \leqslant 2\rho$. Also

$$|\psi'(t_0+\rho)| = \left|\int_{t_0}^{t_0+\rho} \psi''(t)\,dt\right| \geqslant r\rho;$$

and so $|\psi'(t)| \geqslant r\rho$ for $t \geqslant t_0+\rho$. Hence, by Lemma 1, $|I_3| \leqslant 4/r\rho$.

A similar inequality holds for I_1, so that

$$|I| \leqslant 2\rho + \frac{8}{r\rho}.$$

Taking $\rho = 2r^{-\frac{1}{2}}$, we obtain the result.

3.3. LEMMA 3. *Suppose that $\psi'(t)$ is monotonic and that $|\psi''(t)|$ and $|\psi'''(t)|$ are decreasing; and that*

$$\psi'(t_0) = 0, \quad \psi''(t_0) \neq 0,$$

$$|\psi'''(t_0)| \leqslant B|\psi''(t_0)|^{\frac{3}{2}}.$$

Then

$$\left|\int_a^b e^{i\psi(t)}\,dt\right| \leqslant \frac{C}{|\psi''(t_0)|^{\frac{1}{2}}},$$

where $C = C(B)$ depends only on B.

It is not necessary that t_0 should fall in the range of integration, which may be any part of the range in which the conditions are satisfied; but we state the proof in the typical case in which $a < t_0 < b$. We divide I into I_1, I_2, I_3 as in (3.2.1), and write ψ_2 and ψ_3 for $|\psi''(t_0)|$ and $|\psi'''(t_0)|$. Then $|I_2| \leqslant 2\rho$ as before. Also

$$|\psi'(t_0-\rho)| = \left|\int_{t_0-\rho}^{t_0} \psi''(t)\,dt\right| \geqslant \rho\psi_2,$$

and

$$|\psi'(t_0+\rho)| = \left|\int_{t_0}^{t_0+\rho} \psi''(t)\,dt\right| \geqslant \rho|\psi''(t_0+\rho)| = \rho\left|\psi''(t_0)+\int_{t_0}^{t_0+\rho} \psi'''(t)\,dt\right|$$

$$\geqslant \rho\psi_2 - \rho^2\psi_3 = \rho\psi_2\left(1-\rho\psi_2^{\frac{1}{2}}\frac{\psi_3}{\psi_2^{\frac{3}{2}}}\right) \geqslant \rho\psi_2(1-B\rho\psi_2^{\frac{1}{2}}).$$

Hence, if we take $\rho = (2B)^{-1}\psi_2^{-\frac{1}{2}}$,

we obtain $$|I| \leqslant 2\rho + \frac{4}{\rho\psi_2} + \frac{8}{\rho\psi_2} < C\psi_2^{-\frac{1}{2}}.$$

3.4. LEMMA 4. *If $\chi(t)$ is a positive decreasing function, and*

$$\left|\int_a^b e^{i\psi(t)}\,dt\right| \leqslant C$$

for all b, then

$$\left|\int_a^b e^{i\psi(t)}\chi(t)\,dt\right| \leqslant 4C\chi(a).$$

This is an obvious corollary of the second mean value theorem.

Lemmas concerning $K(x)$.

4.1. LEMMA 5. *If $k(\frac{1}{2}+it)$ belongs to* **k**, *then the integral* (1.1.6), *with $c=\frac{1}{2}$, is uniformly convergent in any finite interval $0<a\leqslant x\leqslant b$, and $K(x)$ is continuous in such an interval. Also $K(x)$ is bounded as $x\to\infty$, and is $O(x^{-\frac{1}{2}})$ as $x\to 0$.*

By Lemma 1, with $\psi(t)=t\log t-t-t\log x$,

$$\left|\int_T^{T'} e^{i(t\log t-t)}x^{-it}\,dt\right| \leqslant \frac{4}{\log T-\log x},$$

for $x<T<T'$. Hence the part of (1.1.6) arising from the α in (2.1.1) is uniformly convergent. The uniform convergence of the part arising from β then follows from Lemma 4, and that of the part arising from γ is obvious. This proves the first clause of the lemma.

Next let

$$(4.1.1)\quad K(x)=\frac{1}{2\pi x^{\frac{1}{2}}}\left\{\int_{-\infty}^{-1}+\int_{-1}^{1}+\int_1^{\infty}\right\}k(\tfrac{1}{2}+it)\,x^{-it}\,dt=\frac{1}{2\pi x^{\frac{1}{2}}}(J_1+J_2+J_3).$$

Plainly J_2 is bounded for all x. Also

$$J_3=\alpha\int_1^{\infty}e^{i\psi(t)}\,dt+\int_1^{\infty}e^{i\psi(t)}\beta(t)\,dt+\int_1^{\infty}e^{i\psi(t)}\gamma(t)\,dt.$$

The first integral here satisfies the conditions of Lemma 3 for large x, since $t_0=x$, $\psi''=1/t$, $\psi'''=-1/t^2$. Hence it is $O(x^{\frac{1}{2}})$ for large x. If $x<1$,

$$\psi'(t)=\log t-\log x>\log\frac{1}{x},$$

and the integral is bounded, by Lemma 1. Lemma 4 then shows that the second integral has the same properties; and the third is obviously bounded. Hence J_3 is bounded for small x, and $O(x^{\frac{1}{2}})$ for large x. A similar argument applies to J_1. The second clause of the lemma then follows from (4.1.1).

It should be observed that (1.2.3) now follows from (1.1.6).

4.2. LEMMA 6. *If* $k(\frac{1}{2}+it)$ *belongs to* $\mathbf{k}$, *and*

$$\Phi(\lambda, x, y) = \int_{1/\lambda}^{\lambda} \left(1 - \frac{|\log u|}{\log \lambda}\right) K(xu)\, K(yu)\, du,$$

where $\lambda > 2$ *and* $x > 0$, *then*

$$|y^{\frac{1}{2}}\Phi| < B = B(x, \zeta)\dagger$$

for all positive y *for which* $|y-x| \geqslant \zeta$.

We have

$$(4.2.1)\quad \Phi(\lambda, x, y) = \frac{1}{4\pi^2} \int_{1/\lambda}^{\lambda} \left(1 - \frac{|\log u|}{\log \lambda}\right) du \int_{-\infty}^{\infty} k(\tfrac{1}{2}+it)(xu)^{-\frac{1}{2}-it}\, dt \times \int_{-\infty}^{\infty} k(\tfrac{1}{2}+it')(yu)^{-\frac{1}{2}-it'}\, dt'.$$

Let

$$(4.2.2)\quad \Phi(\lambda, x, y, T) = \frac{1}{4\pi^2} \int_{1/\lambda}^{\lambda} \left(1 - \frac{|\log u|}{\log \lambda}\right) du \int_{-T}^{T} k(\tfrac{1}{2}+it)(xu)^{-\frac{1}{2}-it}\, dt \times \int_{-T}^{T} k(\tfrac{1}{2}+it')(yu)^{-\frac{1}{2}-it'}\, dt'.$$

By Lemma 5, $\Phi(\lambda, x, y)$ is the limit of $\Phi(\lambda, x, y, T)$, and it is therefore enough to prove that

$$(4.2.3)\qquad |y^{\frac{1}{2}}\Phi(\lambda, x, y, T)| < B(x, \zeta).$$

We may invert the order of integration in (4.2.2). Since

$$\int_{1/\lambda}^{\lambda} \left(1 - \frac{|\log u|}{\log \lambda}\right) u^{-1-it-it'}\, du = \frac{4 \sin^2 \frac{1}{2}\mu(t+t')}{\mu(t+t')^2},$$

where

$$\mu = \log \lambda,$$

we have

$$(4.2.4)\quad \Phi(\lambda, x, y, T) = \frac{1}{\pi^2 (xy)^{\frac{1}{2}}} \int_{-T}^{T} k(\tfrac{1}{2}+it)\, x^{-it}\, dt \int_{-T}^{T} k(\tfrac{1}{2}+it')\, y^{-it'} \frac{\sin^2 \frac{1}{2}\mu(t+t')}{\mu(t+t')^2}\, dt'.$$

† Here, and in all our later analysis, x is "fixed". It is to be observed that B depends upon the function $k(\frac{1}{2}+it)$ as well as upon the parameters shown, any number occurring in a datum concerning k being treated as a constant. When we say, in the argument which follows, that a function is bounded, or is $O(1)$, we mean that it is bounded by a B.

We split up the integral into

$$\int_{-1}^{1}\int_{-T}^{T}+\int_{1}^{T}\int_{-1}^{1}+\int_{-T}^{-1}\int_{-1}^{1}+\int_{1}^{T}\int_{1}^{T}+\int_{1}^{T}\int_{-T}^{-1}+\int_{-T}^{-1}\int_{1}^{T}+\int_{-T}^{-1}\int_{-T}^{-1}$$
$$=J_1+J_2+J_3+J_4+J_5+J_6+J_7.$$

Since $k(\frac{1}{2}+it)$ is bounded,

$$J_1=O\left\{\int_{-1}^{1}dt\int_{-T}^{T}\frac{\sin^2\frac{1}{2}\mu(t+t')}{\mu(t+t')^2}\,dt'\right\}=O\left(\int_{-1}^{1}dt\int_{-\infty}^{\infty}\frac{\sin^2\frac{1}{2}\mu z}{\mu z^2}\,dz\right)=O(1),$$

and similarly J_2 and J_3 are bounded. Since J_7 is of the same type as J_4, and J_6 as J_5, it is enough to prove J_4 and J_5 bounded.

The leading term in J_4 is

$$\alpha^2\int_1^T e^{i(t\log t-t)}x^{-it}\,dt\int_1^T e^{i(t'\log t'-t')}y^{-it'}\frac{\sin^2\frac{1}{2}\mu(t+t')}{\mu(t+t')^2}\,dt'.$$

If we make the substitution

$$t=u+v,\quad t'=u-v,$$

we obtain

$$\tfrac{1}{2}\int_1^{\frac{1}{2}T+\frac{1}{2}}(xy)^{-iu}\frac{\sin^2\mu u}{\mu u^2}\,du\int_{1-u}^{u-1}e^{i\psi(v)}\,dv+\int_{\frac{1}{2}T+\frac{1}{2}}^{T}\int_{u-T}^{T-u}=I_1+I_2,$$

where

$$\psi(v)=(u+v)\log(u+v)+(u-v)\log(u-v)-2u+v\log(y/x),$$

and the integrands are the same in I_2 as in I_1. The inner integrals are of the type considered in Lemma 2, with

$$\psi''(v)=\frac{1}{u+v}+\frac{1}{u-v}\geqslant\frac{2}{u}.$$

Hence each inner integral is $O(u^{\frac{1}{2}})$, and the whole is

$$O\left(\int_1^T\frac{du}{\mu u^{\frac{1}{2}}}\right)=O(1).$$

Similarly (using Lemma 4) the terms arising from $\alpha\beta(t)$, $\alpha\beta(t')$, and $\beta(t)\beta(t')$ are bounded; and those involving γ are of the type

$$O\left\{\int_1^T|\gamma(t)|\,dt\int_1^T\frac{dt'}{t'^2}\right\}=O(1).$$

Hence J_4 is bounded.

Passing to J_5, the leading term is

$$\alpha\alpha' \int_1^T e^{i(t\log t-t)} x^{-it}\,dt \int_1^T e^{-i(t'\log t'-t')} y^{it'} \frac{\sin^2 \frac{1}{2}\mu(t-t')}{\mu(t-t')^2}\,dt'.$$

Making the same substitution, we obtain

$$\alpha\alpha' \int_0^{\frac{1}{2}T-\frac{1}{2}} (xy)^{-iv} \frac{\sin^2 \mu v}{\mu v^2}\,dv \int_{v+1}^{T-v} e^{i\chi(u)}\,du, \tag{4.2.5}$$

where

$$\chi(u) = (u+v)\log(u+v) - (u-v)\log(u-v) - 2v + u\log(y/x),$$

together with a similar integral over $-\frac{1}{2}T+\frac{1}{2} \leqslant v \leqslant 0$. We may confine our attention to (4.2.5).

We have

$$\chi'(u) = \log(u+v) - \log(u-v) + \log(y/x),$$

$$\chi''(u) = \frac{1}{u+v} - \frac{1}{u-v}.$$

Then $\chi'(u) = 0$ when $\qquad u = u_0 = \dfrac{x+y}{x-y}\,v,$

and $\chi''(u) < 0$.

First suppose that $y \geqslant x+\zeta$. Then

$$\chi'(u) > \log\left(1+\frac{\zeta}{x}\right) > B,$$

and, by Lemma 1,

$$\left|\int_{v+1}^{T-v} e^{i\chi(u)}\,du\right| < \frac{4}{B} = O(1). \tag{4.2.6}$$

If $y \leqslant x-\zeta$, we must distinguish two cases. If $v \leqslant (x-y)/(4y)$, then

$$u_0 = \frac{x+y}{x-y}\,v \leqslant v+\tfrac{1}{2} < v+1,$$

$$\chi'(u) \leqslant \chi'(v+1) = \log\left\{(2v+1)\,\frac{y}{x}\right\} \leqslant \log\frac{x+y}{2x} < -B,$$

and the integral (4.2.6) is bounded. If $v \geqslant (x-y)/(4y)$, then, since

$$\chi''(u_0) = -\frac{(x-y)^2}{2xyv}, \quad \chi'''(u_0) = \frac{(x+y)(x-y)^3}{4x^2y^2v^2},$$

the conditions of Lemma 3 are satisfied†. Hence the integral (4.2.6) is

$$O\left\{\frac{1}{|\chi''(u_0)|^{\frac{1}{2}}}\right\} = O(v^{\frac{1}{2}}).$$

It follows that the inner integral in (4.2.5) is in any case

$$O(1+|v|^{\frac{1}{2}})$$

and so that (4.2.5) is bounded. This accounts for the dominant term in J_5, and the other terms may be accounted for like the corresponding terms in J_4‡. Hence J_5 is bounded; and this completes the proof.

4.3. LEMMA 7. *If* $k(\frac{1}{2}+it)$ *belongs to* **k** *and*

$$\Psi(\lambda, x, y) = \int_{1/\lambda}^{\lambda} \left(1-\frac{|\log u|}{\log \lambda}\right) \frac{K(xu)\,K_1(yu)}{u}\,du,$$

where $\lambda > 2$ *and* $x > 0$, *then*

$$|\Psi| < B(x, \zeta) \tag{4.3.1}$$

for $0 < x-\zeta \leqslant y \leqslant x+\zeta$; *and* $\Psi(\lambda, x, y)$ *converges* (*boundedly*), *when* $\lambda \to \infty$, *to the limit*

$$0 \ \ (y<x), \qquad \tfrac{1}{2} \ \ (y=x), \qquad 1 \ \ (y>x).$$

As in the proof of Lemma 6, we may replace $\Psi(\lambda, x, y)$ by

$$\Psi(\lambda, x, y, T) = \frac{y^{\frac{1}{2}}}{\pi^2 x^{\frac{1}{2}}} \int_{-T}^{T} k(\tfrac{1}{2}+it)\, x^{-it}\,dt \int_{-T}^{T} k(\tfrac{1}{2}+it')\, \frac{y^{-it'}}{\frac{1}{2}-it'} \frac{\sin^2 \frac{1}{2}\mu(t+t')}{\mu(t+t')^2}\,dt'.$$

We have first to prove this bounded; and, if we divide up the repeated integral as in § 4.2, it appears at once, as in the proof of Lemma 6, that the only parts of the integral which require a detailed discussion are the parts J_4' and J_5' corresponding to the J_4 and J_5 of § 4.2.

In the discussion of J_4 we made no use of the hypothesis $|x-y| \leqslant \zeta$, and J_4' differs from J_4 (apart from the outside factor) only in the occurrence of an additional factor $\frac{1}{2}-i(u-v)$ in the denominator. It is

† The main condition reduces to $(xyv)^{\frac{1}{2}} > B(x+y)$, which is true because $yv > B$.

‡ There is a term of the type

$$O\left\{\int_1^T |\gamma(t)|\,dt \int_1^T |\gamma(t')|\, \frac{\sin^2 \frac{1}{2}\mu(t-t')}{\mu(t-t')^2}\,dt'\right\}$$

which is bounded because $\gamma(t')$ is bounded and $\gamma(t)$ integrable over $(1, \infty)$.

plain then that the argument for J_4 will dispose of J_4' also, if we add an appeal to Lemma 4 at the appropriate places.

The part of J_5' corresponding to (4.2.5) (omitting the outside factor) is

$$(4.3.2) \qquad \int_0^{\frac{1}{2}T-\frac{1}{2}} (xy)^{-iv} \frac{\sin^2 \mu v}{\mu v^2}\, dv \int_{v+1}^{T-v} \frac{e^{i\chi(u)}}{\frac{1}{2}-i(u-v)}\, du.$$

We may replace the last denominator by $-iu$, since the error so introduced is

$$O\left\{\int_0^{\frac{1}{2}T-\frac{1}{2}} \frac{\sin^2 \mu v}{\mu v^2}\, dv \int_{v+1}^{\infty} \frac{v+1}{u(u-v)}\, du\right\} = O\left\{\int_0^{\frac{1}{2}T-\frac{1}{2}} \frac{\sin^2 \mu v}{\mu v^2} \frac{v+1}{v} \log(1+v)\, dv\right\}$$

$$= O\left\{\int_0^1 \frac{\sin^2 \mu v}{\mu v^2}\, dv + \int_1^{\infty} \frac{\log(v+1)}{\mu v^2}\, dv\right\} = O(1).$$

The inner integral in (4.3.2) then becomes

$$i\int_{v+1}^{T-v} \frac{e^{i\chi(u)}}{u}\, du = i\int_{u \leqslant 1/|\xi|} + i\int_{u \geqslant 1/|\xi|} = iI_1 + iI_2,$$

say, where $\xi = \log(x/y)$. Here $|\xi| < B$, and one of the integrals may disappear.

We consider I_2 first. If $y > x$, then $\chi'(u) > -\xi = |\xi|$ and

$$I_2 = |\xi| \int_{1/|\xi|}^{u_1} e^{i\chi(u)}\, du = O\left(|\xi| \frac{1}{|\xi|}\right) = O(1),$$

by Lemmas 4 and 1.

If $y = x$, there is no I_2.

If $y < x$, we argue much as at the end of § 4.2. If $v \leqslant (x-y)/(4y)$, then

$$I_2 = O\left\{\frac{\xi}{|\chi'(v+1)|}\right\} = O\left(\frac{\xi}{|\log\{(x+y)/2x\}|}\right) = O(1),$$

since $\xi = O(x-y)$. If $v \geqslant (x-y)/(4y)$, then

$$I_2 = O\left\{\frac{\xi}{|\chi''(u_0)|^{\frac{1}{2}}}\right\} = O\left[\xi \left\{\frac{2xyv}{(x-y)^2}\right\}^{\frac{1}{2}}\right] = O(v^{\frac{1}{2}}).$$

Substituting these estimates in (the simplified form of) the integral (4.3.2), we obtain a bounded integral.

It remains to estimate the contribution of I_1 to (4.3.2), and here we cannot deal with (4.3.2) as it stands, but must associate with it the corresponding term arising from J_6. The leading term in J_6 is the

conjugate of that in J_5, so that every term in our decomposition is accompanied by its conjugate. We associate the part of (4.3.2) arising from I_1 with its conjugate, and transform the latter by writing $-v$ for v. We thus obtain an inner integral involving

$$e^{i\{(u+v)\log(u+v)-(u-v)\log(u-v)-2v\}}\left(\frac{y}{x}\right)^{iu}\frac{1}{-iu}$$

$$-e^{-i\{(u-v)\log(u-v)-(u+v)\log(u+v)+2v\}}\left(\frac{y}{x}\right)^{-iu}\frac{1}{iu},$$

and a repeated integral

$$2\int_0^{\frac{1}{2}T+\frac{1}{2}}(xy)^{-iv}\frac{\sin^2\mu v}{\mu v^2}\,dv\int_{u\leqslant 1/|\xi|}e^{i\{(u+v)\log(u+v)-(u-v)\log(u-v)-2v\}}\sin\left(u\log\frac{x}{y}\right)\frac{du}{u}.$$

Since $\sin u\xi = O(u|\xi|)$, the inner integral is

$$O\left(|\xi|\int_{u\leqslant 1/|\xi|}du\right)=O(1),$$

and the repeated integral is bounded.

We have thus disposed of the leading terms (involving $\alpha\alpha'$) in J_5' (and J_6'). The discussion of these integrals may then be completed as in the proof of Lemma 6; we use Lemma 4 for the terms in $\alpha\beta'$, ..., while the boundedness of the terms involving a γ is almost evident. This completes the proof that Ψ is bounded.

We have finally to prove that, when x and y are fixed, Ψ tends to a certain limit. In the first place, the argument which precedes is easily adapted to show that

$$\Psi(\lambda, x, y, T_2)-\Psi(\lambda, x, y, T_1)\to 0, \tag{4.3.3}$$

when T_1 and T_2 tend to infinity, uniformly with respect to λ. If $y\neq x$, then (since x and y are now fixed) the difficulties which arose when ξ is small are avoided. The discussion is then similar to that of Lemma 6, the additional term $\frac{1}{2}-it'$ in the denominator leading to convergence instead of boundedness. If $y=x$, we must associate conjugate terms as in the argument which precedes, but the term involving $\sin u\xi$ vanishes. In either case (4.3.3) follows without difficulty. It then follows from (4.3.3) that

$$\Psi(\lambda, x, y, T)\to\Psi(\lambda, x, y), \tag{4.3.4}$$

when $T\to\infty$, uniformly in λ.

The theory of Fejér's integral shows that, when T is fixed and $\mu \to \infty$,

$$\int_{-T}^{T} k(\tfrac{1}{2}+it') \frac{y^{-it'}}{\frac{1}{2}-it'} \frac{\sin^2 \frac{1}{2}\mu(t+t')}{\mu(t+t')^2} dt' \to \tfrac{1}{2}\pi k(\tfrac{1}{2}-it) \frac{y^{it}}{\frac{1}{2}+it}$$

boundedly in t. Hence

$$\lim_{\lambda\to\infty} \Psi(\lambda, x, y, T) = \frac{y^{\frac{1}{2}}}{2\pi x^{\frac{1}{2}}} \int_{-T}^{T} k(\tfrac{1}{2}+it)\, k(\tfrac{1}{2}-it) \left(\frac{y}{x}\right)^{it} \frac{dt}{\frac{1}{2}+it}$$

$$= \frac{1}{2\pi} \int_{-T}^{T} \left(\frac{y}{x}\right)^{\frac{1}{2}+it} \frac{dt}{\frac{1}{2}+it} = \varpi(x, y) + O\left(\frac{1}{T}\right),$$

where $\varpi(x, y)$ is the limit-function of the lemma. Hence

$$\lim_{T\to\infty} \lim_{\lambda\to\infty} \Psi(\lambda, x, y, T) = \varpi;$$

and we can invert the order of the limits on account of the uniformity of (4.3.4). This completes the proof of the lemma.

Proof of Theorem 1.

5.1. We require one further lemma.

LEMMA 8. *Suppose that $F(y)$ is integrable in (a, b), that $g(\lambda, y)$ is bounded for $a \leqslant y \leqslant b$, $\lambda > \lambda_0$, and that*

$$\lim_{\lambda\to\infty} \int_{\alpha}^{\beta} g(\lambda, y)\, dy = 0$$

for every α, β such that $a \leqslant \alpha < \beta \leqslant b$. Then

$$\lim_{\lambda\to\infty} \int_{a}^{b} F(y)\, g(\lambda, y)\, dy = 0.$$

This is a special case of a well-known theorem of Hobson†.

5.2. By Lemma 5, the inner integral in (2.1.3) is uniformly convergent for $1/\lambda \leqslant u \leqslant \lambda$, so that we may invert the integrations. We thus obtain

$$\int_{1/\lambda}^{\lambda} \left(1 - \frac{|\log u|}{\log \lambda}\right) K(xu)\, du \int_{0}^{\infty} K(uy)\, F(y)\, dy = \int_{0}^{\infty} F(y)\, \Phi(\lambda, x, y)\, dy.$$

We write

$$\int_{0}^{\infty} F(y)\, \Phi dy = \int_{0}^{\delta} + \int_{\delta}^{x-\zeta} + \int_{x-\zeta}^{x} + \int_{x}^{x+\zeta} + \int_{x+\zeta}^{\Delta} + \int_{\Delta}^{\infty} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6.$$

† Hobson (12), 422–3.

It follows from Lemma 6 that

$$(5.2.1) \qquad |I_1| \leqslant B\int_0^{\delta} y^{-\frac{1}{2}}|F|\,dy < \epsilon, \quad |I_6| < B\int_{\Delta}^{\infty}|F|\,dy < \epsilon,$$

for $\delta = \delta(\epsilon)$, $\Delta = \Delta(\epsilon)$, $\lambda > 2$.

Next

$$\int_{\alpha}^{\beta}\Phi\,dy = \int_{1/\lambda}^{\lambda}\left(1-\frac{|\log u|}{\log\lambda}\right)K(xu)\,\frac{K_1(\beta u)-K_1(\alpha u)}{u}\,du$$

$$= \Psi(\lambda, x, \beta)-\Psi(\lambda, x, \alpha).$$

If $\alpha < \beta < x$, or $x < \alpha < \beta$, this tends to 0, when $\lambda \to \infty$, by Lemma 7; and so, by Lemma 8,

$$(5.2.2) \qquad \lim_{\lambda\to\infty} I_2 = 0, \quad \lim_{\lambda\to\infty} I_5 = 0,$$

when ζ, δ, and Δ are fixed.

We may suppose ζ small enough to ensure that $F(y)$ is of bounded variation in $(x-\zeta, x+\zeta)$, and then

$$F(y)-F(x-0) = F_1(y)-F_2(y),$$

where $F_1(y)$ and $F_2(y)$ are positive and decreasing and tend to 0 when $y \to x$ from below. Then

$$I_3 = F(x-0)\{\Psi(\lambda, x, x)-\Psi(\lambda, x, x-\zeta)\}$$

$$+\int_{x-\zeta}^{x} F_1(y)\,\Phi(\lambda, x, y)\,dy-\int_{x-\zeta}^{x} F_2(y)\,\Phi(\lambda, x, y)\,dy.$$

The first term tends to $\frac{1}{2}F(x-0)$. The second is

$$F_1(x-\zeta)\int_{x-\zeta}^{\eta}\Phi\,dy = F_1(x-\zeta)\{\Psi(\lambda, x, \eta)-\Psi(\lambda, x, x-\zeta)\},$$

where $x-\zeta < \eta < x$, and (since Ψ is bounded) this is less than ϵ (for all λ in question) if ζ is sufficiently small. A similar argument applies to the third term. Hence

$$\left|\overline{\lim_{\lambda\to\infty}}\, I_3-\tfrac{1}{2}F(x-0)\right| \leqslant 2\epsilon$$

if ζ is sufficiently small. There is a corresponding result for I_4, and these results, together with (5.2.1) and (5.2.2), prove the theorem.

5.3. THEOREM 2. *The conclusion of Theorem* 1 *still holds if* (2.1.1) *is replaced by*

$$k(\tfrac{1}{2}+it) = e^{it^c}\{\alpha+\beta(t)+\gamma(t)\},$$

where $c > 1$ (*the other conditions being unchanged*).

We do not prove this theorem in detail, for the reasons stated in the introduction. The proof is quite similar to that of Theorem 1 when $c < \frac{3}{2}$, but there are additional complications when $c \geqslant \frac{3}{2}$. We shall merely indicate shortly how these affect the proof of the crucial Lemma 6.

The critical integral, corresponding to (4.2.5), is now a multiple of

$$\int_0^{\frac{1}{2}(T-1)} (xy)^{-iv} \frac{\sin^2 \mu v}{\mu v^2}\, dv \int_{v+1}^{T-v} e^{i\chi(u)}\, du, \tag{5.3.1}$$

with

$$\chi(u) = (u+v)^c - (u-v)^c + u \log \frac{y}{x},$$

$$\chi'(u) = c\{(u+v)^{c-1} - (u-v)^{c-1} - \xi\},$$

where

$$\xi = \frac{1}{c} \log \frac{x}{y} > B.$$

If $c = 2$, $\chi'(u)$ is independent of u; we ignore this special case. In general, we have to consider the position of the value u_0 of u defined by $\chi'(u_0) = 0$.

Suppose first that $1 < c < 2$. Then it is easily verified that u_0 falls outside the range of integration when v is small, and that u_0 is of higher order than v when v is large. Thus it is large v that are important in (5.3.1) when $c < 2$. Similarly it is small v that are important when $c > 2$.

There is in any case a standard formula for an integral $\int e^{i\chi}\, du$ when u_0 falls within the range of integration. This approximation, furnished by the "principle of stationary phase", is

$$\int e^{i\chi}\, du \sim C \frac{e^{i\chi(u_0)}}{\sqrt{\{|\chi''(u_0)|\}}},$$

where C is a constant. Let us examine the results of replacing the inner integral in (5.3.1) by such an approximation. Solving for u_0 in terms of v and ξ, we obtain

$$\int e^{i\chi}\, du \sim C \xi^{-\frac{3-c}{2(2-c)}} v^{\frac{1}{2(2-c)}} \exp\left(Di\, \xi^{-\frac{c-1}{2-c}} v^{\frac{1}{2-c}}\right), \tag{5.3.2}$$

where C and D are constants. If $c < \frac{3}{2}$, then

$$0 < \frac{1}{2(2-c)} < 1;$$

and then (5.3.1) has a majorant of the type

$$\int \frac{\sin^2 \mu v}{\mu v^2}\, v^b\, dv \quad (0 < b < 1);$$

when $c \to 1$, $b \to \frac{1}{2}$, in agreement with the analysis of § 4 . 2. In this case the proof of Theorem 2 runs quite parallel to that of Theorem 1.

There is no such simple proof when $c > \frac{3}{2}$. In this case we are compelled to pay attention to the exact form of (5 . 3 . 2). Let us take, to fix our ideas, $c = 3$, so that (5 . 3 . 2) becomes

$$\int e^{ix}\, du \sim Cv^{-\frac{1}{2}} e^{Di\xi^2/v};$$

and let us further simplify the issue by confining our attention to a particular y, so that ξ may be regarded as constant. Then (5 . 3 . 1) is reduced to the form

$$\int_0 \frac{\sin^2 \mu v}{\mu v^2}\, v^{-\frac{1}{2}} e^{piv + qi/v}\, dv. \tag{5 . 3 . 3}$$

The integral (5 . 3 . 3) is substantially the Fejér integral of

$$v^{-\frac{1}{2}} e^{qi/v} \tag{5 . 3 . 4}$$

(the factor e^{piv} being unimportant); and this integral is known to be bounded†, as may be proved directly by partial integration of the factor $v^{-2} e^{qi/v}$. The indication of our analysis is therefore that (5 . 3 . 1) is bounded when $c = 3$. The argument applies equally to general c, except that, when $c < 2$, the emphasis is on *large* values of v.

We have thus a basis for a proof of Lemma 6, modified to fit the class of $k(s)$ under consideration. To complete the proof we have (*a*) to investigate the errors involved in the "stationary phase" approximation, (*b*) to pay regard to the variability of y, and (*c*) to make the corresponding adaptation of the proof of Lemma 7. All this is naturally tedious, and the details may well be suppressed.

Convergence.

6 . 1. In this section we make additional assumptions concerning $k(s)$. The function

$$k_1(s) = 2^{s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{1}{2} - \frac{1}{2}s)} = \sqrt{\left(\frac{2}{\pi}\right)}\, \Gamma(s) \cos \tfrac{1}{2}s\pi$$

is regular in any strip $\sigma_1 < \sigma < \sigma_2$, except for a finite number of simple

† See Hardy (5). The Fourier series, at $v = 0$, of a function which behaves like (5 . 3 . 4) is, in fact, of the type

$$\Sigma n^{-\frac{1}{2}} e^{ri n^{\frac{1}{2}}},$$

and oscillates finitely.

poles at points where $\sigma \leqslant 0$. If t is large and positive, then

$$k_1(\sigma+it) = At^{\sigma-\frac{1}{2}} e^{i(t\log t - t)}\left\{1+\frac{a}{t}+O\left(\frac{1}{t^2}\right)\right\},$$

and $k_1(\sigma-it)$ satisfies the conjugate formula. Here A and a are complex, and a depends upon σ†.

Let
$$k_2(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)} = \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(s)\sin\tfrac{1}{2}s\pi,$$

$$k_3(s) = 2^{s-\frac{3}{2}}\frac{\Gamma(\frac{1}{2}s)}{\Gamma(\frac{3}{2}-\frac{1}{2}s)} = \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(s)\frac{\cos\frac{1}{2}s\pi}{1-s},$$

$$k_4(s) = 2^{s-\frac{3}{2}}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(2-\frac{1}{2}s)} = \sqrt{\left(\frac{2}{\pi}\right)}\,\Gamma(s)\frac{\sin\frac{1}{2}s\pi}{2-s}.$$

These three functions are of the forms

$$k_1(s)\left\{i+O\left(\frac{1}{|s|^2}\right)\right\}, \quad k_1(s)\left\{-\frac{1}{s}+O\left(\frac{1}{|s|^2}\right)\right\}, \quad k_1(s)\left\{-\frac{i}{s}+O\left(\frac{1}{|s|^2}\right)\right\}$$

when t is large and positive, and of the forms

$$k_1(s)\left\{-i+O\left(\frac{1}{|s|^2}\right)\right\}, \quad k_1(s)\left\{-\frac{1}{s}+O\left(\frac{1}{|s|^2}\right)\right\}, \quad k_1(s)\left\{\frac{i}{s}+O\left(\frac{1}{|s|^2}\right)\right\}$$

when t is large and negative.

We now consider functions $k(s)$ which satisfy the following conditions : (i) $k(s)$ is regular in a strip $\sigma_1 < \sigma < \sigma_2$, where $\sigma_1 < 0$, $\sigma_2 > 1$, except perhaps for a finite number of simple poles on the imaginary axis; and (ii)

$$k(s) = k_1(s)\left\{\alpha+\frac{\beta}{s}+O\left(\frac{1}{|s|^2}\right)\right\}$$

for large positive t, and

$$k(s) = k_1(s)\left\{\gamma+\frac{\delta}{s}+O\left(\frac{1}{|s|^2}\right)\right\}$$

for large negative t. We call such a function a function of class $\mathbf{k}'$.

It may be verified that, in the examples of (1.4), each $k(s)$ which is a combination of gamma-functions belongs to $\mathbf{k}'$. For example, the $k(s)$ of (1.5.2) belongs to $\mathbf{k}'$, and

$$\alpha = e^{(\frac{1}{2}\nu-\frac{1}{4})\pi i}, \qquad \beta = -\tfrac{1}{2}i(\nu^2-\tfrac{1}{4})\,e^{(\frac{1}{2}\nu-\frac{1}{4})\pi i},$$

while γ and δ are the conjugates of α and β.

---

† Actually $A = e^{-\frac{1}{4}\pi i}, \quad a = -\frac{1}{2}i(\sigma^2-\sigma+\frac{1}{6})$.

The functions k_1, k_2, k_3, k_4 all belong to $\mathbf{k}'$. Further, if k is any function of $\mathbf{k}'$, we can find constants a_1, a_2, a_3, a_4 such that

$$k = a_1 k_1 + a_2 k_2 + a_3 k_3 + a_4 k_4 + k^*, \tag{6.1.1}$$

where

$$k^* = O\left(\left|\frac{k_1}{s^2}\right|\right) \tag{6.1.2}$$

for large s of the strip†.

The four K corresponding to the special k are

$$K_1(x) = \sqrt{\left(\frac{2}{\pi}\right)} \cos x ‡, \quad K_2(x) = \sqrt{\left(\frac{2}{\pi}\right)} \sin x,$$

$$K_3(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin x}{x}, \quad K_4(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin x - x \cos x}{x^2}.$$

If k is any function of $\mathbf{k}'$, then

$$K = a_1 K_1 + a_2 K_2 + a_3 K_3 + a_4 K_4 + K^*, \tag{6.1.3}$$

where K^* corresponds to the k^* of (6.1.1).

6.2. LEMMA 9. *If $k(s)$ belongs to $\mathbf{k}'$, then $K(x)$ is bounded for all positive x.*

This is true of the four special K, so that it is enough to prove

$$K^*(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} k^*(s)\, ds$$

bounded. If $x \geqslant 1$, we take $c = 1+\delta$, where $0 < \delta < \frac{1}{2}$ and $1+\delta < \sigma_2$. Since $k^*(s)$ is then $O(|s|^{\delta-\frac{3}{2}})$, $K^*(x)$ is bounded, and indeed $O(x^{-1-\delta})$. If $0 < x \leqslant 1$, we take $c = -\delta$, where $\sigma_1 < -\delta < 0$. Then

$$K^*(x) = \frac{1}{2\pi i} \int_{-\delta-i\infty}^{-\delta+i\infty} x^{-s} k^*(s)\, ds + R,$$

the latter term being the sum of the residues at any poles on the imaginary axis. It is plain that R is bounded, and the integral is bounded because $k^*(s) = O(|s|^{-\delta-\frac{3}{2}})$. Hence $K^*(x)$ is bounded for all positive x.

6.3. LEMMA 10. *If $k(s)$ belongs to $\mathbf{k}'$ and*

$$\phi(\lambda, x, y) = \int_{1/\lambda}^{\lambda} K(xu) K(yu)\, du, \tag{6.3.1}$$

---

† We have $a_1 + ia_2 = \alpha$, $a_1 - ia_2 = \gamma$, $-a_3 - ia_4 = \beta$, $-a_3 + ia_4 = \delta$.

‡ This use of K_1 is temporary and will not lead to confusion with the K_1 of (1.2.1).

where $\lambda > 1$ and x is positive and fixed, then

$$|\phi| < B(x, \zeta) \tag{6.3.2}$$

for all positive y for which $|y-x| \geqslant \zeta$.

It is plain, after Lemma 9, that we may replace ϕ by

$$\chi = \int_1^\lambda K(xu)\, K(yu)\, du.$$

We express k and K in the forms (6.1.1) and (6.1.3), but simplify them by writing

$$k^{(1)} = a_1 k_1 + a_2 k_2, \quad k^{(2)} = a_3 k_3 + a_4 k_4, \quad k^{(3)} = k^*$$

(and similarly for K). Then

$$\chi = \sum_{p,\, q=1}^{3} \int_1^\lambda K^{(p)}(xu)\, K^{(q)}(yu)\, du = \Sigma \chi_{p,q}.$$

We prove that each of the nine $\chi_{p,q}$ satisfies (6.3.2). We express this shortly by saying that each of them is bounded.

In the first place, this is obvious for $\chi_{1,1}$.

Next, $\chi_{1,2}$ splits up into four terms, a typical term being

$$\int_1^\lambda \sin xu\, \frac{\sin yu - yu \cos yu}{y^2 u^2}\, du \tag{6.3.3}$$

$$= \int_1^{1/y} \sin xu\, \frac{\sin yu - yu \cos yu}{y^2 u^2}\, du + \int_{1/y}^\lambda \sin xu\, \frac{\sin yu}{y^2 u^2}\, du$$

$$- \int_{1/y}^\lambda \sin xu\, \frac{\cos yu}{yu}\, du.$$

Here we suppose that $1/\lambda < y < 1$; if y lies outside these limits the argument is simpler.

Since $(\sin x - x \cos x)/x^2$ is positive and increasing in $0 < x < 1$, the first term on the right of (6.3.3) is

$$(\sin 1 - \cos 1) \int_{u_1}^{1/y} \sin xu\, du,$$

where $0 < u_1 < 1/y$. The second and third are

$$\int_{1/y}^{u_2} \sin xu \sin yu\, du, \qquad -\int_{1/y}^{u_3} \sin xu \cos yu\, du,$$

where $u_2 > 1/y$, $u_3 > 1/y$. All these are bounded, and the other terms of $\chi_{1,2}$ may be shown to be bounded in the same way. Hence $\chi_{1,2}$ is bounded.

A similar argument applies to $\chi_{2,1}$ and $\chi_{2,2}$. Thus a typical term of $\chi_{2,2}$ is

$$\int_1^\lambda \frac{\sin xu}{xu}\frac{\sin yu}{yu}\,du.$$

Here (supposing again that $1/\lambda < y < 1$) we have

$$\int_1^{1/y} \frac{\sin xu}{xu}\frac{\sin yu}{yu}\,du = \frac{\sin y}{y}\int_1^{u_1} \frac{\sin xu}{xu}\,du,$$

$$\int_{1/y}^\lambda \frac{\sin xu}{xu}\frac{\sin yu}{yu}\,du = \int_{1/y}^{u_2} \frac{\sin xu}{xu}\sin yu\,du,$$

and each of these is bounded.

6.4. A typical term in $\chi_{1,3}$ is

$$\frac{1}{2\pi i}\int_1^\lambda \cos xu\,du \int_{c-i\infty}^{c+i\infty} (yu)^{-s} k^{(3)}(s)\,ds. \tag{6.4.1}$$

Suppose first that $k^{(3)}(s)$ has no pole on the imaginary axis. Then we may take $c = 0$. If we do this, and invert the order of integration, we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} y^{-it} k^{(3)}(it)\,dt \int_1^\lambda u^{-it}\cos xu\,du. \tag{6.4.2}$$

The inner integral is $O(|t|)$ for large t. Suppose, for example, that $t > 0$. Then

$$\left|\int_1^\lambda u^{-it}\cos xu\,du\right| < t$$

if $t > \lambda$, while if $t < \lambda$ it is

$$O(t) + \int_t^\lambda u^{-it}\cos xu\,du$$

$$= O(t) + \frac{1}{x}\left[\sin xu\, u^{-it}\right]_t^\lambda - \frac{it}{x^2}\left[u^{-it-1}\cos xu\right]_t^\lambda$$

$$- \frac{it(it+1)}{x^2}\int_t^\lambda u^{-it-2}\cos xu\,du = O(t).$$

Since $k^{(3)}(it) = O(|t|^{-\frac{5}{2}})$ for large t, (6.4.2) is bounded†.

† In fact, with a B independent of ζ.

The argument requires modification if, as may happen, $k^{(3)}(s)$ has poles on the imaginary axis. It is sufficient to consider the case of one pole, say at $s = i\tau$ and with residue C. We write

$$k^{(3)}(s) = C\Gamma(s-i\tau)+\kappa(s)$$

and

$$K^{(3)}(x) = Cx^{-i\tau}e^{-x}+\mathrm{K}(x),$$

K being related to κ as K to k. Then κ satisfies the conditions imposed above upon $k^{(3)}$. The integral (6.4.1) falls into two, one of which is bounded in virtue of the preceding argument. The other is

$$Cy^{-i\tau}\int_1^\lambda u^{-i\tau}e^{-yu}\cos xu\,du,$$

which is also bounded, since

$$\int_1^\lambda u^{-i\tau}\cos xu\,du$$

is bounded†.

Hence $\chi_{1,3}$ is bounded. Practically the same argument proves $\chi_{2,3}$ to be bounded. A typical term here is

$$\frac{1}{2\pi i}\int_1^\lambda \frac{\sin xu - xu\cos xu}{x^2u^2}\,du\int_{c-i\infty}^{c+i\infty}(yu)^{-s}k^{(3)}(s)\,ds.$$

In place of (6.4.2) we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}y^{-it}k^{(3)}(it)\,dt\int_1^\lambda u^{-it}\frac{\sin xu - xu\cos xu}{x^2u^2}\,du,$$

and the inner integral here is $O(\log|t|)$ for large t.

The integrals $\chi_{3,1}$ and $\chi_{3,2}$ are simpler. Thus a typical term in $\chi_{3,1}$ is

$$\frac{1}{2\pi i}\int_1^\lambda \cos yu\,du\int_{c-i\infty}^{c+i\infty}(xu)^{-s}k^{(3)}(s)\,ds.$$

If we take $c = 1+\delta$, where $0<\delta<\frac{1}{2}$ and $1+\delta<\sigma_2$, then this is

$$\int_1^\lambda O(u^{-1-\delta})\,du\int_{-\infty}^{\infty}O(|s|)^{\delta-\frac{3}{2}}dt,$$

which is plainly bounded. The same argument applies to $\chi_{3,2}$.

† By the argument which proved the inner integral in (6.4.2) to be $O(|t|)$.

6.5. It remains only to prove $\chi_{3,3}$ bounded. We have

$$\chi_{3,3} = -\frac{1}{4\pi^2}\int_1^\lambda du \int_{c-i\infty}^{c+i\infty} (xu)^{-s} k^{(3)}(s)\, ds \int_{c'-i\infty}^{c'+i\infty} (yu)^{-s'} k^{(3)}(s')\, ds'.$$

Suppose first, as in § 6.4, that $k^{(3)}(s)$ has no poles on the imaginary axis, and take $c = 1+\delta$ (with the δ of the last paragraph) and $c' = 0$. The integral is absolutely convergent, and we may invert the order of integration. We thus obtain

$$-\frac{1}{4\pi^2}\int_{1+\delta-i\infty}^{1+\delta+i\infty} x^{-s} k^{(3)}(s)\, ds \int_{-i\infty}^{i\infty} y^{-s'} k^3(s') \frac{\lambda^{1-s-s'}-1}{1-s-s'}\, ds'.$$

This is bounded because $\Re(s') = 0$, $\Re(s+s') > 1$, and $k^{(3)}(s)$ and $k^3(s')$ are $O(|s|^{\delta-\frac{3}{2}})$ and $O(|s'|^{-\frac{3}{2}})$ respectively.

If there is a pole at $s = i\tau$, with residue C, then, as in § 6.4, we have to consider an additional term, which is in this case

$$\frac{C}{2\pi i}\int_1^\lambda (yu)^{-i\tau} e^{-yu}\, du \int_{1+\delta-i\infty}^{1+\delta+i\infty} (xu)^{-s} k^{(3)}(s)\, ds.$$

That this is bounded follows as at the end of § 6.4.

Hence $\chi_{3,3}$ is bounded; and this completes the proof of Lemma 10.

6.6. LEMMA 11. *If $k(s)$ belongs to* $\mathbf{k}'$, *and satisfies* (1.1.5), *and*

$$\psi(\lambda, x, y) = \int_{1/\lambda}^{\lambda} K(xu) \frac{K_1(yu)}{u}\, du,$$

where $\lambda > 1$ and x is positive, then

$$|\psi| < B(x, \zeta)$$

for $0 < x-\zeta < y < x+\zeta$; *and* $\psi(\lambda, x, y)$ *converges* (*boundedly*), *when* $\lambda \to \infty$, *to the limit*

$$0 \ \ (y < x), \quad \tfrac{1}{2} \ \ (y = x), \quad 1 \ \ (y > x).$$

In proving this lemma we use a simpler decomposition of $k(s)$, viz.:

$$k(s) = k^{(1)}(s) + k^{(2)}(s),$$

where $k^{(1)}(s)$ is the same as before, but $k^{(2)}(s)$ takes the place of $k^{(2)}(s)+k^{(3)}(s)$, so that

$$k^{(2)}(s) = O(|s|^{\sigma-\frac{3}{2}}).$$

We have

$$(6.6.1)\quad \psi(\lambda, x, y) = -\frac{1}{4\pi^2}\int_{1/\lambda}^{\lambda}\frac{du}{u}\int_{c-i\infty}^{c+i\infty}(xu)^{-s}k(s)\,ds\int_{c'-i\infty}^{c'+i\infty}\frac{(yu)^{1-s'}}{1-s'}k(s')\,ds'$$

$$= \psi_{1,1}+\psi_{1,2}+\psi_{2,1}+\psi_{2,2},$$

where $\psi_{p,q}$ is the integral derived from ψ by replacing $k(s)\,k(s')$ by $k^{(p)}(s)\,k^{(q)}(s')$. We begin by considering $\psi_{2,2}$. The product of the inner integrals here is

$$O\left\{\int_{-\infty}^{\infty}(1+|t|)^{c-\frac{3}{2}}u^{-c}\,dt\int_{-\infty}^{\infty}(1+|t'|)^{c'-\frac{5}{2}}u^{1-c'}\,dt'\right\} = O(u^{1-c-c'}),$$

provided that $c<\frac{1}{2}$, $c'<\frac{3}{2}$. We take $c+c'=\frac{1}{2}$ when $u\leqslant 1$ and $c+c'=\frac{3}{2}$ when $u>1$, thus obtaining

$$\psi_{2,2} = \int_{1/\lambda}^{1}O(u^{-\frac{1}{2}})\,du+\int_{1}^{\lambda}O(u^{-\frac{3}{2}})\,du.$$

Hence $\psi_{2,2}$ converges (boundedly) when $\lambda\to\infty$.

Next

$$\psi_{1,2} = \frac{1}{\pi i\sqrt{(2\pi)}}\int_{1/\lambda}^{\lambda}\frac{a_1\cos xu+a_2\sin xu}{u}\,du\int_{c'-i\infty}^{c'+i\infty}\frac{(yu)^{1-s'}}{1-s'}k^{(2)}(s')\,ds'.$$

The inner integral is

$$O\left\{\int_{-\infty}^{\infty}(1+|t'|)^{c'-\frac{5}{2}}u^{1-c'}\,dt'\right\} = O(u^{1-c'}),$$

provided that $c'<\frac{3}{2}$. We take $c'=\frac{1}{2}$ if $u\leqslant 1$ and $c'=\frac{5}{4}$ if $u>1$, and it follows that $\psi_{1,2}$ converges boundedly. A similar argument applies to $\psi_{2,1}$.

Finally

$$\psi_{1,1} = \frac{2}{\pi}\int_{1/\lambda}^{\lambda}(a_1\cos xu+a_2\sin xu)\,\frac{a_1\sin yu+a_2(1-\cos yu)}{u}\,du$$

$$= \frac{2}{\pi}\left\{a_1^2\int_{1/\lambda}^{\lambda}\frac{\cos xu\sin yu}{u}\,du+a_1a_2\int_{1/\lambda}^{\lambda}\frac{\cos xu-\cos(x+y)u}{u}\,du\right.$$

$$\left.+a_2^2\int_{1/\lambda}^{\lambda}\frac{\sin xu(1-\cos yu)}{u}\,du\right\},$$

and each term here converges boundedly.

This proves the lemma except as regards the value of the limit. To calculate this directly requires some further examination of the argument,

which presents no great difficulty. We can, however, avoid this calculation by an appeal to Lemma 7. The conditions of that lemma are satisfied, and therefore $\Psi(\lambda, x, y)$ tends to the required limit. But it follows from the "consistency theorem" for Rieszian means that, if $\psi(\lambda, x, y)$ tends to a limit, then $\Psi(\lambda, x, y)$ tends to the same limit, and this gives the result required.

6.7. If now we repeat the argument of § 5.2, but with ϕ and ψ in place of Φ and Ψ, we obtain the following theorem:

THEOREM 3. *If* (i) $k(s)$ *belongs to* $\mathbf{k}'$ *and satisfies* (1.1.5), (ii) $F(y)$ *is integrable in* $(0, \infty)$, (iii) x *is positive and* $F(y)$ *is of bounded variation near* $y = x$, *then*

$$(6.7.1) \quad \int_0^\infty K(xu)\,du \int_0^\infty K(uy)\,F(y)\,dy = \tfrac{1}{2}\{F(x+0)+F(x-0)\},$$

the inner integral being a Lebesgue integral and the outer integral a Cauchy integral at infinity.

It need only be observed that the outer integral is in the first instance defined as

$$\lim_{\lambda\to\infty} \int_{1/\lambda}^{\lambda} \ldots du;$$

but that, since K is bounded, and F integrable, the convention about the lower limit may be dropped.

6.8. We have supposed that $k(s)$ satisfies (1.1.5), so that $K(x)$ is a Fourier kernel. There is, however, no new difficulty in the unsymmetrical case. If $k(s)$ belongs to $\mathbf{k}'$, and $h(s)$ is defined by (1.3.4) and also belongs to $\mathbf{k}'$, then our analysis is still valid when we replace k and K by h and H in the appropriate places. We thus obtain

THEOREM 4. *If* (i) $k(s)$ *belongs to* $\mathbf{k}'$, (ii) $h(s)$ *is defined by* (1.3.4) *and belongs to* $\mathbf{k}'$, *and* (iii) $F(y)$ *satisfies the conditions of Theorem* 3, *then*

$$\int_0^\infty K(xu)\,du \int_0^\infty H(uy)\,F(y)\,dy = \tfrac{1}{2}\{F(x+0)+F(x-0)\}.$$

There are naturally theorems corresponding to Theorems 1 and 2 as Theorem 4 corresponds to Theorem 3.

6.9. We have not attempted to prove any general theorem corresponding to Theorem 2 as Theorem 3 corresponds to Theorem 1. The

most interesting cases are those in which $k(s)$ is

$$e^{-ai(s-\frac{1}{2})^r} \quad (a>0, \ r \text{ even}), \qquad e^{a(s-\frac{1}{2})^r} \quad (a>0, \ r \text{ odd}),$$

or a function behaving like one of these at infinity. The case $r=2$ reduces by substitution [as was pointed out in § 1.5 (5)] to a case of Fourier's theorem. We discuss here the next case $r=3$. The argument would be similar in the general case; but there are material simplifications when $r=3$, because some of the integrals which occur in the analysis can be evaluated in finite form.

In this case

$$K(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{a(s-\frac{1}{2})^3} x^{-s}\, ds.$$

If we take $c>\frac{1}{2}$, the integral is absolutely convergent. If follows that $K(x)$ is continuous except for $x=0$ and $x=\infty$; that it is $O(x^{-\Delta})$, for every Δ, at infinity; and $O(x^{-\frac{1}{2}-\delta})$, for every $\delta>0$, at the origin. More precise results follow from the explicit form of $K(x)$ given in § 1 (6).

LEMMA 12. *If*

$$k(s) = e^{a(s-\frac{1}{2})^3} \quad (a>0), \tag{6.9.1}$$

then

$$|y^{\frac{1}{2}}\phi| < B(x, \zeta)$$

for $|y-x| \geqslant \zeta$.

We first take $c=\frac{1}{2}+\delta$, where $\delta>0$. The integral for $K(x)$ is then absolutely convergent, and we may substitute for $K(xu)$ and $K(yu)$, and invert the integrations, in (6.3.1). We thus obtain

$$\phi(\lambda, x, y) = \frac{1}{2\pi^2(xy)^{\frac{1}{2}+\delta}}\int_{-\infty}^{\infty} e^{a(\delta+it)^3} x^{-it}\, dt \int_{-\infty}^{\infty} e^{a(\delta+it')^3} y^{-it'} \frac{\sin\mu(t+t'-2i\delta)}{t+t'-2i\delta}\, dt',$$

where $\mu = \log\lambda$. Putting $t=u+v$, $t'=u-v$, $\xi=\log(x/y)$, we get

$$\begin{aligned}\phi &= \frac{1}{\pi^2(xy)^{\frac{1}{2}+\delta}}\int_{-\infty}^{\infty} e^{2a(\delta+iu)^3}(xy)^{-iu}\frac{\sin 2\mu(u-i\delta)}{2(u-i\delta)}\, du \int_{-\infty}^{\infty} e^{-6a(\delta+iu)v^2}\left(\frac{y}{x}\right)^{iv} dv \\ &= \frac{1}{\pi(6a\pi)^{\frac{1}{2}}(xy)^{\frac{1}{2}+\delta}}\int_{-\infty}^{\infty} e^{2a(\delta+iu)^3}(xy)^{-iu}\frac{e^{-\xi^2/24a(\delta+iu)}}{(\delta+iu)^{\frac{1}{2}}}\frac{\sin 2\mu(u-i\delta)}{2(u-i\delta)}\, du.\end{aligned}$$

The integral converges uniformly in δ, the integrand being $O(u^{-\frac{1}{2}})$ for small and $O(u^{-\frac{3}{2}})$ for large u. Hence we may put $\delta=0$, so obtaining

$$\phi = \frac{1}{\pi(6a\pi xy)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-2aiu^3}(xy)^{-iu}\frac{e^{i\xi^2/24au}}{(iu)^{\frac{1}{2}}}\frac{\sin 2\mu u}{2u}\, du,$$

where $(iu)^{\frac{1}{2}}$ is $e^{\frac{1}{4}\pi i}u^{\frac{1}{2}}$ for $u>0$ and $e^{-\frac{1}{4}\pi i}(-u)^{\frac{1}{2}}$ for $u<0$. It is plain that the part of this integral for which $|u|>1$ is bounded. In the remaining part we may suppress the factor e^{-2aiu^3}, with error

$$\int_0^1 O(u^3)\,O(u^{-\frac{3}{2}})\,du = O(1).$$

We then separate the sine into its exponential components, and obtain multiples of

$$\int_0^1 \exp i\left(2\mu u - u\log xy + \frac{\xi^2}{24au}\right) u^{-\frac{3}{2}}\,du \tag{6.9.2}$$

and of a similar integral with $-\mu$ in the place of μ.

In (6.9.2) we may restore the infinite limit, with bounded error; and when we do this we obtain

$$\int_0^\infty \exp i\left(pu+\frac{q}{u}\right) u^{-\frac{3}{2}}\,du = \frac{|p|^{\frac{1}{4}}}{q^{\frac{1}{4}}}\int_0^\infty \exp\left\{i|p|^{\frac{1}{2}}q^{\frac{1}{2}}\left(\eta u+\frac{1}{u}\right)\right\} u^{-\frac{3}{2}}\,du,$$

where $p = 2\mu - \log xy$, $q = \xi^2/24a$, $\eta = \operatorname{sgn} p$. This is†

$$\frac{|p|^{\frac{1}{4}}}{q^{\frac{1}{4}}}\,O(|p|^{-\frac{1}{4}}q^{-\frac{1}{4}}) = O(q^{-\frac{1}{2}}) = O(|\xi|^{-1}) < B,$$

and this proves the lemma.

6.10. LEMMA 13. *The result of Lemma* 11 *is still true when* $k(s)$ *is defined by* (6.9.1).

Proceeding as before, we find that $\psi(\lambda, x, y)$ is

$$\frac{y^{\frac{1}{2}-\delta}}{\pi^2 x^{\frac{1}{2}+\delta}}\int_{-\infty}^{\infty} e^{2a(\delta+iu)^3}(xy)^{-iu}\,\frac{\sin 2\mu(u-i\delta)}{2(u-i\delta)}\,du \int_{-\infty}^{\infty} e^{-6a(\delta+iu)v^2}\,\frac{e^{-i\xi v}}{\frac{1}{2}-\delta+iv-iu}\,dv,$$

where $0<\delta<\frac{1}{2}$. The inner integral is a sum of integrals of the type

$$\int e^{i\theta(v)}\chi(v)\,dv,$$

where χ is monotonic and bounded and

$$\theta(v) = -6auv^2 - v\xi, \quad \theta''(v) = -12au.$$

It follows from Lemmas 2 and 4 that the integral is $O(|u|^{-\frac{1}{2}})$, uniformly for

† See Watson (**18**, §§6.2–6.22) for general formulae expressing integrals of this type in terms of Bessel functions.

$0 \leqslant \delta \leqslant \frac{1}{4}$. Hence, as in the proof of Lemma 12, we may put $\delta = 0$, when we obtain

$$(6.10.1) \quad \psi = \frac{y^{\frac{1}{2}}}{\pi x^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-2ai u^3} (xy)^{-iu} \frac{\sin 2\mu u}{2u}\, du \int_{-\infty}^{\infty} e^{-6aiuv^2} \frac{e^{-i\xi v}}{\frac{1}{2}+iv-iu}\, dv.$$

The part of (6.10.1) with $|u| > 1$ is

$$\int_1^{\infty} \sin 2\mu u\, O(u^{-\frac{3}{2}})\, du,$$

and therefore (by the Riemann-Lebesgue theorem) converges boundedly to 0. If $|u| \leqslant 1$,

$$e^{-2ai u^3} (xy)^{-iu} = 1 + O(u),$$

and the $O(u)$ gives an integral which converges boundedly to 0. Finally, if we replace $\frac{1}{2}+iv-iu$ by $\frac{1}{2}+iv$ in the inner integral, the error introduced into it is less than

$$|u| \int_{-\infty}^{\infty} \frac{dv}{|\frac{1}{2}+iv|\,|\frac{1}{2}+iv-iu|} = O\left\{|u| \int_{-2}^{2} \frac{dv}{\sqrt{(\frac{1}{4}+v^2)}}\right\}$$

$$+ O\left\{|u| \int_{|v|\geqslant 2} \frac{dv}{|v|\sqrt{(\frac{1}{4}+v^2)}}\right\} = O|u|;$$

and this again gives an integral which converges boundedly to 0. We may therefore replace ψ by

$$(6.10.2) \qquad \frac{y^{\frac{1}{2}}}{\pi^2 x^{\frac{1}{2}}} \int_{-1}^{1} \frac{\sin 2\mu u}{2u}\, du \int_{-\infty}^{\infty} \frac{e^{-6aiuv^2 - i\xi v}}{\frac{1}{2}+iv}\, dv.$$

We write the inner integral here in the form

$$\int_{-\infty}^{\infty} e^{-6aiuv^2} \frac{e^{-i\xi v}}{iv}\, dv - \tfrac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-i\xi v}}{(\frac{1}{2}+iv)iv}\, dv + \tfrac{1}{2} \int_{-\infty}^{\infty} (1 - e^{-6aiuv^2}) \frac{e^{-i\xi v}}{(\frac{1}{2}+iv)iv}\, dv$$

$$= P(u) + Q(u) + R(u),$$

say. The integrals which define $P(u)$ and $Q(u)$ are principal values at $v = 0$. Since

$$R(u) = O\left\{\int_{|v|<|u|^{-\frac{1}{2}}} \frac{|u|\,|v|^2\, dv}{|v|(\frac{1}{4}+v^2)^{\frac{1}{2}}}\right\} + O\left\{\int_{|v|>|u|^{-\frac{1}{2}}} \frac{dv}{|v|(\frac{1}{4}+v^2)^{\frac{1}{2}}}\right\} = O(|u|^{\frac{1}{2}}),$$

the contribution of $R(u)$ tends boundedly to 0, and we need only consider those of $P(u)$ and $Q(u)$.

A simple application of the calculus of residues shows that

$$Q(u) = 2\pi e^{\frac{1}{2}\xi} - \pi = 2\pi x^{\frac{1}{2}} y^{-\frac{1}{2}} - \pi \quad (\xi < 0),$$

$$Q(u) = \pi \qquad (\xi \geqslant 0).$$

Hence the contribution of $Q(u)$ converges boundedly to the limit

$$(6.10.3) \qquad 1 - \tfrac{1}{2} y^{\frac{1}{2}} x^{-\frac{1}{2}} \ (y > x), \quad \tfrac{1}{2} y^{\frac{1}{2}} x^{-\frac{1}{2}} \ (y \leqslant x).$$

Finally,
$$P(u) = -(1-i)(\tfrac{1}{2}\pi)^{\frac{1}{2}} \int_0^{\xi^2/(24au)} e^{it} t^{-\frac{1}{2}}\, dt$$

if u and ξ are positive†; $P(-u)$ is the conjugate of $P(u)$; and $P(u)$ is odd in ξ. If ξ and u are positive, $P(u)$ is of the forms

$$C + O\left(\frac{u^{\frac{1}{2}}}{\xi}\right) \ (u < \xi^2), \qquad O\left(\frac{\xi}{u^{\frac{3}{2}}}\right) \ (u > \xi^2);$$

and there are similar forms for negative u or ξ. The corresponding contributions to (6.10.2) are of the type

$$O(1) + \int_0^{\xi^2} O\left(\frac{1}{\xi u^{\frac{1}{2}}}\right) du + \int_{\xi^2}^{\infty} O\left(\frac{\xi}{u^{\frac{3}{2}}}\right) du = O(1).$$

It follows that the contribution of $P(u)$ to (6.10.2) is bounded.

If $y < x$, $\xi > 0$, and $\mu \to \infty$, then

$$\frac{y^{\frac{1}{2}}}{\pi^2 x^{\frac{1}{2}}} \int_{-1}^{1} \frac{\sin 2\mu u}{2u} P(u)\, du \to \frac{y^{\frac{1}{2}}}{\pi^2 x^{\frac{1}{2}}} \cdot \tfrac{1}{4}\pi \{P(+0) + P(-0)\}$$

$$= -\frac{y^{\frac{1}{2}}}{4\pi x^{\frac{1}{2}}} \left\{ (\tfrac{1}{2}\pi)^{\frac{1}{2}}(1-i) \int_0^{\infty} e^{it} t^{-\frac{1}{2}}\, dt + (\tfrac{1}{2}\pi)^{\frac{1}{2}}(1+i) \int_0^{\infty} e^{-it} t^{-\frac{1}{2}}\, dt \right\} = -\frac{y^{\frac{1}{2}}}{2x^{\frac{1}{2}}}.$$

If $y > x$ it tends to $\frac{1}{2} y^{\frac{1}{2}} x^{-\frac{1}{2}}$, while if $y = x$, $P(u)$ vanishes. Adding these values to (6.10.3), we obtain the result of the lemma.

We have now only to repeat previous arguments in order to prove

THEOREM 5. *If $k(s)$ is defined by* (6.9.1), *$y^{-\frac{1}{2}-\delta} F(y)$ is integrable in $(0, 1)$ for some $\delta > 0$, and $y^{-\frac{1}{2}} F(y)$ is integrable in $(1, \infty)$‡, and $F(y)$ is of bounded variation near $y = x$, then* (6.7.1) *is true in the same sense as in Theorem* 3.

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† This is easily proved by differentiation with respect to ξ.

‡ We are content to state those conditions of integrability at 0 and at ∞ which follow naturally from our analysis.

A special convergence theorem.

7.1. We conclude by proving the theorem referred to at the end of § 1. Here we assume acquaintance with the transform theory developed by Watson.

THEOREM 6. *Suppose* (i) *that* $k(\frac{1}{2}+it)$ *satisfies* (1.6.1) *and* (1.6.2), *so that*† $x^{-1}K_1(x)$, *defined by* (1.2.3), *with* $c=\frac{1}{2}$, *belongs to* $L^2(0,\infty)$; (ii) *that* $K_1(x)$ *is the integral of* $K(x)$; (iii) *that* $x^{-\frac{1}{2}}K_1(x)$ *is bounded.*

Further suppose that

$$F(x)=\frac{1}{x}\int_0^x \phi(y)\,dy, \tag{7.1.1}$$

where $\phi(y)$ *belongs to* $L^2(0,\infty)$. *Then*

$$\int_0^\infty K(xu)\,du\int_0^\infty K(uy)\,F(y)\,dy=F(x), \tag{7.1.2}$$

for every positive x, *each integral being a Cauchy integral at each limit.*

It follows from (7.1.1) and Schwarz's inequality that $F(x)=o(x^{-\frac{1}{2}})$ both for small and for large x, and from a theorem of Hardy‡ that $F(x)$ and

$$F^*(x)=\frac{1}{x}\int_0^x |\phi(y)|\,dy$$

belong to $L^2(0,\infty)$.

Let $\psi(x)$ be the transform of $\phi(x)$. Then, by Watson's theory, $\psi(x)$ belongs to L^2 and

$$\int_0^x \phi(y)\,dy=\int_0^\infty \frac{K_1(xu)}{u}\,\psi(u)\,du.$$

Let

$$G(u)=\int_u^\infty \frac{\psi(v)}{v}\,dv. \tag{7.1.3}$$

Then

$$\begin{aligned}\int_0^x \phi(y)\,dy &= -\int_0^\infty K_1(xu)\,G'(u)\,du\\ &= \lim_{\delta\to 0,\ \Delta\to\infty}\left\{-\Big[K_1(xu)\,G(u)\Big]_\delta^\Delta + x\int_\delta^\Delta K(xu)\,G(u)\,du\right\}.\end{aligned}$$

† By Watson's theory.

‡ Hardy (6, 8, 9).

Now it follows from (7.1.3) and Schwarz's inequality that $G(u) = o(u^{-\frac{1}{2}})$ when $u \to 0$ or $u \to \infty$; and (*ex hypothesi*) $K_1(xu) = O(u^{\frac{1}{2}})$. Hence the integrated terms vanish in the limit, and

$$(7.1.4) \qquad F(x) = \frac{1}{x}\int_0^x \phi(y)\,dy = \int_0^\infty K(xu)\,G(u)\,du,$$

the integral being a Cauchy integral at each limit.

Again, (7.1.3) may be written

$$G(u) = \int_0^\infty \psi(v)\,\mu(v)\,dv,$$

where $$\mu(v) = 0 \quad (v < u), \qquad \mu(v) = 1/v \quad (v > u).$$

Hence, by Parseval's theorem†,

$$G(u) = \int_0^\infty \phi(v)\,\lambda(v)\,dv,$$

where $\lambda(v)$ is the transform of $\mu(v)$. Now

$$\lambda(v) = \frac{d}{dv}\left\{\int_u^\infty \frac{K_1(vt)}{t^2}\,dt\right\} = \frac{d}{dv}\left\{v\int_{uv}^\infty \frac{K_1(w)}{w^2}\,dw\right\} = \int_{uv}^\infty \frac{K_1(w)}{w^2}\,dw - \frac{K_1(uv)}{uv};$$

and so

$$(7.1.5) \quad G(u) = \int_0^\infty \phi(v)\,dv\int_{uv}^\infty \frac{K_1(w)}{w^2}\,dw - \int_0^\infty \phi(v)\,\frac{K_1(uv)}{uv}\,dv = G_1(u) - G_2(u),$$

say. Here

$$(7.1.6) \quad G_1(u) = \int_0^\infty \frac{K_1(w)}{w^2}\,dw\int_0^{w/u}\phi(v)\,dv = \int_0^\infty \frac{K_1(w)}{w^2}\,\frac{w}{u}\,F\left(\frac{w}{u}\right)dw$$

$$= \int_0^\infty \frac{K_1(uy)}{uy}\,F(y)\,dy,$$

the inversion being justified because $K_1(w)/w$ and $F^*(w/u)$ are L^2, and so

$$\int_0^\infty \frac{|K_1(w)|}{w^2}\,dw\int_0^{w/u}|\phi(v)|\,dv = \frac{1}{u}\int_0^\infty \frac{|K_1(w)|}{w}\,F^*\left(\frac{w}{u}\right)dw$$

is finite. Also

$$G_2(u) = \lim_{\delta\to 0,\ \Delta\to\infty}\left\{\left[\frac{K_1(uv)}{uv}\,.\,vF(v)\right]_\delta^\Delta - \int_\delta^\Delta \frac{K(uv)}{v}\,.\,vF(v)\,dv + \int_\delta^\Delta \frac{K_1(uv)}{uv^2}\,.\,vF(v)\,dv\right\}.$$

† In Watson's theory.

The terms integrated out again vanish in the limit, and so

$$G_2(u) = -\int_0^\infty K(uv)\,F(v)\,dv + G_1(u), \tag{7.1.7}$$

the integral being again a Cauchy integral at both limits. From (7.1.5), (7.1.6), and (7.1.7) we deduce

$$G(u) = \int_0^\infty K(uv)\,F(v)\,dv; \tag{7.1.8}$$

and (7.1.4) and (7.1.8) prove the theorem.

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GENERAL TRANSFORMS

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1. *Generalities concerning Fourier's integrals and transforms.*

The existence of Fourier's cosine integral, by means of which an arbitrary function $f(x)$, subject to appropriate general conditions, is capable of being represented as a repeated integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty \cos xy \cos yz \, f(z) \, dz \, dy,$$

naturally suggests the problem of determining the most general real kernel $\varpi(x)$ such that an arbitrary function $f(x)$ is capable of being represented as a repeated integral

$$f(x) = \int_0^\infty \int_0^\infty \varpi(xy) \, \varpi(yz) \, f(z) \, dz \, dy. \tag{1.1}$$

A class of kernels which contain one parameter is supplied by Hankel's integral theorem involving Bessel functions, in which

$$\varpi(x) = x^{\frac{1}{2}} J_\nu(x) \quad (\nu \geqslant -\tfrac{1}{2}).$$

As is well known, Hankel's integral reduces to Fourier's cosine integral when $\nu = -\frac{1}{2}$ and it reduces to Fourier's sine integral when $\nu = \frac{1}{2}$.

A formal solution of the problem of determining the most general form of the kernel $\varpi(x)$ is quite simple. Suppose equation (1.1) to be satisfied by an arbitrary function $f(x)$, let s be a complex variable and define the functions $F(s)$, $\Omega(s)$ by the formulae

$$F(s) = \int_0^\infty x^{s-1} f(x) \, dx, \qquad \Omega(s) = \int_0^\infty x^{s-1} \varpi(x) \, dx. \tag{1.2}$$

Then, assuming the permissibility of making the requisite changes in the order of the integrations, we have

$$\begin{aligned} F(s) &= \int_0^\infty \int_0^\infty \int_0^\infty x^{s-1}\, \varpi(xy)\, \varpi(yz)\, f(z)\, dz\, dy\, dx \\ &= \int_0^\infty \int_0^\infty y^{-s}\, \Omega(s)\, \varpi(yz)\, f(z)\, dz\, dy \\ &= \int_0^\infty \Omega(s)\, \Omega(1-s)\, z^{s-1} f(z)\, dz \\ &= \Omega(s)\, \Omega(1-s)\, F(s). \end{aligned}$$

Now Mellin's inversion formula,

$$f(x) = \frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} x^{-s}\, F(s)\, ds,$$

obviously shows that $F(s)$ cannot be identically zero when $f(x)$ is not identically zero, and so we conclude that

$$\Omega(s)\, \Omega(1-s) = 1 \tag{1.3}$$

is a formally necessary condition for the truth of the general formula (1.1). The condition is also formally sufficient; to prove this statement, suppose (1.3) to be satisfied, let

$$\int_0^\infty \int_0^\infty \varpi(xy)\, \varpi(yz)\, f(z)\, dz\, dy = f_1(x),$$

and also let

$$\int_0^\infty x^{s-1} f_1(x)\, dx = F_1(s).$$

We get, on repeating the previous analysis,

$$\begin{aligned} F_1(s) &= \int_0^\infty \int_0^\infty \int_0^\infty x^{s-1}\, \varpi(xy)\, \varpi(yz)\, f(z)\, dz\, dy\, dx \\ &= \Omega(s)\, \Omega(1-s)\, F(s) \\ &= F(s). \end{aligned}$$

It follows immediately from Mellin's inversion formula that

$$f_1(x) = f(x),$$

and this establishes the formal sufficiency of the condition (1.3).

Since (1.1) contains the product of two functions of the type $\varpi(x)$, we may change the sign of $\varpi(x)$, if necessary, to ensure that $\Omega(\frac{1}{2}) = +1$,

and then, if we disregard considerations of many-valuedness, we have

$$\log \Omega(s) = -\log \Omega(1-s),$$

so that (1.3) is equivalent to the statement that $\log \Omega(s)$ *is to be an odd function of* $s-\frac{1}{2}$.

We have consequently obtained a complete formal solution of our problem; it may be stated as follows:

Let $\log \Omega(s)$ *be an odd function of* $s-\frac{1}{2}$; *and let*

$$\varpi(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} x^{-s}\,\Omega(s)\,ds.$$

Then an arbitrary function $f(x)$ *can be represented by formula* (1.1), *namely*

$$f(x) = \int_0^\infty \int_0^\infty \varpi(xy)\,\varpi(yz)\,f(z)\,dz\,dy.$$

Conversely, if an arbitrary function $f(x)$ *can be represented by formula* (1.1), *and if*

$$\Omega(s) = \int_0^\infty x^{s-1}\,\varpi(x)\,dx,$$

then $\log \Omega(s)$ *is an odd function of* $s-\frac{1}{2}$.

As is to be expected, this formal result includes Hankel's integral theorem; for, when $\varpi(x) = x^{\frac{1}{2}} J_\nu(x)$, we have

$$\Omega(s) = \int_0^\infty x^{s-\frac{1}{2}} J_\nu(x)\,dx = 2^{s-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s+\frac{3}{4})},$$

by a formula due to Weber†; and this value of $\Omega(s)$ clearly satisfies (1.3).

If we attempt to justify the formal analysis which has just been given, we are immediately confronted by formidable difficulties. Even before we attack the problems raised by the changes in the order of the integrations, we have to face the awkward fact that, in the obvious test-case provided by Fourier's cosine integral, we want to have c between 0 and 1 in order that the integrals defining $\Omega(s)$ and $\Omega(1-s)$ may be convergent, while the integral which expresses $\varpi(x)$ in terms of $\Omega(s)$ does not necessarily converge in the ordinary sense when $c \geqslant \frac{1}{2}$, and so, since both $\Omega(s)$ and $\Omega(1-s)$ occur in the analysis, the introduction of some kind of

† See my *Theory of Bessel functions* (1922), 391.

generalized integral seems inevitable. Again, when we embark upon an investigation on more classical lines, such as those on which Hobson† approaches Fourier's integral theorem, the absence of any obvious useful approximation for

$$\int_0^Y \varpi(xy)\,\varpi(yz)\,dy$$

is a substantial obstacle at the very outset.

In the preceding paper, Hardy and Titchmarsh‡ have found it possible to overcome these difficulties by prescribing somewhat closely the behaviour of $\Omega(s)$ on the critical line $\Re(s) = \frac{1}{2}$. A study of their analysis indicates the likelihood of the impossibility of removing their limitations imposed on $\Omega(s)$ so completely that the only surviving restriction to be imposed on $\Omega(s)$ should be the natural restriction§ that it should satisfy the functional equation (1.3).

When we turn to the related topic of generalizations of the Plancherel theory of transforms, the analysis available is decidedly more powerful, and it is to this topic exclusively that I propose to devote the present paper.

In the theory of Fourier transforms, initiated by Plancherel‖ and developed by Titchmarsh¶, the central theorem for the cosine transform (there is a corresponding theorem for the sine transform) is as follows:

Corresponding to every function $f(x)$ which belongs to†† $L^2(0, \infty)$, *there exists a function $g(x)$, which also belongs to $L^2(0, \infty)$, such that*

$$g(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty \frac{\sin xy}{y} f(y)\,dy, \tag{1.41}$$

almost everywhere in $(0, \infty)$; and the relation between $f(x)$ and $g(x)$ is a reciprocal one, so that

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{d}{dx} \int_0^\infty \frac{\sin xy}{y} g(y)\,dy, \tag{1.42}$$

† *Functions of a real variable*, 2 (1926), 720–728.

‡ *Supra.*

§ Combined, of course, with the consequences of the tacit assumption that $\varpi(x)$ is to be a real function of x.

‖ *Rend. di Palermo*, 30 (1910), 289–335.

¶ *Proc. Camb. Phil. Soc.*, 21 (1922), 463–473; *Proc. London Math. Soc.* (2), 23 (1924), 279–289.

†† This means that the Lebesgue integral $\int_0^\infty \{f(x)\}^2\,dx$ exists; I ignore generalizations of the theorem in which $f(x)$ is a function of class L^p with $p \neq 2$.

almost everywhere in $(0, \infty)$. *Further*†

$$\int_0^\infty \{f(x)\}^2\,dx = \int_0^\infty \{g(x)\}^2\,dx. \tag{1.43}$$

It is possible to generalize this theorem in a perfectly adequate manner. The problem to be attacked is the determination of conditions to be imposed upon the real function $\chi(y)$ to ensure the truth of the following theorem which is the obvious generalization and the exact analogue of the theorem which has just been quoted:

THEOREM I. *Corresponding to every function* $f(x)$ *which belongs to* $L^2(0, \infty)$, *there exists a function* $g(x)$, *which also belongs to* $L^2(0, \infty)$, *such that*

$$g(x) = \frac{d}{dx}\int_0^\infty \frac{\chi(xy)}{y}\,f(y)\,dy, \tag{1.51}$$

almost everywhere in $(0, \infty)$; *and the relation between* $f(x)$ *and* $g(x)$ *is a reciprocal one, so that*

$$f(x) = \frac{d}{dx}\int_0^\infty \frac{\chi(xy)}{y}\,g(y)\,dy, \tag{1.52}$$

almost everywhere in $(0, \infty)$. *Further*

$$\int_0^\infty \{f(x)\}^2\,dx = \int_0^\infty \{g(x)\}^2\,dx. \tag{1.53}$$

The assumption of the truth of Theorem I does not, however, afford a satisfactory means of determining necessary conditions to be imposed on $\chi(y)$; the obstacles which prevent progress are the hypotheses that (1.51) and (1.52) hold not *everywhere* but *almost everywhere*. It is, however, possible to enunciate a modified (and, in some respects, more natural) theorem in which these difficulties do not occur. The modified theorem is as follows:

THEOREM Ia. *Corresponding to every function* $f(x)$ *which belongs to* $L^2(0, \infty)$, *there exists a function* $g(x)$, *which also belongs to* $L^2(0, \infty)$, *such*

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† A theorem such as (1.43) is often called Parseval's theorem. If $g_1(x)$ and $g_2(x)$ are the transforms of two functions, $f_1(x)$ and $f_2(x)$, which are both of class $L^2(0, \infty)$, we may apply Parseval's theorem to the four functions $f_1(x)$, $f_2(x)$, and $f_1(x) \pm f_2(x)$, and deduce the generalized Parseval theorem

$$\int_0^\infty f_1(x)\,f_2(x)\,dx = \int_0^\infty g_1(x)\,g_2(x)\,dx.$$

Cf. Hobson, *Functions of a real variable* (1 ed., 1907), 716–717.

that†

$$\int_0^x g(\xi)\,d\xi = \Big[\int_0^\infty \frac{\chi(\xi y)}{y} f(y)\,dy\Big]_{\xi=0}^{\xi=x}, \tag{1.61}$$

and the relation between $f(x)$ *and* $g(x)$ *is a reciprocal one, so that*

$$\int_0^x f(\xi)\,d\xi = \Big[\int_0^\infty \frac{\chi(\xi y)}{y} g(y)\,dy\Big]_{\xi=0}^{\xi=x}. \tag{1.62}$$

Further,

$$\int_0^\infty \{f(x)\}^2\,dx = \int_0^\infty \{g(x)\}^2\,dx, \tag{1.63}$$

with the obvious extension to transforms of two different functions.

A complete solution of the problem of determining $\chi(y)$ is now contained in

THEOREM II. *If* $\chi(y)/y$ *is a function which belongs to* $L^2(0, \infty)$, *it is necessary for the truth of Theorem* IA *that* $\chi(y)$ *should be such that the formula*

$$\int_0^\infty \frac{\chi(xy)\,\chi(yz)}{y^2}\,dy = \mathrm{Min}(x, z) \tag{1.71}$$

is valid for all values of x *and* z *in* $(0, \infty)$.

Conversely, if (1.71) *is valid for all values of* x *and* z *in* $(0, \infty)$, *then* $\chi(y)/y$ *belongs to* $L^2(0, \infty)$‡ *and Theorem* IA *is true*; *and, as an immediate consequence, Theorem* I *is also true.*

It is quite easy to prove that the condition (1.71) is necessary for the truth of Theorem IA, and the proof will be given in §2. As might be anticipated, the proof that the condition is also sufficient is a much more elaborate proceeding, and the proof occupies §§3–10. In fact, it is a little surprising that it is possible to get all that is required out of a condition§ of such slender appearance as (1.71).

Theorem II is, however, not quite final, since it does not supply a direct method of constructing functions $\chi(y)$. It is consequently necessary to

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† It turns out (*cf.* §8) that the expression on the right of (1.61) may be replaced by

$$\int_0^\infty \frac{\chi(xy)}{y} f(y)\,dy;$$

I decided to retain the expression given in order to exhibit the origin of the equation.

‡ This is an immediate consequence of putting $x = z = 1$ in (1.71).

§ Since this paper was first written, I have been reminded by Prof. Temple of his use of discontinuous integrals (which I had temporarily forgotten) in a somewhat similar problem, *Proc. London Math. Soc.* (2), 31 (1931), 231–242.

link up Theorem II in some way with the functional equation (1.3). It is a fairly straightforward matter to make the transition from the discontinuous integral (1.71) to a form of the functional equation (1.3), and the final result, which will be proved in § 12, is as follows:

THEOREM III. *Let $\chi(y)$ be a function of y such that $\chi(y)/y$ belongs to $L^2(0, \infty)$, so that*

$$\int_0^\infty \chi(y)\, y^{it-\frac{3}{2}}\, dy$$

exists as a "mean-square integral"†; and let the value to which this integral converges in mean be denoted by‡

$$\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it},$$

so that $|\Omega(\frac{1}{2}+it)/(\frac{1}{2}-it)|$ is a function of class $L^2(-\infty, \infty)$, and $\Omega(\frac{1}{2}+it)$ is a complex function of the real variable t such that $\Omega(\frac{1}{2}+it)$ and $\Omega(\frac{1}{2}-it)$ are conjugate complex numbers. Then it is necessary for the truth of Theorem IA *that the functional equation*

$$\Omega(\tfrac{1}{2}+it)\,\Omega(\tfrac{1}{2}-it) = 1 \tag{1.72}$$

should be satisfied.

Conversely, if $\Omega(\frac{1}{2}+it)$ is a complex function of the real variable t, such that $\Omega(\frac{1}{2}+it)$ and $\Omega(\frac{1}{2}-it)$ are conjugate complex numbers, and such that the functional equation (1.72) is satisfied, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} y^{-\frac{1}{2}-it}\,\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it}\,dt$$

exists as a mean-square integral§; and, if the value of this mean-square

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† This means that a function $\Omega(\frac{1}{2}+it)/(\frac{1}{2}-it)$ exists such that

$$\lim_{q\to\infty}\int_{-\infty}^{\infty}\left|\left[\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it}-\int_{\exp(-q)}^{\exp q}\chi(y)\,y^{it-\frac{3}{2}}\,dy\right]\right|^2 dt = 0.$$

‡ Henceforth, except in § 14, $\Omega(\frac{1}{2}+it)$ is supposed to be defined for real values of t only; no harm is done by the retention of a notation appropriate for analytic functions, and the notation serves as a reminder of the formal origin of the function.

§ This means that

$$\lim_{q\to\infty}\int_0^\infty\left[\frac{\chi(y)}{y}-\frac{1}{2\pi}\int_{-q}^{q} y^{-\frac{1}{2}-it}\,\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it}\,dt\right]^2 dy = 0.$$

Observe that the immediate effect of the proviso that $\Omega(\frac{1}{2}+it)$ and $\Omega(\frac{1}{2}-it)$ are to be conjugate complex numbers is to make $\chi(y)$ a real function of y; when combined with the functional equation, it has the important ulterior consequence that it makes $|\Omega(\frac{1}{2}+it)| = 1$.

integral is denoted by $\chi(y)/y$, *then* $\chi(y)/y$ *is a function of class* $L^2(0, \infty)$, *and* $\chi(xy)/y$ *is a kernel for which Theorem* Ia (*and consequently Theorem* I) *is true.*

The proof of Theorem III will be given in §§ 11–12. It is evident that Theorem III establishes a form of the functional equation (1.3) as an ultimate necessary and sufficient condition for the truth of Theorem Ia; moreover, Theorem III is of such a character that it makes possible the construction of an unlimited number of kernels for which Theorem Ia (and consequently Theorem I) is true. A few examples will be given in § 14.

The theorems which have now been enunciated evidently provide a complete generalization of the theory of Fourier transforms; but it is to be observed that the generalization, as it stands, does not include Hankel transforms in the form in which they have been developed by Titchmarsh† (I remark parenthetically that the Fourier sine transform is *not* obtained by taking $\nu = \frac{1}{2}$ in the Hankel transform). It is, however, possible to make a change of variable by means of which Theorems I, Ia, and III are transformed into theorems which provide a generalization of Titchmarsh's theory of Hankel transforms exactly analogous to the generalization of Plancherel's theory of Fourier transforms which is provided by Theorems I, Ia, and III themselves. It is convenient to defer the whole of the discussion of this generalization of Hankel transforms to § 13.

As a conclusion to these introductory remarks, I have various acknowledgments to make. As might have been expected, my discovery of the formal result (1.3) was a consequence of having studied the methods used by Hardy and Titchmarsh in their important paper‡ on self-reciprocal functions; I have since ascertained that the result was known to them long before I discovered it, and they had, in fact, obtained the results of their preceding paper on generalizations of Fourier's integral before I had even started working on transforms.

A substantial part of the analysis of this paper is obtainable by making appropriate changes in two papers by Hardy§ on Hilbert transforms; but for the existence of these papers it is unlikely that I should have embarked upon serious work on the modern theory of functions of a real variable. I have set out the reasoning in this paper more fully than Hardy has done in the papers just cited because my desire to be sure of

† *Proc. Camb. Phil. Soc.*, 21 (1923), 463–473.

‡ *Quart. J. of Math.* (Oxford series), 1 (1930), 196–231.

§ *Messenger of Math.*, 54 (1925), 20–27, 81–88. I remark that the kernel of a Hilbert transform is not of the form $\chi(xy)/y$ nor, as it stands, of the form $e^x\Theta(x+y)$, which will be discussed presently. It may, however, be brought to the latter form by appropriate changes in notation, and it then loses the skew character which it owes to the presence of a minus sign and becomes symmetrical.

the unfamiliar ground on which I am treading made me decide that it was prudent to abandon my usual policy of aiming at being as terse as possible.

My warmest thanks are due to Prof. Hardy for the trouble which he has taken in going through the first draft of this paper and making suggestions for revising it. In particular, he is responsible for making me realize that Theorem Ia (and not Theorem I) is the really fundamental theorem; he has supplied me with the analysis of § 8, and has thereby considerably shortened and simplified my own investigation; and, at his suggestion, I have rewritten § 11 in a form very much briefer than the original.

Prof. Titchmarsh has suggested to me that this paper could have been considerably shortened and simplified by regarding $f(x)$ as a limit to which a sequence of *differentiable* functions converges in mean†. I decided against rewriting the paper on these lines because I felt that the content of the paper would then have no longer been sufficiently my own work to justify my being regarded as the sole author of it.

2. *The necessity of the condition to be satisfied by* $\chi(y)$.

We shall now assume that $\chi(y)$ is a real function of y such that $\chi(y)/y$ is of class $L^2(0, \infty)$, and that $\chi(xy)/y$ is a kernel for which Theorem Ia is true; and we shall deduce from these assumptions that it is essential for $\chi(y)$ to satisfy (1.71).

Let z be a positive constant, and let $f(x)$ be defined in the following manner:

$$f(x) = 1 \quad (0 \leqslant x < z),$$
$$= 0 \quad (x \geqslant z).$$

When $f(x)$ is so defined, it belongs to $L^2(0, \infty)$; and $g(x)$ then satisfies the equation

$$\int_0^x g(\xi)\, d\xi = \left[\int_0^z \frac{\chi(\xi y)}{y}\, dy\right]_{\xi=0}^{\xi=x},$$

in accordance with (1.61). Hence we have

$$\begin{aligned}\int_0^x g(\xi)\, d\xi &= \left[\int_0^z \frac{\chi(\xi y)}{y}\, dy\right]_{\xi=0}^{\xi=x} \\ &= \left[\int_0^{\xi z} \frac{\chi(u)}{u}\, du\right]_{\xi=0}^{\xi=x} \\ &= \int_0^{xz} \frac{\chi(u)}{u}\, du = \int_0^x \frac{\chi(\xi z)}{\xi}\, d\xi.\end{aligned}$$

† *Cf.* Hardy, *Quart. J. of Math.* (Oxford series), 3 (1932), 111–112.

This result shows that the functions $g(x)$ and $\chi(xz)/x$ are "equivalent"†; and consequently the functions of y,

$$g(y)\,\frac{\chi(xy)}{y}, \quad \frac{\chi(yz)}{y}\cdot\frac{\chi(xy)}{y},$$

are also "equivalent". Hence, by a known theorem,

$$\int_0^\infty \frac{\chi(xy)}{y}\,g(y)\,dy = \int_0^\infty \frac{\chi(xy)\,\chi(yz)}{y^2}\,dy,$$

and therefore, by (1.62),

$$\int_0^x f(\xi)\,d\xi = \Big[\int_0^\infty \frac{\chi(\xi y)\,\chi(yz)}{y^2}\,dy\Big]_{\xi=0}^{\xi=x}.$$

In this result, valid for general values of x, give x the special value z; then, by subtracting the special result from the general result, we have

$$\int_z^x f(\xi)\,d\xi = \Big[\int_0^\infty \frac{\chi(\xi y)\,\chi(yz)}{y^2}\,dy\Big]_{\xi=z}^{\xi=x}. \tag{2.1}$$

But it follows from (1.63) that

$$z = \int_0^\infty \{f(\xi)\}^2\,d\xi = \int_0^\infty \{g(\xi)\}^2\,d\xi = \int_0^\infty \frac{\{\chi(\xi z)\}^2}{\xi^2}\,d\xi = \int_0^\infty \frac{\{\chi(yz)\}^2}{y^2}\,dy,$$

and hence, with the help of (2.1),

$$\int_0^\infty \frac{\chi(xy)\,\chi(yz)}{y^2}\,dy = \int_0^\infty \frac{\{\chi(yz)\}^2}{y^2}\,dy + \int_z^x f(\xi)\,d\xi = z + \int_z^x f(\xi)\,d\xi = \int_0^x f(\xi)\,d\xi.$$

This result may obviously be written in the form

$$\int_0^\infty \frac{\chi(xy)\,\chi(yz)}{y^2}\,dy = \mathrm{Min}(x,\,z),$$

and consequently, when $\chi(y)$ is a real function of y such that $\chi(y)/y$ is of class $L^2(0,\,\infty)$, it is *necessary* for the truth of Theorem Ia that $\chi(y)$ should satisfy (1.71).

It now remains to be shown that the condition (1.71) is *sufficient* for the truth of Theorem Ia; we shall assume that (1.71) is satisfied and shall proceed to establish by quite direct and straightforward methods the various results contained in Theorem Ia.

In order to carry out this investigation, it is desirable for two reasons to make a change in the notation employed. One reason is that

† Hobson, *Functions of a real variable*, 1 (2 ed., 1921), 532.

considerable use will be made of integrals which "converge in mean", the term convergence in mean having reference to both limits; and it is much easier to visualize what is happening when the limits are $-\infty$ and ∞ than when they are 0 and ∞. The other reason is that an effect of the change is to exhibit the parallellism between this paper and the two papers by Hardy on Hilbert transforms to which reference has already been made.

3. *The transformation of Theorem* Ia *and Theorem* II.

We shall now carry out the changes in notation which have just been foreshadowed. First write

$$x = e^{2X}, \quad y = e^{2Y}$$

in the enunciation of Theorem Ia, and then put

$$\chi(e^{2X}) \equiv e^X \Theta(X),$$

$$f(e^{2X}) \equiv e^{-X} F(X), \quad g(e^{2X}) \equiv e^{-X} G(X).$$

Observe that

$$\int_0^\infty \{f(x)\}^2 dx = 2\int_{-\infty}^\infty \{F(X)\}^2 dX, \qquad \int_0^\infty \{g(x)\}^2 dx = 2\int_{-\infty}^\infty \{G(X)\}^2 dX,$$

so that $F(X)$ and $G(X)$ are of class $L^2(-\infty, \infty)$ when, and only when, $f(x)$ and $g(x)$ are of class $L^2(0, \infty)$. In the new notation the transform which occurs in Theorem Ia is

$$\int_{-\infty}^X e^\xi G(\xi)\, d\xi = \left[e^\xi \int_{-\infty}^\infty \Theta(\xi + Y)\, F(Y)\, dY\right]_{\xi=-\infty}^{\xi=X},$$

while the condition (1.71) becomes

$$\int_{-\infty}^\infty \Theta(X+Y)\,\Theta(Y+Z)\, dY = \begin{cases} \frac{1}{2}e^{-X-Z+2X} & (X \leqslant Z) \\ \frac{1}{2}e^{-X-Z+2Z} & (X \geqslant Z) \end{cases}$$

$$= \tfrac{1}{2}e^{-|X-Z|}.$$

Replace the capital letters which denote variables and arbitrary functions by the corresponding lower-case letters, for the sake of appearance. It is then evident that Theorem Ia is equivalent to

THEOREM IV. *Corresponding to every function $f(x)$ which belongs to $L^2(-\infty, \infty)$, there exists a function $g(x)$, which also belongs to $L^2(-\infty, \infty)$,*

such that

$$\int_{-\infty}^{x} e^{\xi} g(\xi)\, d\xi = \left[e^{\xi} \int_{-\infty}^{\infty} \Theta(\xi+y)\, f(y)\, dy \right]_{\xi=-\infty}^{\xi=x}, \tag{3.11}$$

and the relation between $f(x)$ and $g(x)$ is a reciprocal one, so that

$$\int_{-\infty}^{x} e^{\xi} f(\xi)\, d\xi = \left[e^{\xi} \int_{-\infty}^{\infty} \Theta(\xi+y)\, g(y)\, dy \right]_{\xi=-\infty}^{\xi=x}. \tag{3.12}$$

Further,

$$\int_{-\infty}^{\infty} \{f(x)\}^2\, dx = \int_{-\infty}^{\infty} \{g(x)\}^2\, dx, \tag{3.13}$$

with the obvious extension to transforms of two different functions.

The theorem which I now propose to prove concerning Theorem IV is

THEOREM V. *If $\Theta(x)$ is a function which belongs to $L^2(-\infty, \infty)$, it is necessary for the truth of Theorem IV that $\Theta(x)$ should be such that the formula*

$$\int_{-\infty}^{\infty} \Theta(x+y)\, \Theta(y+z)\, dy = \tfrac{1}{2} e^{-|x-z|} \tag{3.14}$$

is valid for all real values of x and z.

Conversely, if (3.14) is valid for all real values of x and z, then $\Theta(x)$ belongs (obviously) to $L^2(-\infty, \infty)$, and Theorem IV is true.

By making the requisite changes in notation, we obtain the first part of Theorem V immediately from the corresponding work of § 2. In §§ 4–10 a proof of Theorem IV will be given based on the hypothesis that $\Theta(x)$ satisfies the condition (3.14), and this will complete the proof of Theorem V.

We shall adopt the terminology current in the theory of Bessel functions and describe (3.14) as a *discontinuous integral*, although the only actual discontinuity is in the form of the analytic functions which represent the value of the integral for positive and negative values of $x-z$.

It is to be remarked that, by using the inequality of Schwarz, we have

$$\left[\int_{-\infty}^{\infty} |\Theta(x+y)\, \Theta(y+z)|\, dy \right]^2 \leqslant \int_{-\infty}^{\infty} \Theta^2(x+y)\, dy \,.\, \int_{-\infty}^{\infty} \Theta^2(y+z)\, dy \tag{3.2}$$

$$= \tfrac{1}{4}.$$

4. *Lemmas involving the discontinuous integral.*

Before attacking the actual reciprocity formulae, it is convenient to obtain a number of results which involve the discontinuous integral only. Let

$$(4.1) \qquad I(a, \alpha) = \int_{-\infty}^{\infty} \{e^{a}\Theta(x+a) - e^{\alpha}\Theta(x+\alpha)\}^2\, dx.$$

Then, whatever be the relative magnitudes of a and α, it is easy to prove that

$$(4.11) \qquad I(a, \alpha) = \tfrac{1}{2}|e^{2a} - e^{2\alpha}|.$$

For

$$I(a, \alpha) = e^{2a}\int_{-\infty}^{\infty}\Theta^2(x+a)\,dx + e^{2\alpha}\int_{-\infty}^{\infty}\Theta^2(x+\alpha)\,dx - 2e^{a+\alpha}\int_{-\infty}^{\infty}\Theta(x+a)\,\Theta(x+\alpha)\,dx$$
$$= \tfrac{1}{2}(e^{2a}+e^{2\alpha}) - e^{a+\alpha}\,e^{-|a-\alpha|};$$

and it is evident that $e^{a+\alpha-|a-\alpha|}$ is equal to $\mathrm{Min}\,(e^{2a}, e^{2\alpha})$. Hence (4.11) follows immediately from (3.14).

Next let

$$(4.2) \qquad I(a, \alpha, b, \beta) = \int_{-\infty}^{\infty} \{e^{a}\Theta(x+a) - e^{\alpha}\Theta(x+\alpha)\}\{e^{b}\Theta(x+b) - e^{\beta}\Theta(x+\beta)\}\, dx.$$

It is easy to verify that

$$2I(a, \alpha, b, \beta) = I(a, \beta) + I(\alpha, b) - I(a, b) - I(\alpha, \beta),$$

and hence, by (4.11),

$$(4.21) \qquad 4I(a, \alpha, b, \beta) = |e^{2a} - e^{2\beta}| + |e^{2\alpha} - e^{2b}| - |e^{2a} - e^{2b}| - |e^{2\alpha} - e^{2\beta}|.$$

Next take h and k to be any *positive* numbers (zero being included), while y and z have unrestricted real values, and let

$$(4.3) \qquad J(y, z, h, k) = \int_{-\infty}^{\infty} \{e^{h}\Theta(x+y+h) - e^{-h}\Theta(x+y-h)\} \times \{e^{k}\Theta(x+z+k) - e^{-k}\Theta(x+z-k)\}\, dx,$$

so that

$$4J(y, z, h, k) = 4e^{-y-z}\, I(y+h, y-h, z+k, z-k)$$
$$= e^{-y-z}\,[\,|e^{2y+2h} - e^{2z-2k}| + |e^{2y-2h} - e^{2z+2k}| - |e^{2y+2h} - e^{2z+2k}| - |e^{2y-2h} - e^{2z-2k}|\,],$$

and, therefore,

$$(4.31)\quad J(y, z, h, k) = \tfrac{1}{2}e^{h-k}\sinh|h+k+y-z| + \tfrac{1}{2}e^{k-h}\sinh|h+k-y+z| - \tfrac{1}{2}e^{h+k}\sinh|h-k+y-z| - \tfrac{1}{2}e^{-h-k}\sinh|h-k-y+z|.$$

We postpone the detailed consideration of this result to § 5. Next, with the same limitation on h and k, write

$$(4.4)\quad J^*(y, z, h, k) = \int_{-\infty}^{\infty} |\{e^h\Theta(x+y+h) - e^{-h}\Theta(x+y-h)\} \times \{e^k\Theta(x+z+k) - e^{-k}\Theta(x+z-k)\}|\,dx.$$

Then, by the inequality of Schwarz,

$$[J^*(y, z, h, k)]^2 \leqslant \left[\int_{-\infty}^{\infty} \{e^h\Theta(x+y+h) - e^{-h}\Theta(x+y-h)\}^2\,dx\right] \times \left[\int_{-\infty}^{\infty} \{e^k\Theta(x+z+k) - e^{-k}\Theta(x+z-k)\}^2\,dx\right]$$

$$= J(y, y, h, h)\,J(z, z, k, k)$$

$$= \sinh 2h \sinh 2k.$$

Hence

$$(4.41)\qquad J^*(y, z, h, k) \leqslant \sqrt{(\sinh 2h \sinh 2k)},$$

and, obviously,

$$(4.42)\qquad |J(y, z, h, k)| \leqslant J^*(y, z, h, k) \leqslant \sqrt{(\sinh 2h \sinh 2k)}.$$

5. *Properties of* $J(y, z, h, k)$.

We now examine in detail the function $J(y, z, h, k)$ defined by (4.3). We first observe that, by (4.31), the function involves y and z only in the form of the difference $z-y$. Accordingly we write

$$(5.1)\qquad z-y = w,$$

$$(5.11)\qquad J(y, z, h, k) = \mu(w, h, k).$$

By using (4.31) we can construct various simpler analytical expressions for $\mu(w, h, k)$, the different expressions being dependent on the various relative values of $\pm(h+k)$, $\pm(h-k)$, and w. If we regard h and k as fixed while w varies, it is easy to see that we have two distinct cases to consider, according as $h-k$ is positive or negative, and that in each case there are five different ranges of w to be taken into account.

If we write

$$h+k=H, \quad |h-k|=K,$$

for brevity, the five ranges are

$$w \leqslant -H, \tag{5.21}$$

$$-H \leqslant w \leqslant -K, \tag{5.22}$$

$$-K \leqslant w \leqslant K, \tag{5.23}$$

$$K \leqslant w \leqslant H, \tag{5.24}$$

$$H \leqslant w, \tag{5.25}$$

while the corresponding values of $\mu(w, h, k)$ are

$$0, \tag{5.31}$$

$$\tfrac{1}{2}(e^{w+2k}-e^{-w-2h}), \tag{5.32}$$

$$\tfrac{1}{2}(e^{w+2k}-e^{w-2k}), \quad (h-k \geqslant 0) \tag{5.33}$$

$$\tfrac{1}{2}(e^{-w+2h}-e^{-w-2h}) \quad (h-k \leqslant 0), \tag{5.33'}$$

$$\tfrac{1}{2}(e^{-w+2h}-e^{w-2k}), \tag{5.34}$$

$$0. \tag{5.35}$$

In each of the ranges except the third, the formula which gives the value of $\mu(w, h, k)$ is independent of the sign of $h-k$.

6. *Lemmas concerning integrals involving $f(x)$.*

After the preliminaries of §§ 4 and 5 we are in a position to discuss a number of lemmas concerning integrals which contain a function $f(x)$ which is arbitrary save for the restriction that it must belong to $L^2(-\infty, \infty)$. We first state a general lemma; the substance of it is due to Lebesgue, but the form given here is effectively the modification of the statement of it by W. H. Young which Hardy† found appropriate for his work on Hilbert transforms. This lemma, of which I omit the proof, is as follows:

LEMMA I. *Suppose* (i) *that $f(x)$ and $\Phi(x, h, h', \ldots)$ are functions (of which the second depends on certain parameters $h, h', \ldots$) which are integrable over any finite range of values of x;* (ii) *that*

$$\Phi(x, h, h', \ldots) \to \Phi(x) \tag{6.11}$$

† Hardy, *loc. cit.*, 22–23.

when $h, h', \ldots$ tend to certain limiting values, for almost all values of x in $(-\infty, \infty)$; (iii) *that $f(x)$ belongs to $L^2(-\infty, \infty)$; and* (iv) *that*

$$\int_{-\infty}^{\infty} \{\Phi(x, h, h', \ldots)\}^2 dx \leqslant C < \infty, \tag{6.12}$$

where C is a constant. Then $f(x)\Phi(x)$ is integrable over $(-\infty, \infty)$, and

$$\int_{-\infty}^{\infty} f(x)\Phi(x, h, h', \ldots)dx \to \int_{-\infty}^{\infty} f(x)\Phi(x)dx. \tag{6.13}$$

We next prove

LEMMA II. *Let*

$$f(x, h, k) = \frac{1}{4\sinh h \sinh k}\int_{-\infty}^{\infty} f(x+w)\mu(w, h, k)dw; \tag{6.21}$$

then, as h and k tend independently to zero through positive values,

$$f(x, h, k) \to f(x) \tag{6.22}$$

for almost all values of x.

We first observe that, by (5.31) and (5.35), $\mu(w, h, k)$ vanishes throughout the range of integration, except on the part for which $-H \leqslant w \leqslant H$, so that

$$f(x, h, k) = \frac{1}{4\sinh h \sinh k}\int_{-H}^{H} f(x+w)\mu(w, h, k)dw. \tag{6.23}$$

Also, from (5.32), (5.33) or (5.33′), and (5.34), it follows that

$$\int_{-H}^{H} \mu(w, h, k)dw = 4\sinh h \sinh k,$$

and so we deduce from (6.23) that

$$f(x, h, k) - f(x) = \frac{1}{4\sinh h \sinh k}\int_{-H}^{H} \{f(x+w) - f(x)\}\mu(w, h, k)dw. \tag{6.24}$$

We next write

$$\psi_1(w) = f(x+w) + f(x-w) - 2f(x),$$

$$\psi_2(w) = f(x+w) - f(x-w),$$

and we now have to consider separately the two cases (i) in which $h-k$ is positive, and (ii) in which $h-k$ is negative.

In the first case,

$$\int_{-H}^{H}\{f(x+w)-f(x)\}\mu(w,\,h,\,k)\,dw$$

$$=\int_0^K[e^w\sinh 2k\{f(x+w)-f(x)\}+e^{-w}\sinh 2k\{f(x-w)-f(x)\}]\,dw$$

$$+\int_K^H[\tfrac{1}{2}(e^{-w+2h}-e^{w-2k})\{f(x+w)-f(x)\}+\tfrac{1}{2}(e^{-w+2k}-e^{w-2h})\{f(x-w)-f(x)\}]\,dw$$

$$=\sinh 2k\int_0^K\{\cosh w\,\psi_1(w)+\sinh w\,\psi_2(w)\}\,dw$$

$$+\int_K^H\{\sinh(H-w)\cosh K\,\psi_1(w)+\sinh(H-w)\sinh K\,\psi_2(w)\}\,dw.$$

Hence we have

$$\left|\int_{-H}^{H}\{f(x+w)-f(x)\}\mu(w,\,h,\,k)\,dw\right|$$

$$\leqslant\sinh 2k\cosh K\int_0^K|\psi_1(w)|\,dw+\sinh(H-K)\cosh K\int_K^H|\psi_1(w)|\,dw$$

$$+\sinh 2k\sinh K\int_0^K|\psi_2(w)|\,dw+\sinh(H-K)\sinh K\int_K^H|\psi_2(w)|\,dw$$

$$=\sinh(H-K)\left[\cosh K\int_0^H|\psi_1(w)|\,dw+\sinh K\int_0^H|\psi_2(w)|\,dw\right].$$

In like manner, in the second case,

$$\int_{-H}^{H}\{f(x+w)-f(x)\}\mu(w,\,h,\,k)\,dw$$

$$=\int_0^K[e^{-w}\sinh 2h\{f(x+w)-f(x)\}+e^w\sinh 2h\{f(x-w)-f(x)\}]\,dw$$

$$+\int_K^H[\tfrac{1}{2}(e^{-w+2h}-e^{w-2k})\{f(x+w)-f(x)\}+\tfrac{1}{2}(e^{-w+2k}-e^{w-2h})\{f(x-w)-f(x)\}]\,dw$$

$$=\sinh 2h\int_0^K\{\cosh w\,\psi_1(w)-\sinh w\,\psi_2(w)\}\,dw$$

$$+\int_K^H\{\sinh(H-w)\cosh K\,\psi_1(w)+\sinh(H-w)\sinh K\,\psi_2(w)\}\,dw.$$

Hence we have

$$\left|\int_{-H}^{H}\{f(x+w)-f(x)\}\mu(w,\,h,\,k)\,dw\right|$$

$$\leqslant \sinh 2h\cosh K\int_0^K|\psi_1(w)|\,dw+\sinh(H-K)\cosh K\int_K^H|\psi_1(w)|\,dw$$

$$+\sinh 2h\sinh K\int_0^K|\psi_2(w)|\,dw+\sinh(H-K)\sinh K\int_K^H|\psi_2(w)|\,dw$$

$$=\sinh(H-K)\Big[\cosh K\int_0^H|\psi_1(w)|\,dw+\sinh K\int_0^H|\psi_2(w)|\,dw\Big].$$

Consequently, in both cases,

$$(6.25)\quad |f(x,\,h,\,k)-f(x)|\leqslant\frac{\sinh(H-K)\cosh K}{4\sinh h\sinh k}\int_0^H|\psi_1(w)|\,dw$$

$$+\frac{\sinh(H-K)\sinh K}{4\sinh h\sinh k}\int_0^H|\psi_2(w)|\,dw.$$

Evidently

$$\frac{\sinh(H-K)\cosh K}{4\sinh h\sinh k}=\frac{\sinh(H-K)\cosh K}{4\sinh\tfrac{1}{2}(H+K)\sinh\tfrac{1}{2}(H-K)}$$

$$=\frac{\cosh\tfrac{1}{2}(H-K)\{\sinh(H+K)+\sinh(H-K)\}}{4\sinh\tfrac{1}{2}(H+K)\sinh H}$$

$$\leqslant\frac{\cosh\tfrac{1}{2}(H-K)\sinh(H+K)}{2\sinh\tfrac{1}{2}(H+K)\sinh H}$$

$$=\frac{\cosh\tfrac{1}{2}(H-K)\cosh\tfrac{1}{2}(H+K)}{\sinh H},$$

and

$$\frac{\sinh(H-K)\sinh K}{4\sinh h\sinh k}=\frac{\sinh(H-K)\sinh K}{4\sinh\tfrac{1}{2}(H+K)\sinh\tfrac{1}{2}(H-K)}$$

$$\leqslant\tfrac{1}{2}\cosh\tfrac{1}{2}(H-K).$$

Now make h and k tend to zero (independently) through positive values. Then H tends to zero, the three functions

$$\cosh\tfrac{1}{2}(H-K)\cosh\tfrac{1}{2}(H+K),\quad\frac{H}{\sinh H},\quad\tfrac{1}{2}\cosh\tfrac{1}{2}(H-K)$$

remain bounded, and, by particular cases of a theorem due to Lebesgue†, the two functions

$$\frac{1}{H}\int_0^H |\{f(x+w)+f(x-w)\}-2f(x)|\,dw, \quad \int_0^H |f(x+w)-f(x-w)|\,dw$$

tend to zero almost everywhere. That is to say, the expression on the right of (6.25) tends to zero almost everwhere, and therefore

$$f(x, h, k)\to f(x)$$

almost everywhere. This establishes Lemma II.

We next prove

LEMMA III. *If*

$$\int_{-\infty}^{\infty} \{f(x)\}^2\,dx = M, \tag{6.31}$$

then, for all *positive values of* h *and* k,

$$\int_{-\infty}^{\infty} \{f(x, h, k)\}^2\,dx \leqslant 4M\cosh^2\tfrac{1}{2}(h+k). \tag{6.32}$$

We first observe that, by applying the inequality of Schwarz to (6.23), we have

$$\{f(x, h, k)\}^2 \leqslant \frac{1}{16\sinh^2 h\sinh^2 k}\int_{-H}^{H}\{f(x+w)\}^2\,dw\,.\int_{-H}^{H}\{\mu(w, h, k)\}^2\,dw,$$

and we then calculate the second integral on the right with the help of (5.32), (5.33) or (5.33′), and (5.34).

When $h-k\geqslant 0$, we find that

$$\int_{-H}^{H}\{\mu(w, h, k)\}^2\,dw = \sinh 2k\cosh 2h - 2k\cosh(2h-2k),$$

and, when $h-k\leqslant 0$, we find that

$$\int_{-H}^{H}\{\mu(w, h, k)\}^2\,dw = \sinh 2h\cosh 2k - 2h\cosh(2h-2k).$$

Hence always

$$\int_{-H}^{H}\{\mu(w, h, k)\}^2\,dw = \sinh(H-K)\cosh(H+K)-(H-K)\cosh 2K.$$

† See Hobson, *Functions of a real variable*, 1 (2 ed., 1921), 582.

Now it is easy to see that

$$\frac{\sinh(H-K)\cosh(H+K)-(H-K)\cosh 2K}{16\sinh^2 h\sinh^2 k}$$

$$\leqslant\frac{(H-K)\cosh(H-K)\cosh(H+K)-(H-K)\cosh 2K}{16\sinh^2\tfrac{1}{2}(H+K)\sinh^2\tfrac{1}{2}(H-K)}$$

$$=\{\tfrac{1}{4}(H-K)\coth\tfrac{1}{2}(H-K)\}\{\coth\tfrac{1}{2}(H+K)\}$$

$$\leqslant\{\tfrac{1}{2}H\coth\tfrac{1}{2}H\}\{\coth\tfrac{1}{2}H\},$$

since $\xi\coth\xi$ is an increasing function of ξ and $\coth\xi$ is a decreasing function.

It follows at once that

$$\{f(x,h,k)\}^2\leqslant\tfrac{1}{2}H\coth^2\tfrac{1}{2}H\int_{-H}^{H}\{f(x+w)\}^2dw,$$

and so

$$\int_{-\infty}^{\infty}\{f(x,h,k)\}^2dx\leqslant\tfrac{1}{2}H\coth^2\tfrac{1}{2}H\int_{-\infty}^{\infty}\int_{-H}^{H}\{f(x+w)\}^2dw\,dx$$

$$=\tfrac{1}{2}H\coth^2\tfrac{1}{2}H\int_{-H}^{H}\int_{-\infty}^{\infty}\{f(x+w)\}^2dx\,dw$$

$$=MH^2\coth^2\tfrac{1}{2}H$$

$$\leqslant 4M\cosh^2\tfrac{1}{2}H.$$

This establishes Lemma III.

7. *The existence of transforms.*

Let $f(x)$ be any function which belongs to $L^2(-\infty,\infty)$; and let $f^*(x;a,b)$ be defined as the function which is equal to $f(x)$ when $a\leqslant x\leqslant b$ and which is zero when x does not lie in the interval (a,b), where a and b are any two real (finite) numbers such that $a\leqslant b$.

Now write

$$\phi(x)=\int_{-\infty}^{\infty}\Theta(x+y)f(y)\,dy, \tag{7.1}$$

$$\phi^*(x;a,b)=\int_a^b\Theta(x+y)f(y)\,dy \tag{7.11}$$

$$=\int_{-\infty}^{\infty}\Theta(x+y)f^*(x;a,b)\,dy.$$

It follows from the inequality of Schwarz that the integral defining $\phi(x)$ is absolutely convergent. The integral defining $\phi^*(x;\ a,\ b)$ is, of course, one with a finite range of integration, though it is often convenient to write it in the form of an infinite integral.

Next write

$$(7.2)\qquad \Delta(x,\ h)=\frac{1}{2\sinh h}\{e^h\phi^*(x+h;\ a,\ b)-e^{-h}\phi^*(x-h;\ a,\ b)\}$$

$$=\frac{1}{2\sinh h}\int_a^b\{e^h\Theta(x+y+h)-e^{-h}\Theta(x+y-h)\}f(y)\,dy,$$

and

$$(7.21)\qquad \Gamma(h,\ k)=\int_{-\infty}^{\infty}\Delta(x,\ h)\Delta(x,\ k)\,dx.$$

It follows from (4.41) that

$$(7.22)\qquad \int_{-\infty}^{\infty}\int_a^b\int_a^b|f(y)|.|f(z)|.|\{e^h\Theta(x+y+h)-e^{-h}\Theta(x+y-h)\}$$

$$\times\{e^k\Theta(x+z+k)-e^{-k}\Theta(x+z-k)\}|\,dy\,dz\,dx$$

$$\leqslant\sqrt{(\sinh 2h\sinh 2k)}\int_a^b|f(y)|\,dy\int_a^b|f(z)|\,dz,$$

and so, since $|f(y)|$ and $|f(z)|$ are integrable in any finite interval $(a,\ b)$, the integral on the left of (7.22) is convergent. Consequently, when we write the integral which defines $\Gamma(h,\ k)$ as a triple integral by using (7.2), we may change the order of the integrations and get†

$$(7.23)\qquad \Gamma(h,\ k)=\frac{1}{4\sinh h\sinh k}\int_a^b\int_a^b f(y)f(z)\,J(y,\ z,\ h,\ k)\,dz\,dy$$

$$=\frac{1}{4\sinh h\sinh k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f^*(y;\ a,\ b)f^*(y+w;\ a,\ b)$$

$$\times\mu(w,\ h,\ k)\,dw\,dy$$

$$=\int_{-\infty}^{\infty}f^*(y,\ a,\ b)f^*(y,\ h,\ k;\ a,\ b)\,dy,$$

where $f^*(y,\ h,\ k;\ a,\ b)$ is the function derived from $f^*(y,\ a,\ b)$ by the same formula (6.21) as $f(y,\ h,\ k)$ is derived from $f(y)$.

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† The use of a and b as limits instead of $-\infty$ and ∞ is necessary to ensure the absolute convergence of the triple integral; in the general problem the argument used by Hardy, *Messenger of Math.*, 53 (1924), 136–137, for Fourier transforms is inapplicable; his corresponding argument for Hilbert transforms, *ibid.*, 54 (1925), 26–27, is incorrect. Hardy has recently published a corrected proof of the theorem for Hilbert transforms, *Quart. J. of Math.*, 3 (1932), 102–112.

We shall now show that it is permissible to apply Lemma I to the expression on the right of (7.23). For $f^*(y, a, b)$ belongs to $L^2(-\infty, \infty)$; by Lemma II,

$$f^*(y, h, k;\ a, b) \to f^*(y, a, b)$$

for almost all values of y, when h and k tend to zero independently; and, by Lemma III,

$$\int_{-\infty}^{\infty} \{f^*(y, h, k;\ a, b)\}^2 dy \leqslant 4\cosh^2 \tfrac{1}{2}(h+k) \int_{-\infty}^{\infty} \{f^*(y;\ a, b)\}^2 dy$$

$$\leqslant 4\cosh^2 \tfrac{1}{2}(h+k) \int_{-\infty}^{\infty} \{f(y)\}^2 dy.$$

The conditions of Lemma I are therefore satisfied, and so, when a and b have any fixed values,

$$\Gamma(h, k) \to \int_{-\infty}^{\infty} \{f^*(y;\ a, b)\}^2 dy, \tag{7.24}$$

as h and k tend to zero independently.

We now write $\Delta(x, h)$ and $\Gamma(h, k)$ in full as $\Delta(x, h;\ a, b)$ and $\Gamma(h, k;\ a, b)$ respectively, in order to exhibit their dependence on a and b; and we proceed to show that it is possible to make

$$\int_{-\infty}^{\infty} \{\Delta(x, h;\ -p, q) - \Delta(x, k;\ -p', q')\}^2 dx \leqslant \epsilon, \tag{7.25}$$

where ϵ is an arbitrary positive number, by giving h, k, p, q, p', q' *any* values such that

$$h \leqslant h_0, \quad k \leqslant h_0,$$
$$p \geqslant A, \quad p' \geqslant A,$$
$$q \geqslant B, \quad q' \geqslant B,$$

where A, B, and h_0 are definite positive functions of ϵ.

First observe that, as a consequence of (7.23) combined with Lemma III and the inequality of Schwarz, we have

$$\int_{-\infty}^{\infty} \{\Delta(x, h;\ a, b)\}^2 dx = \Gamma(h, h;\ a, b)$$

$$= \left| \int_{-\infty}^{\infty} f^*(y;\ a, b) f^*(y, h, h;\ a, b)\, dy \right|$$

$$\leqslant \left[\int_{-\infty}^{\infty} \{f^*(y;\ a, b)\}^2 dy \cdot \int_{-\infty}^{\infty} \{f^*(y, h, h;\ a, b)\}^2 dy \right]^{\frac{1}{2}}$$

$$\leqslant 2\cosh h \int_{-\infty}^{\infty} \{f^*(y;\ a, b)\}^2 dy,$$

so that

$$(7.26) \qquad \int_{-\infty}^{\infty} \{\Delta(x, h;\ a, b)\}^2\, dx \leqslant 2\cosh h \int_a^b \{f(y)\}^2\, dy.$$

Next observe that, as a consequence of the definition of $\Delta(x, h;\ a, b)$, we have

$$\Delta(x, h;\ -p, q) = \Delta(x, h;\ -p, -A) + \Delta(x, h;\ -A, B) + \Delta(x, h;\ B, q),$$

so long as $$-p \leqslant -A \leqslant B \leqslant q.$$

Also, since $2|a_r a_s| \leqslant a_r^2 + a_s^2$, we obviously have

$$\left\{\sum_{r=1}^{5} a_r\right\}^2 \leqslant 5 \sum_{r=1}^{5} a_r^2.$$

Therefore, taking $p' > A$ and $q' > B$, we have

$$(7.3) \quad \{\Delta(x, h;\ -p, q) - \Delta(x, k;\ -p', q')\}^2$$
$$\leqslant 5\,[\{\Delta(x, h;\ -p, -A)\}^2 + \{\Delta(x, h;\ B, q)\}^2$$
$$+ \{\Delta(x, k;\ -p', -A)\}^2 + \{\Delta(x, k;\ B, q')\}^2$$
$$+ \{\Delta(x, h;\ -A, B) - \Delta(x, k;\ -A, B)\}^2].$$

Now choose A and B to be the smallest positive numbers such that

$$(7.31) \qquad \int_{-\infty}^{-A} \{f(y)\}^2\, dy \leqslant \frac{\epsilon}{100}, \qquad \int_B^{\infty} \{f(y)\}^2\, dy \leqslant \frac{\epsilon}{100}.$$

This choice fixes A and B. Now that they are fixed, it is evident from (7.24) that we can choose positive numbers, h_1 and k_1, such that we have

$$(7.32) \qquad \left| \Gamma(h, k;\ -A, B) - \int_{-\infty}^{\infty} \{f^*(y;\ -A, B)\}^2\, dy \right| \leqslant \frac{\epsilon}{100}$$

whenever $$0 < h \leqslant h_1, \quad 0 \leqslant k \leqslant k_1.$$

Now choose h_0 so that

$$(7.33) \qquad h_0 = \mathrm{Min}(h_1, k_1, \cosh^{-1} 2).$$

It is then evident from (7.3) that

$$\int_{-\infty}^{\infty}\{\Delta(x,h;\ -p,q)-\Delta(x,k;\ -p',q')\}^2dx$$

$$\leqslant 5\int_{-\infty}^{\infty}\{\Delta(x,h;\ -p,\ -A)\}^2dx+5\int_{-\infty}^{\infty}\{\Delta(x,h;\ B,q)\}^2dx$$

$$+5\int_{-\infty}^{\infty}\{\Delta(x,k;\ -p',\ -A)\}^2dx+5\int_{-\infty}^{\infty}\{A(x,k;\ B,q')\}^2dx$$

$$+5\int_{-\infty}^{\infty}\{\Delta(x,h;\ -A,\ B)-\Delta(x,k;\ -A,\ B)\}^2dx$$

$$\leqslant 10\cosh h\int_{-p}^{-A}\{f(y)\}^2dy+10\cosh h\int_{B}^{q}\{f(y)\}^2dy$$

$$+10\cosh k\int_{-p'}^{-A}\{f(y)\}^2dy+10\cosh k\int_{B}^{q'}\{f(y)\}^2dy$$

$$+5[\Gamma(h,h;\ -A,\ B)-2\Gamma(h,k;\ -A,\ B)+\Gamma(k,k;\ -A,\ B)]$$

$$\leqslant 20\int_{-\infty}^{-A}\{f(y)\}^2dy+20\int_{B}^{\infty}\{f(y)\}^2dy+20\int_{-\infty}^{-A}\{f(y)\}^2dy+20\int_{B}^{\infty}\{f(y)\}^2dy$$

$$+5\left[\Gamma(h,h;\ -A,\ B)-\int_{-\infty}^{\infty}\{f^*(y;\ -A,\ B)\}^2dy\right]$$

$$-10\left[\Gamma(h,k;\ -A,\ B)-\int_{-\infty}^{\infty}\{f^*(y;\ -A,\ B)\}^2dy\right]$$

$$+5\left[\Gamma(k,k;\ -A,\ B)-\int_{-\infty}^{\infty}\{f^*(y;\ -A,\ B)\}^2dy\right]$$

$$\leqslant \tfrac{20}{100}\epsilon+\tfrac{20}{100}\epsilon+\tfrac{20}{100}\epsilon+\tfrac{20}{100}\epsilon+\tfrac{5}{100}\epsilon+\tfrac{10}{100}\epsilon+\tfrac{5}{100}\epsilon$$

$$=\epsilon.$$

This proves (7.25). From (7.25), combined with Fischer's theorem†, it follows that there exists a function $g(x)$, which belongs to $L^2(-\infty,\infty)$, which is defined uniquely almost everywhere, and which is such that

$$\lim\int_{-\infty}^{\infty}\{g(x)-\Delta(x,h;\ -p,q)\}^2dx=0, \tag{7.34}$$

† *Comptes rendus*, 144 (1907), 1022–1024.

the passage to the limit being made by taking†

$$h \to 0, \quad p \to \infty, \quad q \to \infty$$

independently. *Consequently* $\Delta(x, h, -p, q)$ *converges in mean to a function* $g(x)$ *which belongs to* $L^2(-\infty, \infty)$. This function $g(x)$ will be called the general transform of $f(x)$ with respect to the kernel $\Theta(x+y)$.

8. *The equivalence of the definitions of* $g(x)$.

In § 7 we defined $g(x)$ as the function to which $\Delta(x, h; -p, q)$ converges in mean, and it was proved that such a function exists and is of class $L^2(-\infty, \infty)$. This definition, however, is superficially quite different from the definition supplied by (3.11); the next stage in the analysis consequently consists in reconciling these definitions, that is to say, in proving that the function $g(x)$, when defined by the method of § 7, satisfies (3.11).

As a preliminary, we prove that $e^x \phi^*(x; a, b)$ is a continuous function of x. By the inequality of Schwarz we have

$$\begin{aligned}
&|e^X \phi^*(X; a, b) - e^x \phi^*(x; a, b)|^2 \\
&\quad = \left| \int_{-\infty}^{\infty} \{e^X \Theta(X+y) - e^x \Theta(x+y)\} f^*(y; a, b)\, dy \right|^2 \\
&\quad \leqslant \int_{-\infty}^{\infty} \{e^X \Theta(X+y) - e^x \Theta(x+y)\}^2 dy \,.\, \int_{-\infty}^{\infty} \{f^*(y; a, b)\}^2 dy \\
&\quad = \tfrac{1}{2} |e^{2X} - e^{2x}| \int_a^b \{f(y)\}^2 dy \\
&\quad \leqslant \tfrac{1}{2} |e^{2X} - e^{2x}| \int_{-\infty}^{\infty} \{f(y)\}^2 dy,
\end{aligned}$$

and, for any given values of a, b, and x, this can be made arbitrarily small by taking $|X-x|$ to be sufficiently small. This establishes the continuity of $e^x \phi^*(x; a, b)$, *qua* function of x.

Next, from a known theorem‡, which is an immediate consequence of the inequality of Schwarz, it follows that, since $\Delta(x, h; -p, q)$ converges in mean to $g(x)$, the integral

$$\int_{-\infty}^{\infty} \{g(\xi) - \Delta(\xi, h; -p, q)\} E(\xi)\, d\xi$$

† The symbol "lim", without specification of the variables, will subsequently be used to mean this type of triple limit.

‡ *Cf.* Titchmarsh, *Proc. Camb. Phil. Soc.*, 21 (1923), 468.

tends to zero when $p \to \infty$, $q \to \infty$, and $h \to 0$ independently, provided that $E(\xi)$ is any function belonging to $L^2(-\infty, \infty)$. Now define $E(\xi)$ by the formulae

$$E(\xi) = e^{\xi} \quad (\xi \leqslant x),$$

$$E(\xi) = 0 \quad (\xi > x),$$

so that the requisite condition is satisfied.

We thus get

$$\int_{-\infty}^{x} e^{\xi} g(\xi)\, d\xi = \lim \int_{-\infty}^{x} e^{\xi} \Delta(\xi, h;\ -p, q)\, d\xi, \tag{8.1}$$

the integral on the left being convergent because both $g(\xi)$ and $E(\xi)$ are of class $L^2(-\infty, \infty)$.

It is evident, from an application of the inequality of Schwarz to the definition (7.11) of $\phi^*(\xi;\ a, b)$, that $\phi^*(\xi;\ a, b)$ is a function of ξ which is bounded† in $(-\infty, \infty)$. Hence

$$\int_{-\infty}^{x} e^{\xi} \phi^*(\xi;\ a, b)\, d\xi$$

is convergent. We can now deduce that

$$\begin{aligned}
&\lim_{h \to 0} \int_{-\infty}^{x} e^{\xi} \Delta(\xi, h;\ a, b)\, d\xi \\
&\quad = \lim_{h \to 0} \left[\frac{1}{2 \sinh h} \int_{-\infty}^{x} \{ e^{\xi+h} \phi^*(\xi+h;\ a, b) - e^{\xi-h} \phi^*(\xi-h;\ a, b) \}\, d\xi \right] \\
&\quad = \lim_{h \to 0} \left[\frac{1}{2 \sinh h} \int_{-\infty}^{x} e^{\xi+h} \phi^*(\xi+h;\ a, b)\, d\xi \right. \\
&\qquad\qquad \left. - \frac{1}{2 \sinh h} \int_{-\infty}^{x} e^{\xi-h} \phi^*(\xi-h;\ a, b)\, d\xi \right] \\
&\quad = \lim_{h \to 0} \left[\frac{1}{2 \sinh h} \int_{-\infty}^{x+h} e^{\xi} \phi^*(\xi;\ a, b)\, d\xi - \frac{1}{2 \sinh h} \int_{-\infty}^{x-h} e^{\xi} \phi^*(\xi;\ a, b)\, d\xi \right] \\
&\quad = \lim_{h \to 0} \left[\frac{1}{2 \sinh h} \int_{x-h}^{x+h} e^{\xi} \phi^*(\xi;\ a, b)\, d\xi \right] = e^{x} \phi^*(x;\ a, b),
\end{aligned}$$

the last step following from the continuity‡ of $e^x \phi^*(x;\ a, b)$.

† In fact, $$|\phi^*(x;\ a, b)| \leqslant \sqrt{\left[\tfrac{1}{2} \int_{-\infty}^{\infty} \{f(y)\}^2\, dy \right]}.$$

‡ *Cf.* Lebesgue, *Leçons sur l'intégration* (1904), 6.

It now follows from (8.1) that†

$$\begin{aligned}\int_{-\infty}^{x} e^{\xi} g(\xi)\,d\xi &= \lim \int_{-\infty}^{x} e^{\xi}\Delta(\xi, h;\ -p, q)\,d\xi \\ &= \lim_{\substack{p\to\infty \\ q\to\infty}} \left[\lim_{h\to 0}\int_{-\infty}^{x} e^{\xi}\Delta(\xi, h;\ -p, q)\,d\xi\right] \\ &= \lim_{\substack{p\to\infty \\ q\to\infty}} e^{x}\phi^{*}(x;\ -p, q) \\ &= \lim_{\substack{p\to\infty \\ q\to\infty}} e^{x}\int_{-p}^{q}\Theta(x+y)f(y)\,dy \\ &= e^{x}\int_{-\infty}^{\infty}\Theta(x+y)f(y)\,dy.\end{aligned}$$

Since the expression on the left of the last series of equalities is a convergent integral, the expression on the right must tend to zero when $x\to-\infty$. Hence the expression on the right may be written in the form

$$\left[e^{\xi}\int_{-\infty}^{\infty}\Theta(\xi+y)f(y)\,dy\right]_{\xi=-\infty}^{\xi=x},$$

and this shows that the value of $g(x)$, as defined by the method of §7, satisfies (3.11).

We have therefore established the existence of a function $g(x)$ of class $L^2(-\infty, \infty)$ which satisfies (3.11), namely the function to which $\Delta(x, h;\ -p, q)$ converges in mean when $p\to\infty$, $q\to\infty$, and $h\to 0$ independently.

9. *The reciprocity between $f(x)$ and $g(x)$.*

We next have to prove that, when $g(x)$ is defined by the method of §7 as the function to which $\Delta(x, h;\ -p, q)$ converges in mean, then

$$\left[e^{\xi}\int_{-\infty}^{\infty}\Theta(\xi+y)g(y)\,dy\right]_{\xi=-\infty}^{\xi=x} = \int_{-\infty}^{x} e^{\xi}f(\xi)\,d\xi.$$

We take the formula

$$\begin{aligned}(9.1)\quad &\int_{-\infty}^{\infty}\Theta(x+y)\Delta(y, h;\ -p, q)\,dy \\ &= \frac{1}{2\sinh h}\int_{-\infty}^{\infty}\int_{-p}^{q}\Theta(x+y)\{e^{h}\Theta(y+z+h)-e^{-h}\Theta(y+z-h)\}f(z)\,dz\,dy,\end{aligned}$$

† This device of regarding a triple limit as a repeated limit in order to evaluate it will also prove to be of use in §§9–10.

and we proceed to prove that it is permissible to change the order of the integrations on the right. From (3.2) it follows that

$$\int_{-p}^{q}\int_{-\infty}^{\infty}|\Theta(x+y)\{e^{h}\Theta(y+z+h)-e^{-h}\Theta(y+z+h)\}f(z)|\,dy\,dz$$

$$\leqslant e^{h}\int_{-p}^{q}|f(z)|\int_{-\infty}^{\infty}|\Theta(x+y)\,\Theta(y+z+h)|\,dy\,dz$$

$$+e^{-h}\int_{-p}^{q}|f(z)|\int_{-\infty}^{\infty}|\Theta(x+y)\,\Theta(y+z-h)|\,dy\,dz$$

$$\leqslant \tfrac{1}{2}e^{h}\int_{-p}^{q}|f(z)|\,dz+\tfrac{1}{2}e^{-h}\int_{-p}^{q}|f(z)|\,dz.$$

Hence the integral on the right of (9.1) is absolutely convergent, and so the change is permissible. We thus get

$$\int_{-\infty}^{\infty}\Theta(x+y)\,\Delta(y,\,h;\,-p,\,q)\,dy$$

$$=\frac{1}{2\sinh h}\int_{-p}^{q}\int_{-\infty}^{\infty}\Theta(x+y)\{e^{h}\Theta(y+z+h)-e^{-h}\Theta(y+z-h)\}f(z)\,dy\,dz$$

$$=\frac{1}{4\sinh h}\int_{-p}^{q}\{e^{h-|z+h-x|}-e^{-h-|z-h-x|}\}f(z)\,dz$$

$$=\frac{1}{4\sinh h}\int_{-\infty}^{\infty}\{e^{h-|z+h-x|}-e^{-h-|z-h-x|}\}f^{*}(z;\,-p,\,q)\,dz.$$

Now $e^{h-|z+h-x|}-e^{-h-|z-h-x|}$ may be replaced by the following expressions:

(i) $\quad 2e^{z-x}\sinh 2h \quad$ when $\quad z\leqslant x-h,$

(ii) $\quad 2e^{-h}\sinh(x+h-z) \quad$ when $\quad x-h\leqslant z\leqslant x+h,$

(iii) $\quad 0 \quad$ when $\quad z\geqslant x+h.$

Consequently

$$(9.2)\quad \int_{-\infty}^{\infty}\Theta(x+y)\,\Delta(y,\,h;\,-p,\,q)\,dy$$

$$=e^{-x}\cosh h\int_{-\infty}^{x-h}e^{z}f^{*}(z;\,-p,\,q)\,dz$$

$$+\frac{e^{-h}}{2\sinh h}\int_{x-h}^{x+h}\sinh(x+h-z)\,f^{*}(z;\,-p,\,q)\,dz.$$

Now, by the inequality of Schwarz,

$$\left[\frac{e^{-h}}{2\sinh h}\int_{x-h}^{x+h}\sinh(x+h-z)f^*(z;\ -p,\ q)\,dz\right]^2$$

$$\leqslant\frac{e^{-2h}}{4\sinh^2 h}\int_{x-h}^{x+h}\sinh^2(x+h-z)\,dz\int_{x-h}^{x+h}\{f^*(z;\ -p,\ q)\}^2\,dz$$

$$\leqslant\frac{e^{-2h}(\sinh 4h-4h)}{16\sinh^2 h}\int_{-\infty}^{\infty}\{f(z)\}^2\,dz$$

$$\to 0,$$

when $p\to\infty$, $q\to\infty$, and $h\to 0$ independently. Also, from the theorem cited in § 8, it is evident that the expression on the left of (9.2) tends to the limit

$$\int_{-\infty}^{\infty}\Theta(x+y)\,g(y)\,dy$$

in the same circumstances. Hence the first term on the right of (9.2) tends to this limit in these circumstances. That is to say

$$\int_{-\infty}^{\infty}\Theta(x+y)\,g(y)\,dy=\lim\left\{e^{-x}\cosh h\int_{-\infty}^{x-h}e^z f^*(z;\ -p,\ q)\,dz\right\}$$

$$=\lim_{\substack{p\to\infty\\ q\to\infty}}\left[\lim_{h\to 0}\left\{e^{-x}\cosh h\int_{-\infty}^{x-h}e^z f^*(z;\ -p,\ q)\,dz\right\}\right]$$

$$=\lim_{\substack{p\to\infty\\ q\to\infty}}\left[e^{-x}\int_{-\infty}^{x}e^z f^*(z;\ -p,\ q)\,dz\right]$$

$$=e^{-x}\int_{-\infty}^{x}e^z f(z)\,dz.$$

We have therefore proved that

$$\int_{-\infty}^{x}e^{\xi}f(\xi)\,d\xi=e^x\int_{-\infty}^{\infty}\Theta(x+y)\,g(y)\,dy,$$

and, by the argument used in § 8, this result may be written in the form

$$\int_{-\infty}^{x}e^{\xi}f(\xi)\,d\xi=\left[e^{\xi}\int_{-\infty}^{\infty}\Theta(\xi+y)\,g(y)\,dy\right]_{\xi=-\infty}^{\xi=x}.$$

This establishes the reciprocity which was asserted in the enunciation of Theorem IV.

10. *The proof of Parseval's theorem.*

To complete the proof of Theorem IV all that now remains to be done is to prove that

$$\int_{-\infty}^{\infty} \{f(x)\}^2 dx = \int_{-\infty}^{\infty} \{g(x)\}^2 dx. \tag{10.1}$$

Since $\Delta(x, h;\ -p, q)$ converges in mean to $g(x)$, it follows from a known theorem† that, for *any* mode of passage to the limit,

$$\int_{-\infty}^{\infty} \{\Delta(x, h;\ -p, q)\}^2 dx \to \int_{-\infty}^{\infty} \{g(x)\}^2 dx; \tag{10.11}$$

and so, in order to establish (10.1), it is sufficient to prove that, for some mode of passage to the limit,

$$\int_{-\infty}^{\infty} \{\Delta(x, h;\ -p, q)\}^2 dx \to \int_{-\infty}^{\infty} \{f(x)\}^2 dx. \tag{10.12}$$

Now, by (7.23), we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \{\Delta(x, h;\ -p, q)\}^2 dx &= \Gamma(h, h;\ -p, q) \\
&= \int_{-p}^{q} f(y)\, f^*(y, h, h;\ -p, q)\, dy \\
&= \frac{1}{4\sinh^2 h} \int_{-\infty}^{\infty} f^*(y;\ -p, q) \int_{-\infty}^{\infty} f^*(y+w;\ -p, q)\, \mu(w, h, h)\, dw\, dy \\
&= \frac{1}{4\sinh^2 h} \int_{-\infty}^{\infty} f^*(y;\ -p, q) \int_{-2h}^{2h} f^*(y+w;\ -p, q)\, \mu(w, h, h)\, dw\, dy.
\end{aligned}$$

We proceed to show that Lemma I is applicable to the last expression. Whatever be the values of p and q, it follows from Lemma II that

$$\lim_{h\to 0} \left[\frac{1}{4\sinh^2 h} \int_{-2h}^{2h} f^*(y+w;\ -p, q)\, \mu(w, h, h)\, dw\right] = f^*(y;\ -p, q)$$

almost everywhere; and it follows from Lemma III that

$$\begin{aligned}
\int_{-\infty}^{\infty} \left[\int_{-2h}^{2h} f^*(y+w;\ -p, q)\, \mu(w, h, h)\, dw\right]^2 dy \\
&\leqslant 4\cosh^2 h \int_{-\infty}^{\infty} \{f^*(y;\ -p, q)\}^2 dy \\
&\leqslant 4\cosh^2 h \int_{-\infty}^{\infty} \{f(y)\}^2 dy,
\end{aligned}$$

† *Cf.* Titchmarsh, *loc. cit.*, 468.

so that the requisite conditions are satisfied. Hence, by Lemma I,

$$\lim_{h\to 0}\int_{-\infty}^{\infty}\{\Delta(x,\,h;\,-p,\,q)\}^2\,dx=\int_{-\infty}^{\infty}\{f^*(y;\,-p,\,q)\}^2\,dy$$

$$=\int_{-p}^{q}\{f(x)\}^2\,dx.$$

The last integral obviously tends to the limit

$$\int_{-\infty}^{\infty}\{f(x)\}^2\,dx$$

when p and q tend to infinity independently; that is to say

$$\lim_{\substack{p\to\infty\\ q\to\infty}}\left[\lim_{h\to 0}\int_{-\infty}^{\infty}\{\Delta(x,\,h;\,-p,\,q)\}^2\,dx\right]=\int_{-\infty}^{\infty}\{f(x)\}^2\,dx.$$

Hence a mode of passage to the limit has been found for which (10.12) is true. Consequently, since (10.11) holds for any mode of passage to the limit, we can infer the truth of (10.1).

Theorem IV has now been proved in its entirety on the hypothesis that $\Theta(x)$ satisfies (3.14), and so Theorem V has now been established.

11. *A general theorem on Fourier transforms*†.

In order to obtain the connection between the discontinuous integral (3.14) and the functional equation (1.3), with the object of obtaining the connection between the discontinuous integral (1.71) and the same functional equation, and so proving Theorem III, it is convenient to prove a preliminary general theorem concerning the Fourier transform of any function belonging to $L^2(-\infty,\,\infty)$. I shall state this theorem in its general form, without reference either to the discontinuous integral or to

† It seems unlikely that a theorem which is so simple and so easy to prove as the theorem of this section has not been discovered before; and it is quite possible that it has occurred naturally in some problem of Mathematical Physics. However, neither Prof. Hardy nor I can remember having encountered it in the course of our reading. [*Added* 16 *March*, 1933.—The theorem is given both by Bochner, *Das Fouriersche Integrale* (1932), 49, and by Wiener, *The Fourier integral* (1933), 70; both of these books have been published since this paper was accepted for publication. The corresponding theorem for Mellin's inversion formula was given by Hardy, *Messenger of Math.*, 49 (1920), 85–91.]

the functional equation; and I shall first state and prove the theorem in a form involving complex numbers and functions, and then give the equivalent real form of the theorem. The latter form may be of importance in some physical problems.

Although I have called the theorem a general theorem, it is in reality merely a variant of Parseval's theorem for Fourier transforms of two different functions. The theorem is as follows:

THEOREM VI. *Let $f(y)$ be a (real or complex) function which belongs to $L^2(-\infty, \infty)$. Let $G(y)$ be the function to which*

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)\, e^{ity}\, dt$$

converges in mean. Then, for all fixed real values of x and z,

$$(11.1) \qquad \int_{-\infty}^{\infty} f(x+y) f(y+z)\, dy = 2 \int_0^{\infty} G(y)\, G(-y) \cos(x-z)\, y\, dy,$$

both of these integrals being absolutely convergent.

A number of the integrals used in connection with this theorem are mean-square integrals; such integrals will be indicated by appending the letters (M.S.) to them, so that we have, for instance,

$$(\text{M.S.}) \qquad G(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t)\, e^{ity}\, dt,$$

and this means that

$$\lim_{\substack{p\to\infty \\ q\to\infty}} \int_{-\infty}^{\infty} \left| \left[G(y) - \frac{1}{\sqrt{(2\pi)}} \int_{-p}^{q} f(t)\, e^{ity}\, dt \right] \right|^2 dy = 0,$$

where p and q tend to infinity *independently*†. When the lower limit of a mean-square integral is zero, the convergence in mean is to be taken to have reference to the upper limit only unless the contrary is stated.

When x is a constant, we may change the parametric variable in the integral defining $G(y)$ by writing† $t+x$ in place of t. We thus get

$$(\text{M.S.}) \qquad G(y) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} f(t+x)\, e^{i(t+x)y}\, dt,$$

† *Cf.* Hobson, *Functions of a real variable*, 2 (1926), 743–745.

and therefore

$$(11.21;\ \text{M.S.}) \qquad e^{-ixy}\,G(y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t+x)\,e^{ity}\,dt.$$

In like manner, if z is a constant,

$$G(-y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t+z)\,e^{-i(t+z)y}\,dt,$$

and therefore

$$(11.22;\ \text{M.S.}) \qquad e^{izy}\,G(-y) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f(t+z)\,e^{-ity}\,dt.$$

It is evident from (11.21) and (11.22) that $f(x+y)$ and $e^{izy}\,G(-y)$ are the Fourier transforms of $e^{-ixy}\,G(y)$ and $f(y+z)$ respectively; and so, by Parseval's theorem for Fourier transforms of two different functions, we have

$$\int_{-\infty}^{\infty} f(x+y)f(y+z)\,dy = \int_{-\infty}^{\infty} e^{-ixy}\,G(y)\,.\,e^{izy}\,G(-y)\,dy,$$

and, since $G(y)\,G(-y)$ is an even function of y, (11.1) follows immediately from this result, the absolute convergence of the integrals involved being included in Parseval's theorem. This proves Theorem VI.

[Since I do not remember seeing it explicitly in print, it may be worth while pointing out that the formal transformation which gives Parseval's theorem when complex functions are used is obtained in the following manner.

Let the functions $g_1(x)$ and $g_2(x)$ be the transforms of $f_1(x)$ and $f_2(x)$ so that†

$$g_1(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f_1(y)\,e^{-ixy}\,dy, \qquad f_1(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} g_1(y)\,e^{ixy}\,dy,$$

$$g_2(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} f_2(y)\,e^{-ixy}\,dy, \qquad f_2(x) = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty} g_2(y)\,e^{ixy}\,dy.$$

Then

$$\int_{-\infty}^{\infty} f_1(x)\,g_2(x)\,dx = \frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x)f_2(y)\,e^{-ixy}\,dy$$

$$= \int_{-\infty}^{\infty} g_1(y)f_2(y)\,dy = \int_{-\infty}^{\infty} f_2(x)\,g_1(x)\,dx.$$

† *Cf.* Courant and Hilbert, *Methoden der Mathematischen Physik*, 1 (1924), 64.

There is no difficulty in adapting the analysis given by Hobson† for the cosine transform for the purpose of proving rigorously the theorem for the transforms now under consideration.]

To obtain the real form of Theorem VI, write

$$f_e(y) = \tfrac{1}{2}\{f(y)+f(-y)\}, \quad f_o(y) = \tfrac{1}{2}\{f(y)-f(-y)\}, \tag{11.3}$$

so that $f_e(y)$ and $f_o(y)$ may be called the even and the odd parts of $f(y)$ respectively. Let $g_e(y)$ be the cosine transform of $f_e(y)$ and let $g_o(y)$ be the sine transform of $f_o(y)$, so that $g_e(y)$ and $g_o(y)$ are respectively even and odd functions of y. Let

$$g(y) = g_e(y)+g_o(y),$$

so that $g_e(y)$ and $g_o(y)$ are the even and odd parts of $g(y)$ respectively. Then

$$G(y) = g_e(y)+i\,g_o(y), \quad G(-y) = g_e(y)-i\,g_o(y),$$

and consequently it follows from (11.1) that

$$\int_{-\infty}^{\infty} f(x+y)f(y+z)\,dy = 2\int_0^{\infty} [\{g_e(y)\}^2+\{g_o(y)\}^2]\cos(x-z)\,y\,dy, \tag{11.32}$$

and, since the relation between $f(y)$ and $g(y)$ is a reciprocal one,

$$\int_{-\infty}^{\infty} g(x+y)g(y+z)\,dy = 2\int_0^{\infty} [\{f_e(y)\}^2+\{f_o(y)\}^2]\cos(x-z)\,y\,dy, \tag{11.33}$$

all the integrals in (11.32) and (11.33) being absolutely convergent when $f(y)$ is of class $L^2(-\infty, \infty)$.

As an example of Theorem VI, let

$$f(y) = (y^2-1)^{-\nu-\frac{1}{2}} \quad (y>1),$$

$$f(y) = 0 \quad (y\leqslant 1),$$

where $-\frac{1}{4} < \Re(\nu) < 0$.

The corresponding value of $G(y)$ is given by the formulae‡

$$G(y) = -2^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-\nu)\,(\tfrac{1}{2}y)^{\nu}\{Y_\nu(y)-iJ_\nu(y)\} \quad (y>0),$$

$$G(-y) = -2^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}-\nu)\,(\tfrac{1}{2}y)^{\nu}\{Y_\nu(y)+iJ_\nu(y)\} \quad (y>0).$$

† *Functions of a real variable*, 2 (1926), 747.

‡ See my *Theory of Bessel functions* (1922), 170.

Hence, from (11.1), we have

$$(11.4)\quad \int_0^\infty (\tfrac{1}{2}y)^{2\nu}\{J_\nu^2(y)+Y_\nu^2(y)\}\cos(x-z)\,y\,dy = \frac{4}{\{\Gamma(\frac{1}{2}-\nu)\}^2}\int_{1-\mathrm{Min}(x,z)}^\infty \frac{dy}{[\{(x+y)^2-1\}\{(y+z)^2-1\}]^{\nu+\frac{1}{2}}},$$

which I believe to be a new result. It is valid for a more extensive range of values of ν when x and z are restricted to be unequal.

12. *The connection between the discontinuous integrals and the functional equation.*

The results of § 11 enable us to establish the requisite connection between the discontinuous integrals (3.14) and (1.71) and the functional equation (1.72), and so to prove Theorem III.

We first make a change in the notation of § 11; we replace x, y, z, t by X, Y, Z, $2t$, and we write $h(t)$ in place of $\sqrt{(8\pi)}\,G(2t)$. We may then state Theorem VI in the following modified form:

THEOREM VII. *Let $f(Y)$ be a real function† which is of class $L^2(-\infty, \infty)$, so that, by the theory of Fourier transforms,*

$$\int_{-\infty}^\infty e^{2iYt} f(Y)\,dY$$

exists as a mean-square integral for almost all values of t. Define $h(t)$ by the equation

$$(12.11;\ \text{M.S.})\qquad h(t) = 2\int_{-\infty}^\infty e^{2iYt} f(Y)\,dY,$$

so that $|h(t)|$ is of class $L^2(-\infty, \infty)$. Then, for all real values of X and Z,

$$(12.12)\quad \int_{-\infty}^\infty f(X+Y)f(Y+Z)\,dY = \frac{1}{2\pi}\int_{-\infty}^\infty h(t)h(-t)\cos 2(X-Z)t\,dt,$$

both of the integrals in (12.12) being absolutely convergent.

Alternatively, let $h(t)$ be a complex function of t such that $h(t)$ and $h(-t)$ are conjugate complex numbers and such that $|h(t)|$ belongs to $L^2(-\infty, \infty)$,

† The restriction that $f(Y)$ is to be a real function is merely to ensure that $h(t)$ and $h(-t)$ are conjugate complex numbers.

so that, by the theory of Fourier transforms,

$$\int_{-\infty}^{\infty} e^{-2iYt} h(t)\, dt$$

exists as a mean-square integral for almost all values of Y. Define the real function $f(Y)$ by the equation

$$f(Y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2iYt} h(t)\, dt, \tag{12.13; M.S.}$$

so that $f(Y)$ is of class $L^2(-\infty, \infty)$. Then (12.12) holds for all real values of X and Z, both of the integrals involved being, as before, absolutely convergent.

The equations (12.11) and (12.13) are, of course, inverse in the modified form of the Fourier transform which is now under consideration.

Now let $\Theta(Y)$ be a real function which satisfies the equation

$$\int_{-\infty}^{\infty} \Theta(X+Y)\, \Theta(Y+Z)\, dY = \tfrac{1}{2} e^{-|X-Z|},$$

so that $\Theta(Y)$ is of class $L^2(-\infty, \infty)$. Define $\Omega(\frac{1}{2}+it)$ by the equation

$$\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it} = 2 \int_{-\infty}^{\infty} e^{2iYt} \Theta(Y)\, dY, \tag{12.21; M.S.}$$

so that, by the first part of Theorem VII,

$$\frac{1}{2\pi} \int_0^{\infty} \frac{\Omega(\frac{1}{2}+it)\, \Omega(\frac{1}{2}-it)}{t^2+\frac{1}{4}} \cos 2(X-Z)t\, dt = \tfrac{1}{2} e^{-|X-Z|} = \frac{1}{2\pi} \int_0^{\infty} \frac{\cos 2(X-Z)t}{t^2+\frac{1}{4}}\, dt. \tag{12.22}$$

Now
$$\frac{\Omega(\frac{1}{2}+it)\, \Omega(\frac{1}{2}-it)}{t^2+\frac{1}{4}} = \left| \frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it} \right|^2,$$

and consequently the expression on the left is of class $L^1(0, \infty)$. Hence both of the integrals

$$\int_0^{\infty} \int_0^{\infty} \exp(-\alpha^2 w^2)\, \frac{\Omega(\frac{1}{2}+it)\, \Omega(\frac{1}{2}-it)}{t^2+\frac{1}{4}}\, dt\, dw, \qquad \int_0^{\infty} \int_0^{\infty} \exp(-\alpha^2 w^2)\, \frac{1}{t^2+\frac{1}{4}}\, dt\, dw$$

are convergent when α has any positive value, and so we are at liberty to

change the order of the integrations on both sides of the equation

$$\int_0^\infty \int_0^\infty \exp(-a^2 w^2) \frac{\Omega(\frac{1}{2}+it)\,\Omega(\frac{1}{2}-it)}{t^2+\frac{1}{4}} \cos 2wt\, dt\, dw = \int_0^\infty \int_0^\infty \exp(-a^2 w^2) \frac{\cos 2wt}{t^2+\frac{1}{4}}\, dt\, dw,$$

which is derived from (12.22) by writing w in place of $X-Z$ and then integrating with respect to w. We thus get

$$\frac{\sqrt{\pi}}{2} \int_0^\infty \exp\left(-\frac{t^2}{a^2}\right) \frac{\Omega(\frac{1}{2}+it)\,\Omega(\frac{1}{2}-it)}{t^2+\frac{1}{4}}\, dt = \frac{\sqrt{\pi}}{2} \int_0^\infty \exp\left(-\frac{t^2}{a^2}\right) \frac{dt}{t^2+\frac{1}{4}}$$

that is to say,

$$\int_0^\infty \exp\left(-\frac{t^2}{a^2}\right) \frac{\Omega(\frac{1}{2}+it)\,\Omega(\frac{1}{2}-it)-1}{t^2+\frac{1}{4}}\, dt = 0. \tag{12.3}$$

By Lerch's theorem, it follows from (12.3) that†

$$\Omega(\tfrac{1}{2}+it)\,\Omega(\tfrac{1}{2}-it)-1$$

is a null-function. Consequently the functional equation (1.72) is a necessary condition for the truth of Theorem Ia when $\Omega(\frac{1}{2}+it)$ is defined by (12.21) or by the equivalent formula‡

$$\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it} = \int_0^\infty y^{it-\frac{3}{2}} \chi(y)\, dy. \tag{12.41; M.S.}$$

Next suppose that $\Omega(\frac{1}{2}+it)$ is a complex function of t such that $\Omega(\frac{1}{2}+it)$ and $\Omega(\frac{1}{2}-it)$ are conjugate complex numbers and such that the functional equation (1.72) is satisfied. It is then evident that

$$|\Omega(\tfrac{1}{2}+it)| = 1,$$

---

† Prof. Hardy has pointed out to me that this inference can be drawn from the integrability of $\Omega(\frac{1}{2}+it)\,\Omega(\frac{1}{2}-it)/(t^2+\frac{1}{4})$ in $(0, \infty)$ by using a theorem due to Pollard, *Proc. London Math. Soc.* (2), 25 (1926), 451–468, applied directly to (12.22), without appealing to Lerch's theorem. I have retained my proof in order to exhibit a connection between Pollard's theorem and Lerch's theorem.

‡ This is a mean-square integral with reference to *both* limits, so that (12.41) means

$$\lim_{q\to\infty} \int_{-\infty}^{\infty} \left| \left[\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it} - \int_{\exp(-q)}^{\exp(q)} y^{it-\frac{3}{2}} \chi(y)\, dy \right] \right|^2 dt = 0.$$

and so $$\left|\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it}\right|$$

is of class $L^2(-\infty, \infty)$. Now define $\Theta(Y)$ by the equation

$$\Theta(Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2iYt}\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it}\,dt, \tag{12.51; M.S.}$$

and then, by the second part of Theorem VII,

$$\begin{aligned}\int_{-\infty}^{\infty}\Theta(X+Y)\Theta(Y+Z)\,dY &= \frac{1}{2\pi}\int_0^{\infty}\frac{\Omega(\frac{1}{2}+it)\,\Omega(\frac{1}{2}-it)}{t^2+\frac{1}{4}}\cos 2(X-Z)t\,dt\\ &= \frac{1}{2\pi}\int_0^{\infty}\frac{\cos 2(X-Z)t}{t^2+\frac{1}{4}}\,dt\\ &= \tfrac{1}{2}e^{-|X-Z|}.\end{aligned} \tag{12.52}$$

Hence $\Theta(Y)$ satisfies the condition (3.14), and it has been seen that this is a sufficient condition for the truth of Theorem IV. Changing the notation, we see that, if $\Omega(\frac{1}{2}+it)$ satisfies the conditions just stated and if

$$\frac{\chi(y)}{y} = \frac{1}{2\pi}\int_{-\infty}^{\infty} y^{-\frac{1}{2}-it}\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it}\,dt, \tag{12.53; M.S.}$$

then Theorem IA (and consequently also Theorem I) is true. This completes the proof of Theorem III.

We have therefore now obtained a necessary and sufficient condition (namely the functional equation) for the validity of a generalized Fourier transform, and this condition is in a form which enables us to construct generalized transforms in unlimited numbers.

13. *Generalized Hankel transforms.*

We have seen in § 12 that the inverse formulae

$$\Theta(Y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-2iYt}\frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it}\,dt, \qquad \frac{\Omega(\frac{1}{2}+it)}{\frac{1}{2}-it} = 2\int_{-\infty}^{\infty} e^{2iYt}\Theta(Y)\,dY,$$

combined with the functional equation (1.72), give rise to the result that,

if $F(X)$ is of class $L^2(-\infty, \infty)$, and if

$$\int_{-\infty}^{X} e^{\xi} G(\xi)\, d\xi = \left[e^{\xi} \int_{-\infty}^{\infty} \Theta(\xi+Y)\, F(Y)\, dY \right]_{\xi=-\infty}^{\xi=X},$$

then the relation between $F(X)$ and $G(X)$ is a reciprocal one.

Now let λ be a positive constant and make the changes in variables and notations indicated by the following equations:

$$x^{\lambda} = e^{X}, \quad y^{\lambda} = e^{Y}, \quad z^{\lambda} = e^{Z},$$

$$\Theta(X) = x^{-\frac{1}{2}} \chi_{\lambda}(x),$$

$$\tau = 2\lambda t, \quad \Omega(\tfrac{1}{2}+it) = \Omega_{\lambda}(\tfrac{1}{2}+i\tau),$$

$$F(X) = x^{\frac{1}{2}} f(x), \quad G(X) = x^{\frac{1}{2}} g(x).$$

In this new notation the problem before us is the determination of functions $\chi_{\lambda}(x)$ for which the following theorem, analogous to Theorem IV, is true:

THEOREM VIII. *Corresponding to every function $f(x)$ which belongs to $L^2(0, \infty)$, there exists a function $g(x)$, which also belongs to $L^2(0, \infty)$, such that*

$$\int_0^x \xi^{\lambda-\frac{1}{2}} g(\xi)\, d\xi = \left[\xi^{\lambda-\frac{1}{2}} \int_0^{\infty} \frac{\chi_{\lambda}(\xi y)}{y} f(y)\, dy \right]_{\xi=0}^{\xi=x}, \tag{13.11}$$

and the relation between $f(x)$ and $g(x)$ is a reciprocal one, so that

$$\int_0^x \xi^{\lambda-\frac{1}{2}} f(\xi)\, d\xi = \left[\xi^{\lambda-\frac{1}{2}} \int_0^{\infty} \frac{\chi_{\lambda}(\xi y)}{y} g(y)\, dy \right]_{\xi=0}^{\xi=x}. \tag{13.12}$$

Further,

$$\int_0^{\infty} \{f(x)\}^2\, dx = \int_0^{\infty} \{g(x)\}^2\, dx, \tag{13.13}$$

with the obvious extension to transforms of two different functions.

The complete solution of this problem in terms of discontinuous integrals is contained in the following translation of Theorem V into the new notation:

THEOREM IX. *If $\chi_{\lambda}(y)/y$ is a function which belongs to $L^2(0, \infty)$, it is necessary for the truth of Theorem VIII that $\chi_{\lambda}(y)$ should be such that the*

formula

$$\int_0^\infty \frac{\chi_\lambda(xy)\,\chi_\lambda(yz)}{y^2}\,dy = \begin{cases} \dfrac{x^{\frac{1}{2}+\lambda}\,z^{\frac{1}{2}-\lambda}}{2\lambda} & (0 \leqslant x \leqslant z) \\ \dfrac{x^{\frac{1}{2}-\lambda}\,z^{\frac{1}{2}+\lambda}}{2\lambda} & (0 \leqslant z \leqslant x) \end{cases} \tag{13.14}$$

is valid for all values of x and z in $(0, \infty)$.

Conversely, if (13.14) *is valid for all values of x and z in $(0, \infty)$, then $\chi_\lambda(y)/y$ belongs (obviously) to $L^2(0, \infty)$, and Theorem VIII is true. Also the equations obtained by differentiating* (13.11) *and* (13.12) *hold almost everywhere in $(0, \infty)$.*

The result of making the transition from the discontinuous integrals to the functional equation is the following analogue of Theorem III:

THEOREM X. *Let $\chi_\lambda(y)$ be a function of y such that $\chi_\lambda(y)/y$ belongs to $L^2(0, \infty)$, so that*

$$2\lambda \int_0^\infty \chi_\lambda(y)\, y^{i\tau-\frac{3}{2}}\,dy$$

exists as a mean-square integral; and let the value to which this integral converges in mean be denoted by

$$\frac{\Omega_\lambda(\frac{1}{2}+i\tau)}{\lambda-i\tau},$$

so that $|\Omega_\lambda(\frac{1}{2}+i\tau)/(\lambda-i\tau)|$ is a function of class $L^2(-\infty, \infty)$, and $\Omega_\lambda(\frac{1}{2}+i\tau)$ is a complex function of the real variable τ such that $\Omega_\lambda(\frac{1}{2}+i\tau)$ and $\Omega_\lambda(\frac{1}{2}-i\tau)$ are conjugate complex numbers. Then it is necessary for the truth of Theorem VIII that the functional equation

$$\Omega_\lambda(\tfrac{1}{2}+i\tau)\,\Omega_\lambda(\tfrac{1}{2}-i\tau) = 1 \tag{13.2}$$

should be satisfied.

Conversely, if $\Omega_\lambda(\frac{1}{2}+i\tau)$ is a complex function of the real variable τ, such that $\Omega_\lambda(\frac{1}{2}+i\tau)$ and $\Omega_\lambda(\frac{1}{2}-i\tau)$ are conjugate complex numbers, and such that the functional equation (13.2) *is satisfied, then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} y^{-\frac{1}{2}-i\tau}\,\frac{\Omega_\lambda(\frac{1}{2}+i\tau)}{\lambda-i\tau}\,d\tau$$

exists as a mean-square integral; and if the value of this mean-square integral is denoted by $\chi_\lambda(y)/y$, then $\chi_\lambda(y)/y$ is a function of class $L^2(0, \infty)$ and

$\chi_\lambda(\xi y)/y$ is a kernel for which Theorem VIII is true, the equations obtained by differentiating (13.11) *and* (13.12) *holding almost everywhere in* $(0, \infty)$.

14. *Examples of the transforms.*

In conclusion I give examples of the transforms of three different general types, with some special cases of the first type. Each type is a particular case of a generalized Hankel transform; the results for generalized Fourier transforms are, of course, obtainable by taking $\lambda = \frac{1}{2}$. It will be observed that $\Omega_\lambda(s)$ is an analytic function of s in all the examples except the last.

Example 1. Let $f_1(s)$ and $f_2(s)$ be two real polynomials. Let the zeros of $f_1(s)$ be $-\alpha_1, -\alpha_2, -\alpha_3, \ldots$, and let the zeros of $f_2(s)$ be $\beta_1, \beta_2, \beta_3, \ldots$. Also let

$$\Re(\alpha_n) > \tfrac{1}{2}, \quad \Re(\beta_n) > \tfrac{1}{2} \quad (n = 1, 2, 3, \ldots),$$

and let c be a positive constant. Let

$$\Omega_\lambda(s) = \frac{f_1(1-s)\, f_2(1-s)}{f_1(s)\, f_2(s)}\, c^{s-\frac{1}{2}},$$

so that the functional equation is satisfied.

We may then take

$$\chi_\lambda(y) = \frac{1}{2\pi i} \lim_{q\to\infty} \int_{\frac{1}{2}-qi}^{\frac{1}{2}+qi} \frac{f_1(1-s)\, f_2(1-s)}{f_1(s)\, f_2(s)} \frac{c^{\frac{1}{2}}}{\lambda+\frac{1}{2}-s} \left(\frac{y}{c}\right)^{1-s} ds,$$

this integral being convergent.

By Cauchy's theorem, $\chi_\lambda(y)$ is equal to the sum of the residues of the integrand at its poles on the left of the contour when $y < c$, and $\chi_\lambda(y)$ is equal to minus the sum of the residues of the integrand at its poles on the right of the contour when $y > c$. Hence†

$$(14.11) \quad \chi_\lambda(y) = \sum_\alpha \frac{c^{\frac{1}{2}}}{\lambda+\frac{1}{2}+\alpha} \frac{f_1(1+\alpha)\, f_2(1+\alpha)}{f_1'(-\alpha)\, f_2(-\alpha)} \left(\frac{y}{c}\right)^{\alpha+1} \qquad (y < c),$$

$$(14.12) \quad \chi_\lambda(y) = \sum_\beta \frac{c^{\frac{1}{2}}}{\beta-\lambda-\frac{1}{2}} \frac{f_1(1-\beta)\, f_2(1-\beta)}{f_1(\beta)\, f_2'(\beta)} \left(\frac{c}{y}\right)^{\beta-1}$$
$$+ \frac{f_1(\frac{1}{2}-\lambda)\, f_2(\frac{1}{2}-\lambda)}{f_1(\frac{1}{2}+\lambda)\, f_2(\frac{1}{2}+\lambda)} \frac{c^\lambda}{y^{\lambda-\frac{1}{2}}} \qquad (y > c).$$

† It is supposed for simplicity that $f_1(s)$ and $f_2(s)$ have no repeated zeros.

Example 2. As a special case of Example 1, take

$$f_1(s) = s+\alpha, \quad f_2(s) = 1.$$

Then

$$\chi_\lambda(y) = \frac{2\alpha+1}{\alpha+\frac{1}{2}+\lambda}\,\frac{y^{\alpha+1}}{c^{\alpha+\frac{1}{2}}} \quad (y<c), \tag{14.21}$$

$$\chi_\lambda(y) = \frac{\alpha+\frac{1}{2}-\lambda}{\alpha+\frac{1}{2}+\lambda}\,\frac{c^\lambda}{y^{\lambda-\frac{1}{2}}} \quad (y>c). \tag{14.22}$$

Example 3. As another special case of Example 1, take

$$f_1(s) = 1, \quad f_2(s) = s-\beta.$$

Then

$$\chi_\lambda(y) = 0 \quad (y<c), \tag{14.31}$$

$$\chi_\lambda(y) = \frac{\beta+\lambda-\frac{1}{2}}{\beta-\lambda-\frac{1}{2}}\,\frac{c^\lambda}{y^{\lambda-\frac{1}{2}}} + \frac{1-2\beta}{\beta-\lambda-\frac{1}{2}}\,\frac{c^{\beta-\frac{1}{2}}}{y^{\beta-1}} \quad (y>c). \tag{14.32}$$

Example 4. As a limiting case of Example 3, take

$$\beta = \lambda+\tfrac{1}{2}.$$

Then

$$\chi_\lambda(y) = 0 \quad (y<c), \tag{14.33}$$

$$\chi_\lambda(y) = \frac{c^\lambda}{y^{\lambda-\frac{1}{2}}}\left[2\lambda\log\left(\frac{y}{c}\right)-1\right] \quad (y>c). \tag{14.34}$$

Example 5. As a final special case of Example 1, take

$$f_1(s) = s+\alpha, \quad f_2(s) = s-\beta.$$

Then

$$\chi_\lambda(y) = \frac{(2\alpha+1)\,(\beta-\alpha-1)}{(\alpha+\frac{1}{2}+\lambda)(\alpha+\beta)}\,\frac{y^{\alpha+1}}{c^{\alpha+\frac{1}{2}}} \quad (y<c), \tag{14.41}$$

$$\chi_\lambda(y) = \frac{(\alpha-\beta+1)(1-2\beta)}{(\alpha+\beta)\,(\beta-\lambda-\frac{1}{2})}\,\frac{c^{\beta-\frac{1}{2}}}{y^{\beta-1}} + \frac{(\alpha-\lambda+\frac{1}{2})(\beta+\lambda-\frac{1}{2})}{(\alpha+\lambda+\frac{1}{2})(\beta-\lambda-\frac{1}{2})}\,\frac{c^\lambda}{y^{\lambda-\frac{1}{2}}} \quad (y>c). \tag{14.42}$$

Example 6. Let $\qquad \nu_r > -1 \quad (r = 1, 2, 3, \ldots, m),$

let c be a positive constant, and let

$$\Omega_\lambda(s) = c^{s-\frac{1}{2}} \prod_{r=1}^{m} \frac{\Gamma(\frac{1}{2}\nu_r+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu_r-\frac{1}{2}s+\frac{3}{4})},$$

so that the functional equation is satisfied.

We may then take

$$\chi_\lambda(y) = \frac{1}{2\pi i} \lim_{q\to\infty} \int_{\frac{1}{2}-qi}^{\frac{1}{2}+qi} \prod_{r=1}^{m} \frac{\Gamma(\frac{1}{2}\nu_r+\frac{1}{2}s+\frac{1}{4})}{\Gamma(\frac{1}{2}\nu_r-\frac{1}{2}s+\frac{3}{4})} \frac{c^{\frac{1}{2}}}{\lambda+\frac{1}{2}-s} \left(\frac{y}{c}\right)^{1-s} ds,$$

this integral being convergent.

By calculating the sum of the residues of the integrand at its poles on the left of the contour, we find that $\chi_\lambda(y)$ is expressible in terms of generalized hypergeometric functions, thus

$$\chi_\lambda(y) = \sum_{r=1}^{m} \left[\frac{y^{\nu_r+\frac{3}{2}}}{c^{\nu_r+1}(\frac{1}{2}\lambda+\frac{1}{2}\nu_r+\frac{1}{2})} \frac{\prod_{n=1}^{m}{}' (\frac{1}{2}\nu_n-\frac{1}{2}\nu_r)}{\prod_{n=1}^{m} \Gamma(\frac{1}{2}\nu_n+\frac{1}{2}\nu_r+1)} \right.$$

$$\times {}_1F_{2m}\left(\frac{\lambda+\nu_r+1}{2}; \frac{\lambda+\nu_r+3}{2}, \frac{\nu_r-\nu_1}{2}+1, \ldots *\ldots, \frac{\nu_r-\nu_m}{2}+1,\right.$$

$$\left.\left. \frac{\nu_r+\nu_1}{2}+1, \ldots, \frac{\nu_r+\nu_m}{2}+1; (-)^m \frac{y^2}{c^2}\right)\right],$$

the prime indicating that the factor $\Gamma(\frac{1}{2}\nu_r-\frac{1}{2}\nu_r)$ is to be omitted from the product of gamma-functions and the asterisk indicating that $\frac{1}{2}(\nu_r-\nu_r)+1$ is to be omitted from the elements of the hypergeometric function.

Example 7. Lastly take†

$$\Omega_\lambda(\tfrac{1}{2}+i\tau) = \exp\{i\tau^\alpha\} \qquad (\tau \geqslant 0),$$

$$\Omega_\lambda(\tfrac{1}{2}+i\tau) = \exp\{-i(-\tau)^\alpha\} \quad (\tau \leqslant 0),$$

where $\alpha > 1$.

We then have

$$\chi_\lambda(y) = \Re \frac{1}{\pi} \int_0^\infty \exp\{i\tau^\alpha\} \frac{y^{\frac{1}{2}-i\tau}}{\lambda-i\tau} d\tau,$$

the integral being convergent.

We may replace the path of integration by a straight line inclined $\frac{1}{2}\pi/\alpha$ to the real axis passing from the origin to infinity; it is then permissible to replace

$$\frac{y^{\frac{1}{2}-i\tau}}{\lambda-i\tau} \quad \text{by} \quad y^{\frac{1}{2}-\lambda} \int_0^y \eta^{\lambda-i\tau-1} d\eta,$$

† Compare example 6 (p. 122) in the preceding paper by Hardy and Titchmarsh.

interchange the order of the integrations, expand the integrand in powers of $\log\eta$, and integrate term-by-term. We thus get

$$\chi_\lambda(y) = \Re\,\frac{1}{\pi}\int_0^{\infty\exp(\frac{1}{2}\pi i/\alpha)}\int_0^y y^{\frac{1}{2}-\lambda}\eta^{\lambda-i\tau-1}\exp\{i\tau^\alpha\}\,d\eta\,d\tau$$

$$= \Re\,\frac{1}{\pi}\int_0^y \sum_{n=1}^{\infty}\int_0^{\infty\exp(\frac{1}{2}\pi i/\alpha)} y^{\frac{1}{2}-\lambda}\eta^{\lambda-1}\,\frac{(-i\tau\log\eta)^{n-1}}{(n-1)!}\exp(i\tau^\alpha)\,d\tau\,d\eta$$

$$= \Re\,\frac{1}{\pi}\int_0^y y^{\frac{1}{2}-\lambda}\eta^{\lambda-1}\left[\sum_{n=1}^{\infty}\frac{(-i\log\eta)^{n-1}}{(n-1)!}\,\frac{1}{\alpha}\exp\left(\frac{n\pi i}{2\alpha}\right)\Gamma(n/\alpha)\right]d\eta$$

$$= \frac{1}{\pi\alpha}\int_0^y y^{\frac{1}{2}-\lambda}\eta^{\lambda-1}\left[\sum_{n=1}^{\infty}\sin\{\tfrac{1}{2}n\pi(1-\alpha^{-1})\}\,\frac{\Gamma(n/\alpha)}{(n-1)!}\,(\log\eta)^{n-1}\right]d\eta.$$

By taking $\lambda = \frac{1}{2}$, we have

$$\chi(y) = \frac{1}{\pi\alpha}\int_0^y \eta^{-\frac{1}{2}}\left[\sum_{n=1}^{\infty}\sin\{\tfrac{1}{2}n\pi(1-\alpha^{-1})\}\,\frac{\Gamma(n/\alpha)}{(n-1)!}\,(\log\eta)^{n-1}\right]d\eta$$

as a kernel for a generalized Fourier transform.

ON THE CLASSIFICATION OF ALGEBRAS*

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PART I.

Introduction.

In Mathematical Physics many quantities of a non-commutative nature are used. For the most part these do not conform to any algebra that has been specially studied from an algebraic point of view. One reason for this is that the non-commutative quantities in question have very special properties that are far from fundamental algebraically. For example, certain quantities are used only as left-hand multipliers, and others only as right-hand multipliers†. Nevertheless, in some cases a comparatively simple non-commutative algebra is involved, though all such algebras seem to involve an infinite basis. So far, with the exception of a paper by Wedderburn‡, very little is known of algebras with infinite bases.

In this paper an attempt is made towards the classification of non-commutative algebras, so as to include algebras with an infinite basis. A few of the simpler algebras are studied in detail, including some of the algebras used by Mathematical Physicists. Though the treatment is purely algebraic and no special attempt is made to justify various assumptions that have been made in Quantum Theory, it is hoped that some doubtful points may be made clear incidentally.

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* This paper is a revised form of a paper which was originally communicated to the Society on 11 February, 1930, and read on 13 March, 1930, under the title "Algebras in which the equation $px-xp=1$ has a solution".

† P. A. M. Dirac, *Quantum mechanics* (1930).

‡ J. H. M. Wedderburn, "Algebras without a finite basis", *Trans. American Math. Soc.*, 26 (1924), 395–426.

The only algebras which are examined here are those in which the coefficients are taken over the fields either of the real or of the complex numbers. When these two cases are different, an indication is given.

For the benefit of those readers who are unfamiliar with abstract algebra, the following definitions of the words *nilpotent* and *idempotent* may be helpful*.

An element x of an algebra is said to be *nilpotent* if, for some index n,

$$x^n = 0.$$

Further, x is said to be *properly nilpotent* if ax and xa are nilpotent for every element a of the algebra. Dickson proves that an algebra A contains properly nilpotent elements if and only if it possesses a maximal nilpotent invariant sub-algebra N, and then the properly nilpotent elements of A coincide with the non-zero elements of N.

An element x of the algebra is said to be *idempotent* if

$$x^2 = x,$$

and consequently

$$x^n = x,$$

for every positive index n.

An idempotent element e of an algebra A is called a *principal idempotent* (for A) if there does not exist in A an idempotent element u such that $eu = ue = 0$. If the algebra A contains an element m such that $mx = x$ for every element x of the algebra, m is said to be the *modulus* of the algebra, and is the one and only principal idempotent of A. The algebras considered here have a *modulus*, but we use the word *principal idempotent* instead of *modulus* to specify this element, since we already use the latter word in an entirely different significance.

Algebras which do not contain a modulus (using the word in its significance for abstract algebra) would naturally arise in the study of sub-algebras of the algebras considered in this paper. If such a study were attempted, a new word would have to be used to replace the word *modulus* in one of its senses.

Of the various methods of definition for a non-commutative algebra, the following is most often used for finite algebras. A set of linearly independent elements $e_1, e_2, \ldots, e_n$ is defined and n^3 constants λ_{ijk}. The multiplication table $e_i e_j = \Sigma \lambda_{ijk} e_k$ defines the algebra. For the algebra to be associative, certain definite relations must hold between the constants λ_{ijk}.

* See Dickson, *Algebras and their arithmetics* (1923), 43 *et seq.*, for a complete account.

This method is hardly suitable for algebras with an infinite basis, for we should have, not only a triple infinity of constants λ_{ijk}, but also an infinity of equations to be satisfied by these constants, in order that the algebra may be associative. We therefore take an alternative definition*. A finite number of elements is defined. Assuming the associative but not the commutative law, the products obtainable from these are then defined as linearly independent. This is the most general associative algebra obtainable from these elements. Particular algebras are then obtained by taking sets of residues *modulo* certain definite expressions. Thus, the algebra of quaternions is given by sets of residues among the products obtainable from i and j, taken *modulo*

$$i^2+1, \quad j^2+1, \quad ij+ji.$$

In the case of the resulting algebra having an infinite basis, the algebra may be further extended in two ways. Certain types of infinite series in the elements defined may be defined as new elements. Or, if there are no factors of zero, the reciprocal, both right and left hand, of each element, may be defined as a new element.

Definitions.

The algebra defined by the associative products of r elements $e_1, e_2, \ldots, e_r$ will be called the *general polynomial algebra of Class r*. The algebra defined by sets of residues from this algebra taken *modulo* p expressions $\mu_1, \mu_2, \ldots, \mu_p$ will be called *a polynomial algebra of Class r*.

If further elements are defined by infinite series in these elements, the new extended algebra will be called a *corresponding serial algebra of Class r*. If, alternatively, reciprocals of the elements are defined as new elements, the algebra so formed will be called a *corresponding rational algebra of Class r*.

The *degree* of the algebra is defined as the least degree of the expressions $\mu_1, \mu_2, \ldots, \mu_p$ in the elements $e_1, e_2, \ldots, e_r$. $e_1, e_2, \ldots, e_r$ will be described as the *primitive elements*, and $\mu_1, \mu_2, \ldots, \mu_p$ as the *moduli* of the algebra.

Now, clearly, for any given algebra the choice of primitive elements is not unique. In defining quaternions, for example, any two quaternions which are not commutative may be taken as primitive elements. The

* *Cf.* H. Grassmann, *Die Ausdehnungslehre* (1844 and 1862); A. N. Whitehead, *Universal algebra*, Part I (1898), 171; J. B. Shaw, *Synopsis of linear associative algebra* (1907), 19.

Class of the algebra also is dependent on the method of definition. For quaternions the primitive elements may be i, j, and k, and the moduli i^2+1, j^2+1, k^2+1, $ij-k$. Where the definition of an algebra by the methods described is not indicated, the Class will be taken as the least number of primitive elements necessary to define the algebra. And the degree will be taken as the least degree possible by suitable choice of these primitive elements.

With some moduli, or some combinations of moduli, every product of the primitive elements will be congruent to zero. Clearly, if this is the case, no algebra will be defined. It will be said that such moduli, or such combinations of moduli, are *not permissible*. If, on the contrary, the sets of residues with these moduli or combinations of moduli do lead to algebras, these will be said to be *permissible*. One of the problems which have to be solved is that of finding what moduli or combinations of moduli are permissible.

THEOREM I. *Algebras of Class I are commutative.*

THEOREM II. *One modulus is sufficient to define any algebra of Class I.*

Let x be the primitive element, and let $f(x)$ and $g(x)$ be two polynomials in x which are both moduli.

First suppose f and g to be mutually prime. Then $F(x)$ and $G(x)$ can be found such that $F(x)f(x)+G(x)g(x)=1$. If f and g are congruent to zero, so also is $F(x)f(x)+G(x)g(x)$. But, with unity congruent to zero, every number of the algebra is also congruent to zero. Thus no algebra is defined and it is not permissible for f and g to be mutually prime,

Secondly, suppose f and g to have a greatest common factor, $\lambda(x)$. Then $F(x)$ and $G(x)$ can be found such that $F(x)f(x)+G(x)g(x)=\lambda(x)$. The two moduli $f(x)$ and $g(x)$ can be replaced by the single modulus $\lambda(x)$.

THEOREM III. *Any algebra of Class I, taken over the complex numbers, which is not the general algebra, is the direct sum of a number of algebras of complex numbers, together with properly nilpotent elements.*

Suppose the modulus $f(x)$ to be of degree n, and suppose first that it has no repeated factors.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $f(\xi)=0$.

Put $$\phi_r = \frac{f(x)}{(x-\alpha_r)f'(\alpha_r)}.$$

Then $$\phi_r^2 = \phi_r,$$

$$\phi_r \phi_s = 0 \quad (r \neq s).$$

Any element of the algebra is expressible in the form $\Sigma a_r \phi_r$. Thus the algebra is the direct sum of a number of algebras of complex numbers.

Secondly, suppose f to have repeated factors.

Let $$f = (x-\alpha_1)^{r_1} (x-\alpha_2)^{r_2} \dots (x-\alpha_p)^{r_p}.$$

The following elements divided into p sets form a basis:

$$f/(x-\alpha_1), \quad f/(x-\alpha_1)^2, \quad \dots, \quad f/(x-\alpha_1)^{r_1},$$

$$f/(x-\alpha_2), \quad f/(x-\alpha_2)^2, \quad \dots, \quad f/(x-\alpha_2)^{r_2},$$

$$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots$$

$$f/(x-\alpha_p), \quad f/(x-\alpha_p)^2, \quad \dots, \quad f/(x-\alpha_p)^{r_p}.$$

The product of any two of these elements in different rows vanishes. The algebra reduces to the direct sum of p sub-algebras. We shall show that each of these consists of an algebra of complex numbers together with properly nilpotent elements.

It is sufficient to consider the sub-algebra with basis

$$f/(x-\alpha_1), \quad f/(x-\alpha_1)^2, \quad \dots, \quad f/(x-\alpha_1)^{r_1}.$$

Since the complete algebra has a principal idempotent element, this sub-algebra has also a principal idempotent element y.

A basis of the sub-algebra is given by

$$y, \quad y(x-\alpha_1), \quad y(x-\alpha_1)^2, \quad \dots, \quad y(x-\alpha_1)^{r_1-1}.$$

These satisfy $$y^2 = y, \quad [y(x-\alpha_1)]^{r_1} = 0.$$

THEOREM IV. *An algebra of Class I, taken over the real numbers, is the direct sum of algebras of real numbers, and of complex numbers together with properly nilpotent elements.*

Let $f(x)$ be the modulus, and let $x-\alpha_1$ be a real linear factor repeated r times. Then the elements $f(x)/(x-\alpha_1), \dots, f(x)/(x-\alpha_1)^r$ form a sub-algebra as in Theorem III.

Suppose that $f(x)$ has an irreducible quadratic factor x^2+px+q, repeated r times. We have a sub-algebra with basis

$$f/(x^2+px+q), \quad xf/(x^2+px+q), \quad \dots, \quad f/(x^2+px+q)^r, \quad xf/(x^2+px+q)^r.$$

We shall suppose for simplicity that a linear transformation on the primitive element x has reduced the factor to the form x^2+1.

We have a sub-algebra with basis

$$f/(x^2+1), \quad xf/(x^2+1), \quad \ldots, \quad f/(x^2+1)^r, \quad xf/(x^2+1)^r.$$

There will be an idempotent element y of the sub-algebra,

$$y^2 = y, \quad y(x^2+1)^r = 0.$$

A basis of the sub-algebra is given by

$$y, \quad yx, \quad y(x^2+1), \quad yx(x^2+1), \quad \ldots, \quad yx(x^2+1)^{r-1}.$$

Put

$$z = yx\left[1+\frac{1}{2}(1+x^2)+\frac{1\cdot 3}{2\cdot 4}(1+x^2)^2 \ldots + \frac{1\cdot 3 \ldots (2r-3)}{2\cdot 4 \ldots (2r-2)}(1+x^2)^{r-1}\right].$$

Then $z^2 = -y$.

A basis of the sub-algebra is given by

$$y, \quad y(x^2+1), \quad y(x^2+1)^2, \quad \ldots, \quad y(x^2+1)^{r-1},$$

$$z, \quad z(x^2+1), \quad z(x^2+1)^2, \quad \ldots, \quad z(x^2+1)^{r-1}.$$

The algebra is the algebra of complex numbers with the addition of the nilpotent element $w = y(x^2+1)$ satisfying $yw = w$, $w^r = 0$.

The complete algebra is the direct sum of such algebras.

These two theorems are in complete parallel with the ordinary theory of partial fractions in real and complex variables, and are closely allied with the method of Frobenius for reducing matrices to canonical form.

Algebras of Class II are much more general in character than those of Class I. The following theorems show that all finite algebras over the real or the complex numbers containing no properly nilpotent elements are algebras of Class II. It is not true that all finite algebras with nilpotent elements are of Class II, as the following example shows.

Consider the algebra with basis $e_1, e_2, \ldots, e_n$, satisfying

$$e_1^2 = e_1, \quad e_1 e_p = e_p,$$

$$e_p^2 = 0, \quad e_p e_q = 0 \quad (p \neq 1, \ q \neq 1).$$

It would take $n-1$ primitive elements to define this algebra, and the class is $n-1$.

THEOREM V. *The algebras of quaternions, of real matrices, complex matrices, and quaternion matrices are algebras of Class II.*

For quaternions take i and j as primitive elements.

For complex matrices of order n^2 let

$$\omega = \cos\frac{\pi}{2n} + i\sin\frac{\pi}{2n},$$

and let

$$x = \begin{bmatrix} \omega & & & & \\ & \omega^5 & & & \\ & & \omega^9 & & \\ & & & \ddots & \\ & & & & \omega^{4n-3} \end{bmatrix}, \qquad y = \begin{bmatrix} 01 & & & \\ 001 & & & \\ 0001 & & & \\ \dots & \dots & \dots & \\ 00 & \dots & 01 & \\ 100 & \dots & 0 & \end{bmatrix}.$$

Then, with x and y as the primitive elements and with appropriate moduli, the algebra is defined.

For real matrices take y as for complex matrices, and let

$$x = \text{diag}\,(1, 2, \dots, n).$$

For quaternion matrices, x and y are chosen as for complex matrices, but here i has its quaternion significance, and to y is added the factor

$$\cos\frac{\pi}{2n} + j\sin\frac{\pi}{2n}.$$

THEOREM VI. *The direct sum of any number of algebras of quaternions, real matrices, complex matrices, or quaternion matrices is an algebra of Class II.*

Let the algebra A be the direct sum of the algebras $A_1, A_2, \dots, A_n$. Take

$$x = x_1 + x_2 + \dots + x_n,$$

$$y = y_1 + y_2 + \dots + y_n,$$

where x_p and y_p are numbers in the algebra A_p. If A_p is a real matrix algebra, let x_p be a diagonal matrix of the form

$$x_p = \text{diag}\,(n_p, n_p+1, \dots, n_p+r),$$

where n_p is the least positive integer which is not the coefficient of a diagonal element of a real matrix algebra A_q of suffix $q < p$.

If A_p is a quaternion, complex matrix, or quaternion matrix algebra, then x_p is defined as follows.

For a quaternion algebra x_p is made equal to im_p.

For a complex matrix or quaternion matrix algebra

$$x_p = \text{diag}\,[im_p,\ i(m_p+1),\ \ldots,\ i(m_p+r)],$$

where m_p is the least positive integer such that im_p is not the coefficient of any diagonal element in x_q for $q < p$.

Now, if n is the largest coefficient of a diagonal element in the elements x_p for real matrix algebras and im is the largest coefficient of a diagonal element for quaternion, complex matrix, or quaternion matrix algebras, then x satisfies

$$x(x-1)(x-2)\ldots(x-n)(x^2+1)(x^2+2^2)\ldots(x^2+m^2) = 0.$$

If any factor is omitted, this product becomes, not zero, but a diagonal element in one of the sub-algebras. Thus any quantity composed of diagonal matrices in the various sub-algebras, for quaternions involving i but not j, may be expressed as a real polynomial in x.

Each y_p is defined as in Theorem III. Since the principal idempotent in each sub-algebra may be expressed as a polynomial in x, each y_p may be expressed as a polynomial in x and y. Hence it will be seen that x and y may be used as primitive elements to define the algebra.

COROLLARY. *Any algebra with a finite basis, containing no properly nilpotent elements, is an algebra of Class II.*

For it may be expressed as the direct sum of a number of algebras of quaternions, real matrices, complex matrices, and quaternion matrices*.

The algebras of Class II are too extensive to be treated exhaustively; hence further classification is introduced by considering the degree of the algebra. This has already been defined as the least degree of the moduli in the primitive elements. The degree cannot be unity, since this would be equivalent to reducing the number of primitive elements by one. The next simplest case, namely that of algebras of Class II and Degree II, will be examined in detail. The second part of this paper is devoted to the examination of particular cases of algebras of Class II and Degree II, whilst in the third part an exhaustive account of such algebras is attempted.

* L. E. Dickson, *Algebras and their arithmetics* (1923), 53, 78.

Particular attention is given to the possibility of obtaining division algebras, since, owing to the unique properties of these algebras, they are of exceptional importance. For example, the factor theorem, which states that, if $xy = 0$, then either $x = 0$ or $y = 0$, is true only in a division algebra. The theory of determinants with non-commutative elements, as developed by A. R. Richardson*, is applicable only to division algebras.

In this paper the coefficients in the algebras are taken over the fields either of real numbers or of complex numbers; consequently the division algebras of Dickson† do not appear, since the coefficients for those algebras are taken over the field of rationals only. A corresponding extension of the fields of the Dickson algebras reduces these to matrix algebras. In fact, Frobenius‡ has shown that the only division algebras with a finite basis over the field of real numbers are the field itself, the algebra of complex numbers, and the algebra of real quaternions. The division algebras which arise in this paper, apart from these, have, of course, an infinite basis.

PART II.

On certain algebras of Class II and Degree II.

In an endeavour to conform to usage in Quantum Theory, for the first algebra that we shall discuss, we call the two primitive elements p and x. The quadratic modulus is taken to be

$$px - xp - 1.$$

It is necessary to show that this is a permissible modulus. The variety of algebras of Class II in which the equation $px - xp - 1 = 0$ can be satisfied is given by the variety of other moduli that are permissible taken simultaneously with the modulus $px - xp - 1$. We shall show that no other modulus is permissible. Hence there is defined a unique polynomial algebra of Class II. Extensions of this algebra, in the form of corresponding serial and rational algebras, will be discussed.

THEOREM VII. *The modulus $px - xp - 1$ is permissible.*

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* A. R. Richardson, *Proc. London Math. Soc.* (2), 28 (1928), 395.

† L. E. Dickson, "New division algebras", *Trans. American Math. Soc.*, 28 (1926), 207–234; "Construction of division algebras", *Trans. American Math. Soc.*, 32 (1930), 319–334.

‡ "Ueber lineare Substitutionen und bilinearen Formen", *Journal für Math.*, 84 (1878), 1–63.

The quantity $px-xp-1$ will be a permissible modulus, provided that the equation $px-xp-1=0$ is not incompatible with the laws of algebra already assumed, that is, the associative and distributive laws. To prove the theorem we have merely to find quantities, known to satisfy these laws of algebra, which also satisfy the given equation.

Consider the infinite matrices

$$p=\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 2 & & & \\ & & 0 & 3 & & \\ & & & 4 & & \\ & & & & 5 & \\ & & & & & \ddots \end{bmatrix}, \qquad x=\begin{bmatrix} 0 & & & & & \\ 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \ddots & \end{bmatrix}.$$

The products px, xp have respectively 1, 2, 3, 4, ..., and 0, 1, 2, 3, ..., down the leading diagonal and zero elsewhere. Hence $px-xp=1$, and our example proves the theorem.

THEOREM VIII. *In the polynomial algebra defined by the modulus* $px-xp-1$ *every quantity can be expressed in the form* $\Sigma a_{mn}p^m x^n$.

If $px-xp=1$, then, very readily,

$$px^i-x^ip=ix^{i-1},$$

$$p^jx-xp^j=jp^{j-1}.$$

We prove the theorem by mathematical induction. Assume that it is true for polynomial expressions of degree not greater than r in either p or x. We then prove it for a polynomial expression of degree $r+1$.

Consider any term in the expression. This is a multiple of a product obtained from p's and x's. If the first element of the product is p and the last is x, then it can be obtained by multiplying a product of degree not greater than r in x and p separately, on the left by p and on the right by x. If we substitute for the latter in the form $\Sigma a_{mn}p^m x^n$, we obtain an expression for the former in the same form.

If the degree of the term in p exceeds r, but the product does not begin with p, then it must begin with a product of the form x^ip. Substitute for x^ip the equivalent px^i-ix^{i-1}, and we obtain two terms of which the degree of the second in x or p does not exceed r. Similarly, if the product is of degree $r+1$ in x, and ends with xp^j, substitute for this

$p^j x - jp^{j-1}$. The resulting terms may be expressed in the form $\Sigma a_{mn} p^m x^n$, as has been shown.

Mathematical induction completes the proof.

If a quantity of the algebra is expressed in this form, it is said to be in *normal form*.

THEOREM IX. *The expression in the form* $\Sigma a_{mn} p^m x^n$ *is unique.*

If the expression in this form is not unique, then subtraction gives an expression

$$\phi(p, x) = \Sigma b_{mn} p^m x^n = 0.$$

Now $$p(p^m x^n) - (p^m x^n) p = n p^m x^{n-1},$$

so that $$p\phi - \phi p = \Sigma n b_{mn} p^m x^{n-1} = 0.$$

Further, this quantity $\Sigma n b_{mn} p^m x^{n-1}$ will not be identically zero, unless ϕ does not involve x, in which case the operation will have been superfluous.

Repeat this process until the degree in x is zero. Then we have

$$f(p) = \Sigma c_m p^m = 0;$$

in this expression all the coefficients c_m do not vanish.

But, since $p^m x - x p^m = m p^{m-1}$,

$$f(p) x - x f(p) = \Sigma m c_m p^{m-1} = 0.$$

Repeat the process until the degree in p is zero. We shall have a constant, not zero since it is a multiple of the coefficient of the highest power of p in $f(p)$, equated to zero. Since this is impossible, our theorem is true.

THEOREM X. *No second modulus is compatible with the modulus* $px - xp - 1$.

If there was a second modulus, then, by using the equation

$$px - xp - 1 = 0,$$

we could express it in the form

$$\Sigma b_{mn} p^m x^n.$$

The *reductio ad absurdum* of Theorem IX shows that this is not a permissible modulus.

Our polynomial algebra is thus defined uniquely. We proceed to discuss some of the properties of the algebra.

THEOREM XI. *No number of the algebra, save the constants, has an inverse.*

If this were not true, then there would be numbers $P = \Sigma a_{mn} p^m x^n$, and $Q = \Sigma b_{mn} p^m x^n$, such that $PQ = 1$.

Now let that term in P of highest degree, first in p and then in x, be $a_{m_1 n_1} p^{m_1} x^{n_1}$, and that term in Q with the same property be $b_{m_2 n_2} p^{m_2} x^{n_2}$.

Then, since

$$p^{m_1} x^{n_1} . p^{m_2} x^{n_2} = p^{m_1+m_2} x^{n_1+n_2} + \text{terms of lower degree},$$

it is readily seen that the term in PQ with highest degree, first in p and then in x, is

$$a_{m_1 n_1} b_{m_2 n_2} p^{m_1+m_2} x^{n_1+n_2}.$$

This does not vanish, and it can be unity only if m_1, m_2, n_1, n_2, are all zero, *i.e.* if both P and Q are constants. Hence the theorem is proved.

The same reasoning leads to

THEOREM XII. *There are no factors of zero.*

The second normal form.

A polynomial expression can also be expressed uniquely in the form $\Sigma b_{nm} x^n p^m$. This will be called the second normal form.

The question arises* as to what is the connection between the constants a_{mn} and b_{nm}, if a quantity is expressible in the two normal forms as $\Sigma a_{mn} p^m x^n$ and $\Sigma b_{nm} x^n p^m$. To elucidate this and similar problems we have the following theorems.

THEOREM XIII.

$$p^m x^n = x^n p^m + \Sigma \lambda_i^{nm} x^{n-i} p^{m-i},$$

where

$$\lambda_i^{nm} = \frac{n(n-1)(n-2)\dots(n-i+1)\, m(m-1)\dots(m-i+1)}{i!}$$

$$= \frac{n!\ m!}{(n-i)!\ (m-i)!\ i!}.$$

We shall assume an expansion in this form, which is very easily established, and obtain the values of the constants λ_i^{nm}.

---

* It was suggested by a Referee.

Now, if $$p^m x^n = x^n p^m + \Sigma \lambda_i^{nm} x^{n-i} p^{m-i},$$

we have, clearly, the following values of certain λ's:

$$\lambda_0^{nm} = 1, \quad \lambda_0^{0m} = 1, \quad \lambda_i^{0m} = 0 \quad (i \neq 0),$$

$$\begin{aligned} p^m x^n &= p^m x^{n-1} x \\ &= x^{n-1} p^m x + \Sigma \lambda_i^{n-1\,m} x^{n-i-1} p^{m-i} x \\ &= x^n p^m + m x^{n-1} p^{m-1} + \Sigma \lambda_i^{n-1\,m} [x^{n-i} p^{m-i} + (m-i) x^{n-i-1} p^{m-i-1}]. \end{aligned}$$

Equating coefficients, we have

$$\lambda_i^{nm} = \lambda_i^{n\ 1\,m} + (m-i+1) \lambda_{i-1}^{n-1\,m}.$$

Using $\binom{n}{r}$ to represent the coefficient of x^r in $(1+x)^n$, we have

$$\begin{aligned} \lambda_i^{nm} &= \lambda_i^{n-2\,m} + 2(m-i+1) \lambda_{i-1}^{n-2\,m} + (m-i+1)(m-i+2) \lambda_{i-2}^{n-2\,m} \\ &= \lambda_i^{n-r\,m} + \Sigma \binom{r}{s} (m-i+1)(m-i+2) \dots (m-i+s) \lambda_{i-s}^{n-r\,m}. \end{aligned}$$

Putting $r = n$ and remembering that

$$\lambda_0^{0m} = 1, \quad \lambda_i^{0m} = 0 \quad (i \neq 0),$$

we have $$\lambda_i^{mn} = \binom{n}{i} (m-i+1) \dots (m-1) m \lambda_0^{0m}$$

$$= \frac{n!\ m!}{i!\ (n-i)!\ (m-i)!}.$$

COROLLARY. *Since*

$$(-x) p - p(-x) = 1,$$

it follows that $$x^n p^m = p^m x^n + \Sigma (-1)^i \lambda_i^{nm} p^{m-i} x^{n-i}.$$

THEOREM XIV.

$$p^m x^n p^h x^k = x^{n+k} p^{m+h} + \Sigma\, {}^{mn}\phi_t^{hk} x^{n+k-t} p^{m+h-t},$$

where

$${}^{mn}\phi_t^{hk} = \frac{(m+h)_t (n+k)_t}{t!} \left[1 - \frac{hnt}{(m+h)(n+k) \cdot 1} + \frac{h(h-1)\, n(n-1)\, t(t-1)}{(m+h)(m+h-1)(n+k)(n+k-1)\, 1 \cdot 2} + \dots \right],$$

$(m+h)_t$ being written for the product

$$(m+h)\,(m+h-1)\ldots(m+h-t+1).$$

We have

$$p^m x^n p^h x^k = p^m \Big[\sum_{i=0}^{n} (-1)^i \lambda_i^{nh} p^{h-i} x^{n-i}\Big] x^k$$

$$= \sum_{i=0}^{n} (-1)^i \lambda_i^{nh} p^{m+h-i} x^{n+k-i}$$

$$= \sum_{i=0}^{n} \sum_{j=0}^{n+k} (-1)^i \lambda_i^{nh} \lambda_j^{m+h-i\,n+k-i} x^{n+k-i-j} p^{m+h-i-j}.$$

Hence clearly

$${}^{mn}\phi_t^{hk} = \sum_{i+j=t} (-1)^i \lambda_i^{nh} \lambda_j^{m+h-i\,n+k-i}$$

$$= \frac{(m+h)_t\,(n+k)_t}{t!}\Big[1 - \frac{hnt}{(m+h)\,(n+k)\,1}$$

$$+ \frac{h(h-1)\,n(n-1)\,t(t-1)}{(m+h)\,(m+h-1)\,(n+k)\,(n+k-1)\,1\,.\,2} - \cdots\Big].$$

There is another way of finding the value of ${}^{mn}\phi_j^{hk}$, which leads to an interesting result connecting the sums of generalized hypergeometric series.

We have

$$p^m x^n p^h x^k = \Big[\sum_{i=0}^{n} \lambda_i^{nm} x^{n-i} p^{m-i}\Big]\Big[\sum_{j=0}^{h} \lambda_j^{kh} x^{k-j} p^{h-j}\Big]$$

$$= \Sigma\, \lambda_i^{nm} \lambda_j^{kh} x^{n-i} p^{m-i} x^{k-j} p^{h-j}.$$

Since $$p^{m-i} x^{k-j} = \Sigma\, \lambda_s^{m-i\,k-j} x^{k-j-s} p^{m-i-s},$$

we have $${}^{mn}\phi_t^{hk} = \sum_{i+j+s=t} \lambda_i^{nm} \lambda_j^{kh} \lambda_s^{m-i\,k-j}$$

$$= \sum_{i+j+s=t} \frac{n_i m_{i+s} h_j k_{j+s}}{i!\;j!\;s!}.$$

Hence

$$\frac{(m+h)_t\,(n+k)_t}{t!}\Big[1 - \frac{hnt}{(m+h)(n+k).1}$$

$$+ \frac{h(h-1)\,n(n-1)\,t(t-1)}{(m+h)\,(m+h-1)\,(n+k)\,(n+k-1)\,1\,.\,2} - \cdots\Big]$$

$$= \sum_{i+j+s=t} \frac{n_i m_{i+s} h_j k_{j+s}}{i!\;j!\;s!}. \qquad (A)$$

In this result put $k = 0$. Then

$$\frac{(m+h)_t n_t}{t!}\left[1-\frac{ht}{(m+h).1}+\frac{h(h-1)t(t-1)}{(m+h)(m+h-1)1.2}-\dots\right]=\frac{n_t m_t}{t!}.$$

Thus

$$1-\frac{ht}{(m+h).1}+\frac{h(h-1)t(t-1)}{(m+h)(m+h-1)1.2}-\dots=\frac{m_t}{(m+h)_t}. \tag{B}$$

Alternatively, putting $n = 0$, we obtain

$$1+\frac{mt}{(h-t+1).1}+\frac{m(m-1)t(t-1)}{(h-t+1)(h-t+2)1.2}+\dots=\frac{(m+h)_t}{h_t}. \tag{C}$$

These results (B) and (C) are particular cases of the formula for the sum of the hypergeometric series when $z = 1$, and are variants of Vandermonde's theorem.

We use these results to obtain the right-hand side of (A) in a simpler form.

$$\sum_{i+j+s=t}\frac{n_i m_{i+s} h_j k_{j+s}}{i!\ j!\ s!}$$

$$=\frac{m_t k_t}{t!}\left[1+\frac{nt}{(k-t+1).1}+\frac{n(n-1)t(t-1)}{(k-t+1)(k-t+2)1.2}+\dots\right.$$

$$+\frac{ht}{(m-t+1).1}\left\{1+\frac{n(t-1)}{(k-t+1).1}+\frac{n(n-1)(t-1)(t-2)}{(k-t+1)(k-t+2)1.2}+\dots\right\}$$

$$+\frac{h(h-1)t(t-1)}{(m-t+1)(m-t+2)1.2}\left\{1+\frac{n(t-2)}{(k-t+1).1}\right.$$

$$\left.\left.+\frac{n(n-1)(t-2)(t-3)}{(k-t+1)(k-2+2)1.2}+\dots\right\}+\dots\right]$$

$$=\frac{m_t k_t}{t!}\left[\frac{(n+k)_t}{k_t}+\frac{ht(n+k-1)_{t-1}}{(m-t+1)1(k-1)_{t-1}}+\dots\right]$$

$$=\frac{m_t(n+k)_t}{t!}\left[1+\frac{hkt}{(m-t+1)(n+k).1}\right.$$

$$\left.+\frac{h(h-1)k(k-1)t(t-1)}{(m-t+1)(m-t+2)(n+k)(n+k-1)1.2}+\dots\right].$$

This result together with (A) now gives

THEOREM XV.

$$(m+h)_t\left[1-\frac{hnt}{(m+h)(n+k).1}+\frac{h(h-1)n(n-1)t(t-1)}{(m+h)(n+h-1)(n+k)(n+k-1)1.2}-\dots\right]$$

$$=m_t\left[1+\frac{hkt}{(m-t+1)(n+k).1}\right.$$

$$\left.+\frac{h(h-1)k(k-1)t(t-1)}{(m-t+1)(m-t+2)(n+k)(n+k-1)1.2}+\dots\right].$$

This result has, however, already been obtained from the theory of generalized hypergeometric series*, but it is interesting in respect of the number of well known formulae which can be deduced from it. Thus, if we put $m/(m+h)=z$ and make m and h tend to infinity, or alternatively put $k/(n+k)=z$ and make k and n tend to infinity, we obtain well known results connecting the hypergeometric series for z and $1-z$. It can be used to prove Saalschütz's formula†

$${}_3F_2(a,\ b,\ c;\ d,\ e)=\frac{\Gamma(d)\,\Gamma(1-e+a)\,\Gamma(1-e+b)\,\Gamma(1-e+c)}{\Gamma(1-e)\,\Gamma(d-a)\,\Gamma(d-b)\,\Gamma(d-c)},$$

where a, b, c are negative integers and $d+e=a+b+c+1$.

For terms in which the degree in p is equal to the degree in x, the form $\Sigma a_n(px)^n$ is very convenient, since the terms are commutative.

It should be noticed that

$$p(px)^n=(px+1)^n p,$$

$$x(px)^n=(px-1)^n x.$$

The following theorems show the connection between this form and the normal forms.

THEOREM XVI.

$$p^n x^n=px(px+1)(px+2)\dots(px+n-1)\ddagger,$$

$$x^n p^n=(px-1)(px-2)\dots(px-n).$$

* W. N. Bailey, *Proc. London Math. Soc.* (2), 28 (1928), 242–254, (5.31).

† Saalschütz, "Eine Summationsformel", *Zeitschrift für Math. und Phys.*, 35 (1890), 186–188.

‡ Or $p^n x^n=(xp+1)(xp+2)\dots(xp+n)$, ...

Cf. Turnbull, *Matrices and invariants*, 119.

These results are readily proved by induction. Remembering that

$$(px+r)\,x = x(px+r+1),$$

we have

$$\begin{aligned} p^n x^n &= p(p^{n-1}x^{n-1})\,x \\ &= p\,.\,px(px+1)\,(px+2)\ldots(px+n-2)\,x \\ &= px(px+1)\,(px+2)\ldots(px+n-1). \end{aligned}$$

The proof of the second result is similar.

THEOREM XVII. *If* $\phi_0(n)$, $\phi_1(n)$, $\phi_2(n)$, ... *are polynomials in* n, *defined by the equations*:

$$\phi_0(n) = 1,$$

$$\phi_1(n) = \sum_{r=1}^{n} r\phi_0(r),$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$\phi_{s+1}(n) = \sum_{r=1}^{n} r\phi_s(r),$$

then

$$(px)^n = \phi_0(n)\,p^n x^n - \phi_1(n-1)\,p^{n-1}x^{n-1} + \phi_2(n-2)\,p^{n-2}x^{n-2} \ldots + (-1)^r \phi_r(n-r)\,p^{n-r}x^{n-r} + \ldots,$$

and

$$(xp)^n = \phi_0(n)\,p^n x^n + \sum_{r=1}^{n} \phi_r(n-r)\,p^{n-r}x^{n-r}.$$

The result is also obtained by induction.

$$\begin{aligned} (px)^{n+1} &= (px)^n px \\ &= \phi_0(n)\,[p^{n+1}x^{n+1} - np^n x^n] \\ &\quad + \Sigma(-1)^r \phi_r(n-r)\,[p^{n-r+1}x^{n-r+1} - (n-r)\,p^{n-r}x^{n-r}]. \end{aligned}$$

The theorem follows provided that

$$\phi_r(n+1-r) = \phi_r(n-r) + (n+1-r)\,\phi_{r-1}(n+1-r),$$

and this follows from the definition of the polynomials ϕ_r.

The proof of the second result is similar.

The values of $\phi_r(n)$ for low values of r are as follows:

$$\phi_0(n) = 1,$$

$$\phi_1(n) = \sum_{r=1}^{n} r = \frac{n(n+1)}{2},$$

$$\phi_2(n) = \sum_{r=1}^{n} \frac{r^2(r+1)}{2} = \frac{n(n+1)(n+2)(3n+1)}{24},$$

$$\phi_3(n) = \sum_{r=1}^{n} r\phi_2(r) = \frac{n^2(n+1)^2(n+2)(n+3)}{48},$$

$$\phi_4(n) = \sum_{r=1}^{n} r\phi_3(r) = \frac{n(n+1)(n+2)(n+3)(n+4)(15n^3+30n^2+5n-2)}{5760}.$$

An obvious extension of a polynomial algebra with an infinite basis is obtained by defining infinite series as new elements of the algebra.

In the algebra with modulus $px-xp-1$, the products $p^m x^n$ are all linearly independent. Hence (the dimensions of the later terms being distinct from those of the earlier terms) the infinite series $\Sigma a_{mn} p^m x^n$ has a definite significance without questions of convergence or divergence arising. If two such series were multiplied, a question of convergence would arise, as indeed it would in the multiplication of two infinite matrices. Without making any attempt to obtain best possible conditions, we give a sufficient condition that multiplication should be possible according to the usual rules. If, in the series of the form $\Sigma a_{mn} p^m x^n$ for existent terms, $m-n$ is bounded above (alternatively below) products are always convergent, and in the product $m-n$ is still bounded above (below).

Theorem VIII does not hold for infinite series. Consider a series in the second normal form $\Sigma b_{nm} x^n p^m$. Each term translated into the first normal form is equivalent to a finite series. On summing these series, the coefficient of any $p^m x^n$ may be a divergent series. It would thus seem that all types of infinite series are not comparable. In particular, series in the second normal form are not comparable with series in the first normal form. It would appear to be necessary to restrict our infinite series to some specific form. One of the normal forms suggests itself.

THEOREM XVIII. *Elements being defined corresponding to infinite sequences in the second normal form, the algebra becomes simply isomorphic,*

with the infinite matrix algebra of the representation:

$$x = \begin{bmatrix} 0 & & & & & & \\ 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & \cdot \end{bmatrix}, \qquad p = \begin{bmatrix} 0 & 1 & & & & & \\ & & 2 & & & & \\ & & & 3 & & & \\ & & & & 4 & & \\ & & & & & 5 & \\ & & & & & & \cdot \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \end{bmatrix}.$$

The representation of the term $x^n p^m$ has only zeros in the top left-hand rectangle of n rows and m columns. It will be seen that any series $\Sigma a_{nm} x^n p^m$ is represented by a definite matrix. Also, representing the series corresponding to the matrix with unity in the i-th row and j-th column and zero elsewhere by e_{ij}, we have

$$e_{11} = 1 - xp + \frac{x^2 p^2}{2!} - \frac{x^3 p^3}{3!} + \ldots,$$

$$e_{ij} = \frac{1}{(j-1)!} x^{i-1} e_{11} p^{j-1}.$$

From these series, for any given matrix, the corresponding series may be found. Thus the algebras are simply isomorphic. They are, of course, not division algebras, since there are factors of zero. For instance, although

$$x e_{11} = e_{21},$$

yet we have

$$e_{11} x = 0.$$

This last result, obvious from the properties of matrices, can also be shown from the series.

Since

$$\frac{x^r p^r x}{r!} = \frac{x^{r+1} p^r + r x^r p^{r-1}}{r!} = \frac{x^{r+1} p^r}{r!} + \frac{x^r p^{r-1}}{(r-1)!},$$

so that

$$\Sigma (-1)^r \frac{x^r p^r x}{r!} = 0,$$

x has a left-hand inverse, namely $e_{12} + e_{23} + e_{34} + \ldots$. But this left-hand

inverse is not a right-hand inverse. Representing it by y, we have

$$yx = 1, \quad xy = 1 - e_{11}.$$

There is no right-hand inverse of x. Similarly p has a right-hand inverse but no left-hand inverse.

There is another representation, namely

$$x = \begin{bmatrix} 0 & 1 & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix}, \qquad p = \begin{bmatrix} 0 & & & & & \\ & -1 & & & & \\ & & -2 & & & \\ & & & -3 & & \\ & & & & \cdot & \\ & & & & & \cdot \end{bmatrix}.$$

This corresponds to the algebra of infinite series in the first normal form.

We have $$e_{11} = 1 + px + \frac{p^2x^2}{2!} + \frac{p^3x^3}{3!} + \ldots.$$

Like results follow; x has a right-hand and p a left-hand inverse.

Instead of extending our original algebra thus by means of infinite series, we can extend it by defining new quantities which are both right-hand and left-hand inverses of polynomial expressions in x and p.

If P is a polynomial expression in x and p, we define P^{-1} so that both the equations

$$PP^{-1} = 1 \quad \text{and} \quad P^{-1}P = 1$$

are satisfied.

As we have seen, these equations are independent, and one cannot be deduced from the other.

Hence we deduce the algebra of rational expressions of p and x. If P, Q, R, S represent four polynomial expressions, then PQ^{-1} is defined to be equal to $R^{-1}S$, if

$$RP = SQ.$$

$PQ^{-1} + RQ^{-1}$ is defined as equal to $(P+R)Q^{-1}$.

THEOREM XIX. *If P and Q are polynomials in x and p, then non-zero polynomials R and S can be found such that*

$$RP = SQ.$$

Suppose that the degree of P and Q in either p or x does not exceed r. Take R and S to be polynomials of degree $3r$ in x and p, with undetermined coefficients; there will be $2(3r+1)^2$ coefficients.

$RP-SQ$ is a polynomial of degree $4r$. There will be $(4r+1)^2$ coefficients which are linearly dependent upon the $2(3r+1)^2$ undetermined coefficients.

Since $(4r+1)^2 < 2(3r+1)^2$, the $2(3r+1)^2$ coefficients can be chosen, not all vanishing, so that $RP-SQ$ vanishes identically. This proves the theorem.

COROLLARY I. *R and S can be chosen so that $PR = QS$.*

The proof is similar.

COROLLARY II.

$$R^{-1}(P+Q) = R^{-1}P + R^{-1}Q.$$

Choose P_1, Q_1, R_1, R_2, S_1, and S_2, so that

$$RP_1 = PR_1, \quad RQ_1 = QR_2, \quad R_1S_1 = R_2S_2.$$

Then

$$\begin{aligned} R^{-1}P + R^{-1}Q &= P_1S_1(R_1S_1)^{-1} + Q_1S_2(R_2S_2)^{-1} \\ &= (P_1S_1 + Q_1S_2)(R_1S_1)^{-1} = R^{-1}(P+Q), \end{aligned}$$

since

$$R(P_1S_1 + Q_1S_2) = (P+Q)(R_1S_1).$$

THEOREM XX. *The operations of addition, subtraction, multiplication, and division are closed for expressions PQ^{-1} or $R^{-1}S$, P, Q, R, and S being supposed to be polynomials in p and x.*

To express PQ^{-1} in the form $R^{-1}S$, we have only to find R and S such that Theorem XIII is satisfied. Conversely $R^{-1}S$ can be expressed in the form PQ^{-1}.

Also $PQ^{-1} = PT(QT)^{-1}$, for, if $PQ^{-1} = R^{-1}S$, then $RP = SQ$, and also $RPT = SQT$. Hence

$$R^{-1}S = PT(QT)^{-1} = PQ^{-1}.$$

If we wish to add PQ^{-1} and RS^{-1}, then we choose T and U so that

$$QT = SU.$$

Then

$$\begin{aligned} PQ^{-1} + RS^{-1} &= PT(QT)^{-1} + RU(SU)^{-1} \\ &= (PT+RU)(QT)^{-1}. \end{aligned}$$

Subtraction would lead to $(PT-RU)(QT)^{-1}$.

To prove the theorem for multiplication, we have first to show that the associative law holds. It is sufficient to show that

$$(PQ^{-1})\,R = P(Q^{-1}\,R).$$

Let $$PQ^{-1} = Q_1^{-1}\,P_1, \quad i.e. \quad Q_1\,P = P_1\,Q.$$

$$(PQ^{-1})\,R = Q_1^{-1}\,P_1\,R.$$

Let $$Q^{-1}\,R = R_2\,Q_2^{-1}, \quad i.e. \quad RQ_2 = QR_2.$$

Then $$P(Q^{-1}\,R) = PR_2\,Q_2^{-1}.$$

We have to show that

$$Q_1^{-1}\,P_1\,R = PR_2\,Q_2^{-1},$$

i.e. $$P_1\,RQ_2 = Q_1\,PR_2.$$

But $$P_1\,RQ_2 = P_1\,QR_2 = Q_1\,PR_2.$$

Thus $$(PQ^{-1})\,R = P(Q^{-1}\,R).$$

If $PQ^{-1} = Q_1^{-1}\,P_1$, multiplication of PQ^{-1} and RS^{-1} would give

$$(PQ^{-1})\,(RS^{-1}) = Q_1^{-1}\,(P_1\,R)\,S^{-1}.$$

If $Q^{-1}(P_1\,R) = TU^{-1}$, this would give

$$(PQ^{-1})\,(RS^{-1}) = TU^{-1}\,S^{-1}$$

$$= T(SU)^{-1}.$$

Since $(PQ^{-1})^{-1} = QP^{-1}$, the case of division is reduced to that of multiplication.

THEOREM XXI. *The algebra of rational expressions in p and x is a division algebra.*

Every quantity PQ^{-1} has an inverse QP^{-1}. To prove that the algebra is a division algebra we have only to show that there are no factors of zero.

We have proved this for polynomial expressions.

The completion of the theorem follows directly from Theorem XIX and from the method of obtaining the product of two rational expressions.

It should be pointed out here that infinite series are not permitted in this division algebra. In particular, exponentiation is not allowed. As we have already seen, the definition of a series very much akin to the exponential series, namely

$$1+px+\frac{p^2x^2}{2!}+\frac{p^3x^3}{3!}+\frac{p^4x^4}{4!}+\ldots,$$

leads to factors of zero, thus destroying the property of the algebra being a division algebra.

Nevertheless, defining the series

$$y=1+px+\frac{(px)^2}{2!}+\frac{(px)^3}{3!}+\ldots,$$

we have $yx=exy$, and, as we shall see, under certain conditions the algebra of infinite series in x and y gives a division algebra. It seems possible that, in the algebra $px-xp=1$, certain types of infinite series may be defined without destroying the property of being a division algebra. Before considering this point further we shall discuss certain other algebras of Class II and Degree II.

The algebra with modulus $yx-\mu xy$, where μ is a scalar other than 1 or zero, x and y being the primitive elements.

THEOREM XXII. *The modulus $yx-\mu xy$ is permissible.*

THEOREM XXIII. *Every number of the algebra may be put in the form $\Sigma a_{mn}y^m x^n$.*

In every term containing a product of x's and y's we have only to rearrange the x's and y's so that it is in the form $a_{mn}y^m x^n$ and add a factor $1/\mu$ for each pair of elements x and y such that in the original product the x precedes the y.

THEOREM XXIV. *If μ is not a root of unity, any further modulus will lead to a modulus of the type $y^m x^n$. Every number of the algebra $\Sigma a_{mn}y^m x^n$, such that $a_{00}=0$, will be a factor of zero.*

Put the modulus in the form $f=\Sigma g_m x^m$, where the terms g_m represent polynomials in y.

Then
$$f = \Sigma g_m x^m = 0,$$
$$yf = \Sigma y g_m x^m,$$
$$fy = \Sigma \mu^m y g_m x^m.$$

If the degree of f in x is r, then
$$f_1 = \mu^r yf - fy = 0,$$
and f_1 is of degree $r-1$ at most. f_1 will not vanish identically unless
$$f = g_r x^r.$$

By repeating this operation, we obtain an equation of the form
$$F = g_n x^n = 0.$$

If the degree of g_n in y is s, then
$$F_1 = \mu^s Fx - xF = 0.$$

Repetition of this process leads us to
$$y^m x^n = 0.$$

THEOREM XXVII. *If μ is a primitive n-th root of unity, then further moduli of the form $F(x^n, y^n)$ are permissible.*

In particular it should be noted that, with three moduli $yx - \mu xy$, $x^n - 1$, $y^n - 1$, we obtain the algebra of matrices of order n^2, if $\mu^n = 1$.

THEOREM XXVIII. *In the algebra with modulus $yx - \mu xy$, let $f_r(x)$ $(r = 0, 1, 2, \ldots)$ denote series of the form*
$$x^{n_r}(a_{0r} + a_{1r}x + a_{2r}x^2 + \ldots) \quad (a_{0r} \neq 0),$$
where n_r is an integer, positive, negative, or zero. Then, if elements of the algebra of the form
$$\sum_{n=0}^{\infty} f_r(x)\, y^{r+n}$$
are defined, n being an integer, positive, negative, or zero, the algebra so formed is a division algebra.

This theorem is given by Wedderburn*. The reciprocal of each such element is obtained readily by the formal inversion of series. It should

* J. H. M. Wedderburn, "Algebras without a finite basis", *Trans. American Math. Soc.*, 26 (1924), 395–426 (9.8).

be noted that here we have at once a division algebra and an algebra in which infinite series are permissible. Contrast this with the rational and serial algebras obtained with the modulus $px-xp-1$. Exponentiation of some elements is possible in the algebra, but not for every element.

Thus $$\exp(x^{-1}) = 1+\frac{x^{-1}}{1!}+\frac{x^{-2}}{2!}+\frac{x^{-3}}{3!}+\dots$$

is not defined as an element of the algebra.

Wedderburn further states that, if $yx = \theta(x)\,y$, where $\theta(x)$ is a function of x whose iterated powers are known for positive and negative indices, a division algebra is defined under similar conditions.

As a further example he cites $yx = x^{-1}y$.

If these statements were true, it would simplify our problem of finding an extension of the algebras with modulus $px-xp-1$, so as to include infinite series and yet retain the property of being a division algebra. If we take instead of $yx = \theta(x)\,y$, the obviously equivalent form $yx = x\theta(y)$ and put $\theta(y) = y+1$, we have $yx = xy+x$.

Hence $$x^{-1}yx = xx^{-1}y+1.$$

Putting $x^{-1}y = p$,

$$px-xp = 1,$$

and we have a suitable extension of our algebra with modulus $px-xp-1$.

Unfortunately, not only is the general statement not true, but it breaks down even in the example cited. The definition of the division algebra depends on the formal inversion of series. This again depends on the fact that infinite series are allowable only in increasing powers of x and y.

No formal inversion of the series $\sum_{-\infty}^{\infty} a_n y^n$ can be obtained.

Consider now $$f(y) = \sum_{1}^{\infty} a_n y^n,$$

$$xf(y)+f(y)\,x = \left[\sum_{1}^{\infty} a_n\left(y^n+\frac{1}{y^n}\right)\right]x.$$

This series cannot be formally inverted. In fact, if

$$f = \sum_{1}^{\infty} y^n,$$

we have $$xf+fx+x = \sum_{-\infty}^{\infty} y^n x.$$

Then $(1-y)(xf+fx+x)$ formally multiplied reduces to zero. This cannot be a division algebra unless we assume that $xf+fx+yx=0$; and this assumption seems somewhat artificial.

Again, let us consider the example which concerns us more, namely

$$yx = x(y+1).$$

If we define series of the form $\sum_1^\infty a_n y^n = f$,

then
$$x^{-1}fx = \Sigma a_n (y+1)^n,$$

$$x^{-r}fx^r = \Sigma a_n (y+r)^n.$$

Unless these series are expressible in the form $\Sigma b_n y^n$, then the general expression in the algebra may involve many types of series $\Sigma_n \Sigma_r a_{rn}(y+r)^n$. The determination of the formal inverse of this would present a somewhat formidable problem.

If, on the other hand, those series only are defined of the form $\Sigma a_n y^n$, such that $\Sigma a_n (y+r)^n$ is expressible in the form $\Sigma b_n y^n$, it is clear that we must restrict the scalars a_n so that the function of a complex variable $\Sigma a_n z^n$ has the whole z-plane as its domain of convergence. But with this restriction, the formal inverse of many quantities of the algebra will not have been defined as numbers of the algebra. To avoid this difficulty the following generalization is suggested.

Let S be a group of functions of a real (or complex) variable z, subject to the following restrictions:

(1) It is closed to addition, subtraction, multiplication, and division, except by zero.

(2) If $f(z)$ is a member of the group, so is $f(z+1)$, $f(z-1)$.

(3) 1 and z are members of the group.

Corresponding to each member of the group we define members of the algebra, to satisfy the same laws of addition and subtraction as the functions of z, corresponding to unity, the unity of the algebra, and corresponding to z, the quantity y.

Elements of the algebra $\sum_{r=0}^\infty f_r x^{\alpha+r}$ are then defined, the f_r being numbers of the algebra corresponding to functions in the group S, and α being any finite integer, positive, negative, or zero.

Formal inversion of such series is straightforward, and a division algebra is defined.

Putting $px = y$, we thus obtain an appropriate extension of the polynomial algebra defined by the modulus $px - xp - 1$.

It would be interesting to see what place, if any, in this definition would be taken by the factors of zero defined by infinite series in the first and second normal forms.

We have $$px = y.$$

Hence $$p^n x^n = px(px+1)\dots(px+n-1)$$

$$= y(y+1)\dots(y+n-1).$$

Consequently

$$\sum_1^\infty \frac{p^n x^n}{n!} = \Sigma \frac{y}{n}(1+y)\left(1+\frac{y}{2}\right)\dots\left(1+\frac{y}{n-1}\right)$$

$$= \prod_1^\infty \left(1+\frac{y}{r}\right).$$

But this product clearly diverges, and it is not expressible as a function of y; hence it cannot have any place in our definition.

PART III.

Algebras of Class II and Degree II.

This part is devoted to an examination of the algebras of Class II and Degree II. We take x and y to represent the primitive elements. Since one of the moduli must be of Degree II, we take this as the most general quadratic expression in x and y, namely

$$ax^2+by^2+h(xy+yx)+2fy+2gx+c+k(xy-yx),$$

the coefficients a, b, c, f, g, h, k being arbitrary scalars. By means of a change in the primitive elements we obtain the simplest forms for the modulus. For simplicity we assume that algebras are taken over the complex numbers, and do not discriminate between algebras that can be shown to be equivalent by means of a transformation involving complex

numbers. We then find what further moduli are compatible with this, and we give a brief account of the resulting algebras.

With x and y as primitive elements the modulus is

$$M = ax^2+by^2+h(xy+yx)+2fy+2gx+c+k(xy-yx).$$

Associate with this modulus the plane curve of the second degree,

$$S \equiv a\xi^2+b\eta^2+2h\xi\eta+2f\eta+2g\xi+c = 0.$$

THEOREM XXIX. *If a linear transformation is made on the primitive elements x and y, and the same linear transformation is made on the scalar variables ξ and η, then the coefficients in the transform of the modulus M are the same as the coefficients in the transform of the curve $S = 0$, with the exception of the coefficient k of $xy-yx$, which is multiplied by the determinant of the transformation.*

Suppose the transformation to be

$$x = \alpha x'+\beta y'+\gamma,$$

$$y = \alpha_1 x'+\beta_1 y'+\gamma_1.$$

Then

$$\xi = \alpha\xi'+\beta\eta'+\gamma,$$

$$\eta = \alpha_1\xi'+\beta_1\eta'+\gamma_1.$$

Further,

$$k(xy-yx) = k\,[(\alpha x'+\beta y'+\gamma)(\alpha_1 x'+\beta_1 y'+\gamma_1) - (\alpha_1 x'+\beta_1 y'+\gamma_1)(\alpha x'+\beta y'+\gamma)]$$

$$= k\begin{vmatrix}\alpha & \beta\\ \alpha_1 & \beta_1\end{vmatrix}(x'y'-y'x');$$

$$M = ax^2+by^2+h(xy+yx)+2fy+2gx+c+k(xy-yx)$$

$$= a(\alpha x'+\beta y'+\gamma)^2+b(\alpha_1 x'+\beta_1 y'+\gamma_1)^2$$

$$+h\,[(\alpha x'+\beta y'+\gamma)(\alpha_1 x'+\beta_1 y'+\gamma_1)+(\alpha_1 x'+\beta_1 y'+\gamma_1)(\alpha x'+\beta y'+\gamma)]$$

$$+2f(\alpha_1 x'+\beta_1 y'+\gamma_1)+2g(\alpha x'+\beta y'+\gamma)+c+k(xy-yx)$$

$$= a'x'^2+b'y'^2+h(x'y'+y'x')+2f'y'+2g'x'+c'+k(xy-yx).$$

Also

$$S \equiv a\xi^2+b\eta^2+2h\xi\eta+2f\eta+2g\xi+c$$

$$= a(\alpha\xi'+\beta\eta'+\gamma)^2+b(\alpha_1\xi'+\beta_1\eta'+\gamma_1)^2$$

$$+2h(\alpha\xi'+\beta\eta'+\gamma)(\alpha_1\xi'+\beta_1\eta'+\gamma_1)$$

$$+2f(\alpha_1\xi'+\beta_1\eta'+\gamma_1)+2g(\alpha\xi'+\beta\eta'+\gamma)+c$$

$$= a'\xi'^2+b'\eta'^2+2h'\xi'\eta'+2f'\eta'+2g'\xi'+c'.$$

It is readily seen that the coefficients a', b', c', f', g', h' are the same in both instances.

THEOREM XXX. *By a suitable choice of primitive elements, any modulus of the second degree may be put into one of the eight forms*:

(A) $xy+yx+1+k(xy-yx)$, (B) $y^2+x+k(xy-yx)$,

(C) $xy+yx+k(xy-yx)$, (D) $y^2+y+k(xy-yx)$,

(E) $y^2+k(xy-yx)$, (F) $x+k(xy-yx)$,

(G) $1+k(xy-yx)$, (H) $xy-yx$.

The curve C must represent either an ellipse, a hyperbola, a parabola, two straight lines, two parallel straight lines, two coincident straight lines, one straight line, a straight line at infinity, or the identity $0=0$. Since we may make use of complex transformations, we do not distinguish between ellipse and hyperbola.

According to which of these is represented, by a suitable choice of coordinates we may express the equation in one of the forms:

$$2\xi\eta+1=0, \quad \eta^2+\xi=0,$$

$$2\xi\eta=0, \quad \eta^2+\eta=0,$$

$$\eta^2=0, \qquad \xi=0,$$

$$1=0, \qquad 0=0.$$

By making the corresponding transformation on the primitive elements, our theorem is proved.

COROLLARY. *In the five cases* (B), (D), (E), (F), *and* (G), *if* $k \neq 0$, *a simple linear transformation will bring it to unity. We may thus consider thirteen possible forms*; (A), (C), *and* (H), *as in Theorem II, and*

(B)	$y^2+x+xy-yx,$	(B_1)	$y^2+x,$
(D)	$y^2+y+xy-yx,$	(D_1)	$y^2+y,$
(E)	$y^2+xy-yx,$	(E_1)	$y^2,$
(F)	$x+xy-yx,$	(F_1)	$x,$
(G)	$1+xy-yx,$	(G_1)	$1.$

It should be noted that the modulus (G_1) is not permissible, and that, with modulus (F_1), as indeed with the modulus (B_1), the algebra reduces to an algebra of Class I. We shall not, therefore, consider these cases further, but we proceed to examine the other ten.

THEOREM XXXI. *The rational algebras defined by the moduli* (B), (D), (E), (F), (G) *are simply isomorphic with the rational algebra defined by the modulus* $px-xp-1$.

The modulus (G) is already of this form.

Consider now (E),

$$y^2+xy-yx=0.$$

Multiply on the right and on the left by y^{-1}. We have

$$1+y^{-1}x-xy^{-1}=0.$$

By putting $y^{-1}=-p$ the isomorphism is established.

In the case (F) we have

$$x+xy-yx=0.$$

Multiply on the left by x^{-1}. We have

$$1+x(x^{-1}y)-(x^{-1}y)x=0.$$

By equating $x^{-1}y$ to p the isomorphism is established.

In the case (D) we have

$$y^2+y+xy-yx=0.$$

Multiplying on the left and on the right by y^{-1}, we have

$$1+y^{-1}+y^{-1}x-xy^{-1}=0.$$

This is readily reducible to the form (F).

Lastly we have the case (B),

$$y^2+x+xy-yx=0.$$

Now put $z=y^2+x$. We have

$$yz-zy=yx-xy=y^2+x=z.$$

Hence $$z+zy-yz=0,$$

and this reduces to the form (F).

Permissible moduli.

With the modulus $px-xp-1$, no further moduli are permissible. Nevertheless with the other moduli considered in Theorem XXXI, although these give rise to the same rational algebra, yet certain further moduli are permissible to form polynomial algebras, finite or infinite, with properly nilpotent elements.

With the modulus (E),

$$yx-xy=y^2,$$

every element of the polynomial algebra may be put in the form $\Sigma a_{mn}x^m y^n$. Hence a second modulus would assume the form

$$P=\Sigma a_{mn}x^m y^n=0.$$

Quite readily we have

$$yx^n-x^n y=x^{n-1}y^2+x^{n-2}y^2x+\ldots+y^2x^{n-1}$$

$$=nx^{n-1}y^2+\text{terms involving lower powers of } x.$$

If P is of degree n in x, it may be put into the form

$$P=x^n f_n(y)+\text{terms involving lower powers of } x$$

$$=0;$$

$$P_1=yP-Py=nx^{n-1}y^2f_n(y)+\text{terms involving lower powers of } x$$

$$=0.$$

Similarly $$P_2 = yP_1 - P_1 y = 0.$$

Eventually $$P_n = yP_{n-1} - P_{n-1} y = n!\ y^{2n} f_n(y) = 0.$$

Hence y must satisfy a characteristic equation. Suppose the equation of least degree satisfied by y to be

$$F(y) = 0;$$

then $$F(y)\,x - xF(y) = y^2 F'(y) = 0,$$

where $F'(y)$ represents the polynomial obtained by differentiating $F(y)$ formally.

Hence $F(y)$ must be a factor of $y^2 F'(y)$. This is possible only if $F(y)$ is a monomial expression, *i.e.*

$$y^n = 0.$$

Further moduli are permissible if, and only if, y is nilpotent. If this is the case, it is readily seen that y is also properly nilpotent. A great variety of other moduli are then permissible. For the algebra to be finite, x must also satisfy a characteristic equation

$$f(x) = 0.$$

Now we have $$yx^n - x^n y = nyx^{n-1} y.$$

This formula is most easily obtained from a consideration of the rational algebra considered in Theorem III. If $p = -y^{-1}$, then $px - xp = 1$.

Hence $px^n - x^n p = nx^{n-1}$, and very readily

$$-x^n y + yx^n = nyx^{n-1} y.$$

This formula must necessarily hold for polynomial algebras as well as the rational algebras.

Hence we have

$$yf(x) - f(x)\,y = yf'(x)\,y = 0.$$

Also $$yf(x)\,y = 0.$$

Let $f_1(x)$ be the highest common factor of $f(x)$ and $f'(x)$. Then

$$yf_1(x)\,y = 0.$$

Again $$y^2 f_2(x)\,y^2 = 0,$$

if $f_2(x)$ is the highest common factor of $f_1(x)$ and $f_1'(x)$.

If $f(x)$ has a factor repeated r times, but no factor repeated more than r times, then we have

$$f_r(x) = 1$$

and

$$y^{2r} = 0.$$

Hence $2r > n$, when y^n is the lowest power of y congruent to zero.

Similar algebras with nilpotent elements can be obtained from the modulus (F) $x+xy-yx = 0$.

Any element of the algebra may be put in the form

$$\Sigma a_{mn} x^m y^n.$$

Hence any second modulus can be expressed in the form

$$P = \Sigma f_r(x) y^r = 0,$$

the expressions $f(x)$ representing polynomials in x. Hence

$$P_1 = Px - xP = \Sigma r f_r(x) y^r = 0,$$

$$P_2 = P_1 x - xP_1 = \Sigma r^2 f_r(x) y^r = 0.$$

Similarly

$$\Sigma r^3 f_r(x) y^r = 0,$$

$$\Sigma r^i f_r(x) y^r = 0.$$

Then clearly we must have

$$Q = f_r(x) y^r = 0,$$

$$Qy = f_r(x) y^{r+1} = 0,$$

$$yQ = f_r(x+1) y^{r+1} = 0,$$

and, if $g_1(x)$ is the highest common factor of $f_r(x)$ and $f_r(x+1)$, then

$$g_1(x) y^{r+1} = 0.$$

Continuation of this process clearly gives us

$$y^n = 0.$$

Thus, in this case also, further moduli are permissible provided that y is nilpotent, in which case it is also properly nilpotent. A variety of further moduli are then permissible.

If $$f(x)\,y^r = 0,$$

then $$f_1(x)\,y^{r+1} = 0,$$

where f_1 is the highest common factor of $f(x)$ and $f(x+1)$.

For finite algebras, x must satisfy a characteristic equation

$$F(x) = 0,$$

and, if n is the lowest power of y equal to zero, then

$$F(x), \quad F(x+1), \quad \ldots, \quad F(x+n-1)$$

must have a common factor.

For the modulus (D), we have

$$yx - xy = y^2 + y.$$

For further moduli to be permissible, y^2+y must be nilpotent and hence it must be properly nilpotent. Algebras similar to those considered above result.

For the modulus (B),

$$yx - xy = y^2 + x.$$

Put $$z = y^2 + x.$$

Then the polynomial algebra defined by y and z is the same as that defined by y and x.

The modulus (B) becomes

$$yz - zy = z.$$

This case has already been considered.

THEOREM XXXII. *Under the conditions described in Part II, Theorem XXVI, the moduli* (A) *and* (C) *lead to the same division algebra.*

The modulus (C), $xy+yx+k(xy-yx)$ gives immediately

$$xy = \mu yx \quad (\mu \neq 1).$$

The modulus (A), $xy+yx+1+k(xy-yx)$ leads to

$$xy = \mu yx + \nu \quad (\mu \neq 1).$$

Put $$z = y + \frac{\nu}{\mu - 1} x^{-1}.$$

We have $$xz = xy + \frac{\nu}{\mu - 1} = \mu yx + \frac{\nu\mu}{\mu - 1} = \mu zx.$$

Thus either of these will lead to the same division algebra defined in Part II, Theorem XXVI.

Further permissible moduli which can be taken with the modulus (C) have been described in Part II.

It remains to discuss those which can be taken with the modulus (A). Taking the modulus in the form

$$yx = \mu xy + \nu,$$

suppose, first, that μ is not a root of unity.

Clearly, if $$z = xy + \frac{\nu}{\mu - 1},$$

then $$zx = \mu xz.$$

Any further modulus gives an equation

$$f(x, y) = 0.$$

Multiplying by an appropriate power of x, we obtain an equation of the form

$$f(x, z) = 0.$$

As shown in Part II, this leads to an equation

$$x^r z^s = 0.$$

Hence $$y^r x^r z^s = 0.$$

But $y^r x^r$ may be put in the form

$$(xy)^r + a_1 (xy)^{r-1} + a_2 (xy)^{r-2} + \ldots + a_r.$$

Hence z satisfies a characteristic equation $f(z) = 0$. Suppose further that $f(z) = 0$ is the equation of lowest degree satisfied by z.

Then $$xf(z)\, y = 0,$$

i.e. $$zf\left(\frac{z}{\mu}\right) = 0.$$

Hence $f(z)$ must divide $zf(z/\mu)$. Clearly $f(z)$ must be a monomial expression, *i.e.* $z^n = 0$.

Thus z must be properly nilpotent. In this case we are led to similar algebras with properly nilpotent elements to those of the algebras with modulus (C), $xy-\mu yx$.

If μ is a primitive n-th root of unity, the case is again similar to that of the modulus (C) $xy-\mu yx$. Putting z as before equal to $xy+\nu/(\mu-1)$, and so obtaining $xz = \mu zx$, $zy = \mu yz$, we have readily that x^n and y^n are commutative with z, and hence, with every number of the algebra, z^n is commutative with x and y. Clearly z^n is a polynomial in x^n and y^n; in fact,

$$z^n = x^n y^n + \left(\frac{\nu}{\mu-1}\right)^n.$$

Moduli of the form $f(x^n, y^n)$ are permissible. Moduli x^n+1 and y^n+1 give us the algebra of matrices of order n^2.

The modulus (H), $xy-yx$.

The algebras with modulus $xy-yx$ are commutative. Without further moduli, or with moduli which involve one primitive element only, the algebra is the direct product of two algebras of Class I. Further moduli of the form $f(x, y)$ are however permissible. These algebras have properties very similar to algebras of Class I.

The moduli (D_1) *and* (E_1), y^2+y *and* y^2.

The algebras resulting from these two moduli have very similar properties, and we therefore consider them together. As moduli in an algebra of Class I they have already been considered. They lead respectively to the direct sum of two algebras of real or complex numbers, and to an algebra of real or complex numbers together with one (properly) nilpotent element.

With algebras of Class II, however, the case is different. We have, in fact,

THEOREM XXXIII. *The algebra of Class II with modulus* (D_1) *or* (E_1) *has a sub-algebra simply isomorphic with the general polynomial algebra of Class II.*

The equation $$y^2+y = 0,$$

or $$y^2 = 0,$$

allows us to express every element of the algebra in such a form, as the sum of products of x's and y's, that the product of two consecutive y's without an x intervening does not occur. Further, all such products are linearly independent.

Hence, if $z = xy$, all different products of z and x are linearly independent, and we have a sub-algebra simply isomorphic with the general algebra of Class II. Every element of the algebra, in fact, may be expressed in the form

$$f_1(x, z) + yf_2(x, z).$$

It might be expected that all moduli permissible in the algebra with primitive elements x and z would be permissible with the moduli considered. But this is not so, as the following example shows.

Consider the modulus $xz - zx - 1$, first with $y^2 + y$. We have

$$x^2y - xyx = 1.$$

Multiplying on the right by y, we have

$$-x^2y - xyxy = y,$$

so that

$$-x^3y - x^2yxy = xy,$$

i.e.

$$+x^2z + xz^2 + z = 0,$$

and this is known to be incompatible with $xz - zx = 1$ (Part II, Theorem X).

Secondly, consider the modulus $xz - zx - 1$, together with the modulus y^2. We have

$$x^2y - xyx = 1.$$

Multiplying on the right by y, we have

$$-xyxy = y,$$

so that

$$x^2yxy + xy = 0,$$

i.e.

$$xz^2 + z = 0.$$

This again is incompatible with $xz - zx = 1$.

Nevertheless, a very wide range of moduli are permissible with either of these. Let $f(x, z)$ be any polynomial expression in x and z. Then $f(x, z) \cdot x$ is a permissible modulus with either of the moduli $y^2 + y$ or y^2.

With so wide a range of permissible moduli an exhaustive account seems impossible. We seem in little better position to give an exhaustive

account than when considering all algebras of Class II, irrespective of degree. Instead of attempting to do this we shall consider the case when a second modulus is also of Degree II. If this second modulus reduces to any of the forms (A), (B), (C), (D), (E), (F), (G), or (H), the algebra will already have been considered. There remains the case when both moduli reduces to the form (D_1) or (E_1).

With an appropriate choice of primitive elements we have either

$$(1) \quad x^2 = 0, \quad y^2 = 0,$$

$$(2) \quad x^2 = 0, \quad y^2+y = 0,$$

$$(3) \quad x^2+x = 0, \quad y^2+y = 0,$$

or

$$(4) \quad x^2+x = 0, \quad y^2 = 0.$$

Cases (2) and (4) are obviously equivalent.

Put $z_1 = xy$. Then clearly, in any of these cases, every element of the algebra may be expressed in the form

$$f_1(z_1)+yf_2(z_1)+f_3(z_1)\,x+yf_4(z_1)\,x,$$

f_1, f_2, f_3, and f_4 being polynomial expressions in z_1 only.

THEOREM XXXIV. *A simple isomorphism exists between each of the three algebras of Class II with moduli*

$$(1) \quad x^2, y^2, \qquad (2) \quad x^2, y^2+y, \qquad (3) \quad x^2+x, y^2+y,$$

and a sub-algebra of the direct product of a matrix algebra of order four and a general polynomial algebra of Class I.

If the general element of the direct product of these algebras is represented by

$$\begin{bmatrix} f_1(z) & f_2(z) \\ f_3(z) & f_4(z) \end{bmatrix},$$

the sub-algebras in question are obtained by the restrictions:

(1) that $f_2(z)$ and $f_1(z)-f_4(z)$ are divisible by z,

(2) that $f_2(z)-f_3(z)$ and $f_1(z)-f_4(z)$ are divisible by z,

(3) that $f_2(z)$ is divisible by z, and $f_3(z)$ is divisible by $1-z$.

We consider the three cases separately.

Case (1). $x^2 = 0, \quad y^2 = 0.$

Let A represent the algebra resulting.

Put $$z = z_1 + z_2 = xy + yx,$$

$$z_1 = xy, \quad z_2 = yx.$$

Then $$z_1 z_2 = z_2 z_1 = 0.$$

If $f(z)$ represents any polynomial expression in z, not including a scalar,

$$f(z) = f(z_1) + f(z_2).$$

Further, $$yf(z_2) = f(z_1)\, y = 0,$$

$$yf(z_1) = f(z_2)\, y.$$

Consider the direct product of a matrix algebra of order 4 and a general algebra of Class I with primitive element z. Call this algebra A_1. The general element of the algebra may be expressed in the form

$$\begin{bmatrix} f_1(z) & f_2(z) \\ f_3(z) & f_4(z) \end{bmatrix}.$$

Associate z with the expression $xy+yx$ in the algebra A, the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ with the element y, and $\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$ with the element x. It will be seen that a simple isomorphism exists between the algebra A and the sub-algebra of A_1 obtained by the restriction that $f_2(z)$ is divisible by z, and also $f_1(z) - f_4(z)$.

The general element of A_1,

$$\begin{bmatrix} f_1(z) & f_2(z) \\ f_3(z) & f_4(z) \end{bmatrix},$$

subject to these restrictions, corresponds to

$$f_1(z_1) + f_4(z_2) - \lambda + f_2(z)\, z^{-1} x + y f_3(z),$$

if λ represents the scalar of f_1 and f_4.

The scalar λ is subtracted, since it appears twice in the expression $f_1(z_1)$ and in $f_4(z_2)$.

Case (2). $x^2 = 0, \quad y^2 + y = 0.$

A simple linear transformation on y brings it to the form

$$x^2 = 0, \quad y^2 = 1.$$

Call this algebra A, and, as before, let A_1 be the direct product of a matrix algebra of order 4 and a general algebra of Class I with primitive element z. Now associate z with $xy+yx$. Associate y with the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and x with the matrix $\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$.

If the general element of the algebra A_1 is

$$\begin{bmatrix} f_1(z) & f_2(z) \\ f_3(z) & f_4(z) \end{bmatrix},$$

a simple isomorphism exists between the algebra A and the sub-algebra of A_1 obtained by the restriction that $f_1(z) - f_4(z)$, and also $f_2(z) - f_3(z)$, are divisible by z.

The general element of this sub-algebra is

$$\begin{bmatrix} \lambda + F_1(z) & \mu + F_2(z) \\ \mu + F_3(z) & \lambda + F_4(z) \end{bmatrix},$$

λ and μ being scalars, and F_1, F_2, F_3, and F_4 being divisible by z. This corresponds to

$$\lambda + \mu y + F_1(xy) + y F_3(z) + F_2(z) z^{-1} x + F_4(yx),$$

where $z = xy + yx$.

Case (3). $x^2 = x, \quad y^2 = y.$

Represent the algebra by A and the direct product of a matrix algebra of order 4 and a general algebra of Class I with primitive element z by A_1, let x correspond to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and y to $\begin{bmatrix} z & z \\ 1-z & 1-z \end{bmatrix}$. Then z corresponds to

$$xyx + (1-x)(1-y)(1-x).$$

If the general element of A_1 is

$$\begin{bmatrix} f_1(z) & f_2(z) \\ f_3(z) & f_4(z) \end{bmatrix},$$

then a simple isomorphism exists between A and the sub-algebra of A_1 obtained by the restriction that $f_2(z)$ is divisible by z, and $f_3(z)$ by $1-z$. Supposing z to be equal to

$$xyx+(1-x)(1-y)(1-x);$$

with this restriction the general element of A_1 corresponds to

$$xf_1(z)+(1-x)f_4(z)+(xy-xyx)z^{-1}f_2(z)$$
$$+[(1-x)(1-y)-(1-x)(1-y)(1-x)](1-z)^{-1}f_3(z).$$

Since this paper was read, a paper has been published by Kermack and McCrea, *Proc. Edin. Math. Soc.* (2), 2 (1931), 205–240, which also deals with the equation $px-xp=1$. The two papers naturally overlap to a certain extent, but they also diverge to a remarkable degree.

ON DIXON'S PROPERTIES OF THE CUBIC SURFACE

By S. Bishara.

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1. It is a known property of the twenty-seven lines of a cubic surface that they can be arranged in determinantal form as follows:

I	II	III
$a_1\ a_2\ a_3$	$b_4\ a_4\ c_{56}$	$c_{14}\ c_{15}\ c_{16}$
$b_1\ b_2\ b_3$	$b_5\ a_5\ c_{64}$	$c_{24}\ c_{25}\ c_{26}$
$c_{23}\ c_{31}\ c_{12}$	$b_6\ a_6\ c_{45}$	$c_{34}\ c_{35}\ c_{36}$

These possess the property that each determinantal form gives the nine lines common to a pair of conjugate Steinerian trihedra. The members of two of these forms define three double-sixes and each constituent is a member of two of these double-sixes. Thus, if we take the eighteen lines of I and II, a_1 belongs to the two double-sixes

$$a_1\ a_2\ a_3\ a_4\ a_5\ a_6 \qquad a_1\ a_2\ a_3\ c_{56}\ c_{64}\ c_{45}$$
$$b_1\ b_2\ b_3\ b_4\ b_5\ b_6, \qquad c_{23}\ c_{31}\ c_{12}\ b_4\ b_5\ b_6.$$

Cayley has proved* that, of all the cubic surfaces containing the nine lines of one of the above sets, four are nodal cubic surfaces. Consequently, there are three sets of tetrads of nodal points corresponding to the three sets of lines I, II, III. A. L. Dixon has made an intensive study† of the relationship between these configurations of lines, the Schur quadrics defined by the double-sixes above referred to, and the nodal cubics dealt with by Cayley. Dixon's methods are mainly algebraic. The results obtained by Dixon, in so far as they related to envelopes, were dealt with by W. P. Milne‡ chiefly from the standpoint of synthetic geometry. Milne

* Cayley, *Collected Math. Papers*, 6, 129–134.

† A. L. Dixon, *Proc. London Math. Soc.* (2), 26 (1927), 351–362, and A. L. Dixon, *Journal London Math. Soc.*, 1 (1926), 170–175.

‡ W. P. Milne, *Journal London Math. Soc.*, 4 (1929), 246–250.

has given simpler proofs of many of Dixon's results and has extended his investigations further, but many of Dixon's most elegant properties could not be established by Milne's method. The object of this paper is to apply the methods of synthetic geometry to those theorems not dealt with by Milne. In this way, many simpler proofs and new results have been obtained. The chief new results embodied in the present paper are as follows:

(i) *The three quartic surfaces which are generated by the parabolic curves of the three pencils of cubic surfaces through one of the sets I, II, III are identical.*

(ii) *The quartic surface thus obtained is a desmic quartic surface.*

(iii) *Its twelve nodes are the above three tetrads of nodal points discussed by Cayley.*

(iv) *These three tetrads are quadruply in perspective.*

2. The following is a proof that the parabolic curves of a pencil of cubic surfaces through the nine lines of a pair of conjugate Steinerian trihedra describe a quartic surface.

Consider the pencil of cubic surfaces

$$\lambda_1 u_1 v_1 w_1 - \lambda_2 u_2 v_2 w_2 = 0 \tag{1}$$

having in common the nine lines of mutual intersection of the trihedra $u_1 v_1 w_1 = 0$ and $u_2 v_2 w_2 = 0$. Let these trihedra be denoted by T_1 and T_2 and their vertices by τ_1 and τ_2 respectively. Take a member of the pencil (1). Its Hessian is the locus of points whose polar quadrics with respect to this cubic are cones. The polar quadric of an arbitrary point P is

$$\lambda_1 Q_1 - \lambda_2 Q_2 = 0, \tag{2}$$

where Q_1 is the polar quadric of P with respect to T_1 and Q_2 is its polar quadric with respect to T_2. (2) degenerates into a cone if

$$\lambda_1^4 \Delta - \lambda_1^3 \lambda_2 \Theta + \lambda_1^2 \lambda_2^2 \Phi - \lambda_1 \lambda_2^3 \Theta' + \lambda_2^4 \Delta' = 0. \tag{3}$$

Since Q_1 and Q_2 are cones whose vertices are τ_1 and τ_2 respectively, Δ and Δ' both vanish. $\Theta = 0$ is the condition that Q_2 as a locus shall be apolar to Q_1 as an envelope. This is satisfied either if Q_2 passes through the vertex of Q_1 or if Q_1 breaks up into two planes. All points satisfying

the first condition lie in a plane P_1, viz. the polar plane of τ_1 with respect to T_2. Points which satisfy the second condition describe the three planes of T_1. Hence we find that the locus of points which are such that their polar quadrics with respect to T_1 and T_2 satisfy the invariant relation $\Theta = 0$ is $u_1 v_1 w_1 P_1 = 0$. Similarly, the locus of points whose polar quadrics with respect to T_1 and T_2 satisfy $\Theta' = 0$ is $u_2 v_2 w_2 P_2 = 0$, where P_2 is the polar plane of τ_2 with respect to T_1. The Hessian is therefore expressible in the form

$$\lambda_1^2 u_1 v_1 w_1 P_1 - \lambda_1 \lambda_2 \Phi + \lambda_2^2 u_2 v_2 w_2 P_2 = 0, \tag{4}$$

and from the identity

$$(\lambda_1^2 u_1 v_1 w_1 P_1 - \lambda_1 \lambda_2 \Phi + \lambda_2^2 u_2 v_2 w_2 P_2) - (\lambda_1 u_1 v_1 w_1 - \lambda_2 u_2 v_2 w_2)(\lambda_1 P_1 - \lambda_2 P_2)$$
$$\equiv \lambda_1 \lambda_2 (u_1 v_1 w_1 P_2 - \Phi + u_2 v_2 w_2 P_1), \tag{5}$$

it follows that the parabolic curves of all members of the pencil lie upon the quartic surface

$$u_1 v_1 w_1 P_2 - \Phi + u_2 v_2 w_2 P_1 = 0.$$

3. Consider the cubic surface containing the twenty-seven lines of the sets I, II, III. Each of these sets leaves on the surface eighteen lines which form three associated double-sixes.

By taking the lines of II and III we have the double-sixes

$$\left.\begin{array}{lll} b_5\ a_5\ c_{46}\ c_{16}\ c_{26}\ c_{36} & b_6\ a_6\ c_{54}\ c_{14}\ c_{24}\ c_{34} & b_4\ a_4\ c_{65}\ c_{15}\ c_{25}\ c_{35} \\ b_6\ a_6\ c_{45}\ c_{15}\ c_{25}\ c_{35}, & b_4\ a_4\ c_{56}\ c_{16}\ c_{26}\ c_{36}, & b_5\ a_5\ c_{64}\ c_{14}\ c_{24}\ c_{34} \end{array}\right\} \text{[A]},$$

and by taking III and I we have

$$\left.\begin{array}{lll} c_{24}\ c_{25}\ c_{26}\ a_3\ b_3\ c_{12} & c_{34}\ c_{35}\ c_{36}\ a_1\ b_1\ c_{23} & c_{14}\ c_{15}\ c_{16}\ a_2\ b_2\ c_{13} \\ c_{34}\ c_{35}\ c_{36}\ a_2\ b_2\ c_{13}, & c_{14}\ c_{15}\ c_{16}\ a_3\ b_3\ c_{12}, & c_{24}\ c_{25}\ c_{26}\ a_1\ b_1\ c_{23} \end{array}\right\} \text{[B]},$$

and, finally, from I and II we have

$$\left.\begin{array}{lll} b_1\ b_2\ b_3\ c_{56}\ c_{64}\ c_{45} & c_{23}\ c_{31}\ c_{12}\ b_4\ b_5\ b_6 & a_1\ a_2\ a_3\ a_4\ a_5\ a_6 \\ c_{23}\ c_{31}\ c_{12}\ a_4\ a_5\ a_6, & a_1\ a_2\ a_3\ c_{56}\ c_{64}\ c_{45}, & b_1\ b_2\ b_3\ b_4\ b_5\ b_6 \end{array}\right\} \text{[C]}.$$

Let the Schur quadrics of the double-sixes [A], [B], [C] be denoted by A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 respectively. Denote by U_1 the quadric containing the first row of II and the first column of III, by U_2 the quadric containing the second row of II and the second column of III, and similarly for U_3. In the same way we define the quadrics V_1, V_2, V_3 which are determined from the rows of III and the corresponding

columns of I, and W_1, W_2, W_3 which are defined similarly by the rows of I and columns of II. We notice that U_1 contains the six lines which are common to the two double-sixes whose Schur quadrics are A_2 and A_3 and similarly for the others. Let Γ_1, Γ_2, Γ_3 be respectively the three quartic surfaces generated by the parabolic curves corresponding to the pencils of cubic surfaces through I, II, and III respectively.

Now take any straight line l which intersects a_1, a_2, a_3 and take the member Ψ of the pencil I which contains this line *in toto*. l is a bitangent to the Hessian of this cubic surface, and hence a bitangent to Γ_1. Further, it touches Γ_1 at the points at which it touches the Hessian. These two results follow from the fact that Γ_1 is expressible in the form $\Gamma_1 \equiv H - \Psi\Pi$, where H is the Hessian of Ψ and Π is linear in x, y, z, t. Let us distinguish the eighteen lines on the surface Ψ other than the nine lines of I by dashed letters. In this notation l is either b_4', b_5', or b_6', and in each case it is a line of the two double-sixes

$$\begin{array}{ll} c_{23}\ c_{31}\ c_{12}\ b_4'\ b_5'\ b_6' & a_1\ a_2\ a_3\ a_4'\ a_5'\ a_6' \\ a_1\ a_2\ a_3\ c_{56}'\ c_{64}'\ c_{45}', & b_1\ b_2\ b_3\ b_4'\ b_5'\ b_6'. \end{array}$$

The parabolic points of l therefore lie upon the Schur quadrics of these two double-sixes. These quadrics are, in fact, C_2 and C_3, since a Schur quadric is uniquely determined by means of three pairs of corresponding lines, and these are taken to be the first three pairs for each of the double-sixes. Hence we obtain the following result:

All lines which intersect a_1, a_2, a_3 are bitangents to the quartic surface Γ_1 and the points of contact lie upon a twisted quartic which is the intersection of two Schur quadrics, viz. C_2 and C_3. Also the quadrics C_2, C_3, and W_1 have a common curve of intersection.

Γ_1 contains upon its surface the quadri-quadrics B_2B_3, B_3B_1, B_1B_2, and C_2C_3, C_3C_1, C_1C_2. Similarly, Γ_2 contains the quadri-quadrics A_2A_3, A_3A_1, A_1A_2, and C_2C_3, C_3C_1, C_1C_2. Γ_1 and Γ_2 have therefore in common the three quadri-quadrics C_2C_3, C_3C_1, C_1C_2 and, besides, each passes through the parabolic curve of the cubic surface which contains the twenty-seven lines of I, II, III. They therefore intersect in a curve of order twenty-four, whereas they ought to meet in a curve of order sixteen. We deduce that Γ_1 and Γ_2 coincide and similarly with Γ_3 and Γ_1. The three loci Γ_1, Γ_2, Γ_3 are therefore identical. We shall now denote this surface by Γ. Hence

Γ is a quartic surface which is the locus of the parabolic curves of the pencils of cubic surfaces through I, II, or III, and contains upon it the quadri-

quadrics A_2A_3, A_3A_1, A_1A_2; B_2B_3, B_3B_1, B_1B_2; C_2C_3, C_3C_1, C_1C_2. *Further, all generators of both systems of the nine quadrics* U_1, U_2, U_3; V_1, V_2, V_3; W_1, W_2, W_3 *are bitangents to it.*

4. An algebraic consequence of the coincidence of the surfaces Γ_1, Γ_2, Γ_3 is that the nine Schur quadrics A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 have their point equations connected by the identical relation:

$$A_2A_3+A_3A_1+A_1A_2 \equiv B_2B_3+B_3B_1+B_1B_2 \equiv C_2C_3+C_3C_1+C_1C_2,$$

since each expression when equated to zero represents the equation to Γ.

5. Through the nine lines of I can be drawn four cubic surfaces each possessing a nodal point. Let the four nodes be denoted by L_1, L_2, L_3, L_4. From each of these points can be drawn a transversal to each triad of lines given by a row or a column of I. Hence through L_1, L_2, L_3, L_4 pass the six quadrics V_1, V_2, V_3; W_1, W_2, W_3 containing these six triads of lines. Now, if a cubic surface has a node at a point, its Hessian has also a node at this point. Hence we deduce that L_1, L_2, L_3, L_4 are nodes on Γ, since $\Gamma \equiv H-\Psi\Pi = 0$. In the same way we prove that the four points M_1, M_2, M_3, M_4 corresponding to II, and N_1, N_2, N_3, N_4 corresponding to III, are all nodes on Γ. Further, the M's and N's form an associated system of eight points, being the intersection of the three quadrics U_1, U_2, U_3. Similarly the N's and L's lie on V_1, V_2, V_3; and the L's and M's lie on W_1, W_2, W_3. Now consider the nodal cubic of the pencil I which has a node at L_1. Through L_1 can be drawn a straight line lying wholly on the nodal surface and meeting a_1, a_2, a_3. This line has its parabolic points coincident at L_1, and hence it touches the Schur quadrics C_2 and C_3 at L_1, *i.e.* it touches the quadri-quadric C_2C_3 at L_1. Hence the four lines that can be drawn from L_1, L_2, L_3, L_4 to meet a_1, a_2, a_3 all touch the quadri-quadric C_2C_3 at the respective points L. Similarly, the four lines that can be drawn from M_1, M_2, M_3, M_4 to meet b_4, b_5, b_6 all touch the quadri-quadric C_2C_3 at the respective points M. Plainly the above lines through the L's and those through the M's are generators of opposite systems of the quadric W_1, and hence intersect each other. Hence we see that the quadri-quadric C_2C_3, besides passing through the L's and M's, is touched by two sets of lines, each set consisting of four lines, and such that every line of one set intersects every line of the other set; one set consists of the four tangents at L_1, L_2, L_3, L_4 and these meet a_1, a_2, a_3, and the other set consists of the four tangents at M_1, M_2, M_3, M_4 and these meet b_4, b_5, b_6. Now, if two quadrics meet in a curve which is touched by two lines which intersect, the line joining

the points of contact passes through one or other of the vertices of the common self-conjugate tetrahedron of the two quadrics. Hence every line joining an L to an M must pass through a vertex of the self-conjugate tetrahedron of C_2 and C_3. The L's and M's are therefore in desmic position and have as their centres of perspective the vertices of the self-conjugate tetrahedron of C_2 and C_3. In fact, all members of the net of quadrics passing through the L's and M's, since these are in desmic position, have a common self-conjugate tetrahedron, and, therefore, in particular W_1, W_2, W_3, C_1, C_2, C_3, being members of this net, have a common self-conjugate tetrahedron whose vertices are the vertices of the cones which can be made to pass through L_1, L_2, L_3, L_4; M_1, M_2, M_3, M_4.

Hence any three associated double-sixes of a cubic surface define three Schur quadrics and three quadrics containing the eighteen lines. These six quadrics belong to the same net whose eight basic points are the vertices of two tetrahedra in desmic position; and all members of this net possess a common self-conjugate tetrahedron.

It remains to show that the vertices of the common self-conjugate tetrahedron of the net of quadrics through L_1, L_2, L_3, L_4; M_1, M_2, M_3, M_4 coincide with N_1, N_2, N_3, N_4. Consider the pencil of cubic surfaces through the nine lines of III. We notice that the six quadrics which contain a row or a column of III have four points in common, viz. N_1, N_2, N_3, N_4. The three quadrics defined by the rows intersect in four further points, viz. L_1, L_2, L_3, L_4. Those of the columns intersect in four further points, viz. M_1, M_2, M_3, M_4. Hence the L's and M's do not depend in any way upon which particular cubic of the pencil III we choose, *i.e.*, taking any member of the pencil III, the remaining eighteen lines of the surface define three associated double-sixes whose Schur quadrics always pass through the same eight points as defined above. Now take in particular the member of the pencil III which has a node at N_1. One of the three associated double-sixes which are formed out of the remaining lines of the surface will have its corresponding lines coincident, viz. the double-six defined by the six lines of the surface which lie upon its nodal cone. The Schur quadric of this degenerate double-six coincides with the nodal cone, and hence we obtain a cone whose vertex is N_1 and which passes through the L's and M's. We deduce that the L's, M's, and N's are three tetrahedra which form a desmic system.

Hence the three tetrads of nodal points corresponding to the three sets of nodal cubics through I, II, III form a desmic system, i.e. are quadruply in perspective.

Γ has no further nodes, since any node of the surface which does not lie on C_1, C_2, C_3 must lie on their Jacobian. Since C_1, C_2, C_3 have a common self-conjugate tetrahedron their Jacobian breaks up into the six edges of this tetrahedron. The vertices of the tetrahedron are N_1, N_2, N_3, N_4, which are nodes on Γ. Hence the existence of a further node on the surface would imply that the edge of the self-conjugate tetrahedron on which the node lies would be a line on the surface; and this is not the case. Γ is therefore a desmic surface.

6. The following is an independent proof that, if a quartic surface has eight nodes which are the vertices of two tetrahedra in desmic position, then any further node of the surface, if it exists, must coincide with one or other of the four centres of perspective of these two desmic tetrahedra and hence the surface could have no more than four further nodes.

The eight nodes may be regarded projectively as the eight corners of a parallelepiped. Any plane containing a pair of its parallel edges will intersect the quartic surface in a quartic curve having four nodes. Hence, since no three of these nodes are collinear, the quartic curve must break up into two conics passing through the four nodes. Suppose that the surface has a ninth node S. S will lie on an edge of the tetrahedron whose vertices are the centre of the parallelepiped and the three points in which the plane at infinity meets the lines joining the centres of parallel faces. Let S lie on one of the edges in the plane at infinity. Now take one of the two faces of the parallelepiped that contain this edge. This plane intersects the surface in a pair of conics, one of which has a node. One of the conics must therefore degenerate into a pair of straight lines meeting at S and passing through the four corners of the parallelepiped which lie in this plane. This can happen only if S lies at one or other of the points in which the parallel edges of the face under consideration meet the edge at infinity on which S lies. Hence S must lie at one or other of the vertices of the common self-conjugate tetrahedron of the quadrics through the corners of the parallelepiped.

7. We conclude with the following remarks:

(i) Since M_1, M_2, M_3, M_4 lie on W_1, W_2, W_3, it follows that from each of these points can be drawn a transversal to each of the triads a_1, a_2, a_3; b_1, b_2, b_3; c_{23}, c_{31}, c_{12}, and, further, that the three transversals from each point are coplanar. For, taking the cubic surface of the pencil I which passes through M_1, we notice that it contains upon it the three lines which are drawn from M_1 to meet the above triads. These lines must lie

in a plane, for otherwise the surface will have a node at M_1. In fact, only the Hessian of this surface has a node at M_1.

Hence, if the nine lines of mutual intersection of two trihedra are divided into three triads such that the lines of each triad do not intersect each other, then there exist four points from each of which can be drawn three transversals to the respective triads and such that these lie in a plane.

(ii) Since $L_1 L_2 L_3 L_4$ is a tetrahedron inscribed in each of the Schur quadrics B_1, B_2, B_3; C_1, C_2, C_3 and is self-conjugate with respect to each of A_1, A_2, A_3, we deduce that each of the first six quadrics regarded as a locus is apolar to each of A_1, A_2, A_3 regarded as an envelope. The converse property also holds true.

Hence any two Schur quadrics, chosen one from each of the sets A, B, C, are such that their invariants Θ and Θ' both vanish.

My thanks are due to Prof. W. P. Milne for his suggestions with regard to this paper.

SOME QUARTIC FORMS CONTAINING PLANES

By L. ROTH.

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In the present paper we are concerned with one aspect of a difficult problem—the number of planes which a quartic form in [4] may contain. We consider the case of a form V having no other singularities than isolated nodes; and the facility of the method lies in the fact that, if V has three planes in general position, it is rational and may be represented on a space [3] by a simple process of projection. Only one of the types obtained—the quartic form with five planes—has been considered before*, in connection with transformations of space†; this also appears as a special case of the polo-form with twenty nodes, which has been previously discussed by the present writer‡. It is shown below that the maximum number of planes, no two of which are cospatial, which V may contain is six; and it is hoped later to make these results the point of departure of a more general investigation.

1. In the following sections we proceed to determine the maximum number of planes, no two of which are cospatial§, that V may contain, if V is assumed to possess no other singularities than a finite number of nodes. It is clear that a form V containing *three* such planes must be rational; for from an arbitrary point P of V a single transversal may be drawn to these planes, meeting a fixed space in P'; and, with the exception

* Since the above was written I have discovered that the quartic form containing three planes has been discussed by d'Amico, *Atti Acc. Gioenia in Catania* (4), 18 (1906), but I have been unable to consult this work.

† Veneroni, *Rend. Ist. Lombardo* (2), 34 (1901), 640; J. A. Todd, *Proc. Camb. Phil. Soc.*, 26 (1930), 230.

‡ "A quartic form in four dimensions", *Proc. London Math. Soc.* (2), 30 (1929), 118.

§ Two such planes will, in future, be termed "skew".

of certain singular elements, the correspondence between P and P' is birational*. The present investigation is based on this correspondence.

Suppose, in the first place, that V contains a plane π; the pencil of spaces through π meets V elsewhere in cubic surfaces which define in π a pencil of cubic curves. The nine base-points of this pencil are at once seen to be nodes of V, since any line through one of these points meets V in at most two points elsewhere.

If V contains two skew planes π_1, π_2 intersecting at P, then P must be a node of V; for the tangent space at P must contain both π_1 and π_2 and is therefore indeterminate. Hence V possesses seventeen nodes, one of which is common to π_1 and π_2; and each plane contains nine.

If, however, π_1 and π_2 are cospatial, meeting in a line l, then the planes have in common three nodes lying on l; and the total number of nodes is thus fifteen.

2. *Three skew planes.* Now let V contain three skew planes π_1, π_2, π_3; each intersection of two planes is a node and there are twenty-four nodes in all. Let ω be an arbitrary space meeting the planes π_r in lines l_r $(r = 1, 2, 3)$. Then, as in § 1, a birational correspondence may be set up between the points of V and the points of ω. This correspondence has the following singular elements:

(1) The generators of the regulus ρ which meet l_r have no representative on V; ρ meets V in the lines l_r and a residual quintic γ, lying in ω. This curve γ is self-corresponding.

(2) There is a scroll Σ of lines on V which meet π_r; this will correspond to a base-curve in ω.

2.1. Consider the curve in ω which corresponds to a *plane section* ϖ of V; this is the section by ω of the scroll σ whose generators meet π_r and ϖ; the order of the curve is thus $4.3-3=9$ (since ϖ meets each plane π_r in one point). To find the order of the curve in which σ meets π_1, say, take any line l in π_1; the secants of l, π_2, and π_3 generate a quadric form containing π_2 and π_3; hence the number of such secants which meet ϖ is $4.2-2=6$. It follows that *plane sections of V correspond to curves c_9 meeting each line l_r in six points.*

* This is analogous to the well-known representation of the cubic surface, Baker, *Principles of geometry*, 3 (1923), 192.

2.2. The surface corresponding to a *space section* α of V is the section by ω of the form generated by secants of π_r and α. Since α contains a line of each plane π_r, the order of this form is $3.4-3.2=6$. Through any point of π_1, say, can be drawn $4-2=2$ such secants; hence the planes π_r are double on the form. Thus, *space sections of V correspond to surfaces f_6 containing l_r as double lines.*

To a plane through l_1 corresponds a cubic surface on V, residual to π_1; thus π_1 corresponds to the surface f_5 containing l_1 and the base-curve, and having l_2 and l_3 for double lines.

Now it is easily shown that any surface f_6 must contain six generators of the regulus ρ conjugate to that defined by l_r; whence it would seem that the corresponding space section must contain six points of γ (§ 2). But, since γ is a quintic, one generator λ of the six must be the same for all surfaces f_6, *and therefore forms part of the base-curve.*

2.3. Since any two surfaces f_6 have for free intersection a c_9, it follows that the residual base-curve ψ is of order 14; this, then, is the section by ω of the scroll Σ on V whose generators meet π_r (§ 2). The fixed generator λ meets γ in a point of ψ and corresponds to the point in which Σ meets γ. An arbitrary c_9 must meet ψ in fourteen points; and it is verified that it has four free intersections with an arbitrary f_6.

3. The space section f_4 in which ω meets V is self-corresponding; it passes through l_r, ψ, and γ (but not λ) and forms with ρ a degenerate f_6. A plane section of V in ω is also self-corresponding; it meets ψ in fourteen points and each l_r in one point. To find the genus of ψ we consider the intersection of f_4 with an arbitrary f_6; it consists of l_r, ψ, and a plane quartic meeting ψ in fourteen points. James's formula* then gives 111 for the number of apparent nodes of the intersection residual to l_r. *Hence ψ is of genus* 9. To find the number of points in which ψ meets l_r, we use Salmon's formula† for the intersection of the double curve on f_6 with the residual curve on f_4; it thus appears that ψ meets each line l_r in nine points. Hence λ is the only generator of ρ which meets ψ at a point not on l_r, and it is clear that *every* point of λ represents the generator of Σ which passes through the point common to λ and ψ. The points common to ψ and l_r are the section by ω of the curve in which Σ meets π_r, apart from the generator which each such plane contains. Since this curve is of the same genus as Σ, it has nodes at seven of the

* James, *Proc. Camb. Phil. Soc.*, 21 (1923), 451.

† Salmon-Rogers, *Analytic geometry of three dimensions*, 1 (1914), 371.

nodes in each plane, and quadruple points at the nodes in which it is met by the other two planes. Hence

If V contains three skew planes, it has twenty-four nodes, each of the planes meeting the others in a node. The lines of V which meet all three planes generate a scroll Σ of order 14 *and genus* 9, *meeting each plane in a curve of order* 9 *and one generator; seven of the nodes in each plane are double on Σ; those in which two planes intersect are quintuple on Σ.*

4. *Other lines on V.* Let l be a line of V which does not meet π_r; the secants of l and π_r generate a cubic scroll having π_r for conic planes, whence it follows that l corresponds to a twisted cubic c_3 in ω, having l_r for chords and therefore meeting ψ in five points.

If l meets π_1, the scroll of secants breaks up into a plane and a quadric which meets π_2 and π_3 in lines and π_1 in a conic. Hence l corresponds to a conic meeting l_1 in two points, l_2, l_3 in one point each, and ψ in three points.

If l meets π_1 and π_2, the scroll of secants is a plane, and l therefore corresponds to secants of l_1 and l_2 which meet ψ.

5. *Representation by quartic surfaces.* If the quadric ρ lies on V, any surface f_6 corresponding to a space section breaks up into ρ and a surface f_4 containing l_r. A plane section of V now corresponds to a curve consisting of two generators of ρ and a c_7 of genus 3; hence f_4 contains, besides l_r, a base-curve $\mathbf{c}_6$ of genus 4, which lies on the residual quadric ρ' in ω. This curve is met by c_7 in twelve points; the corresponding scroll F_{12} of lines meeting π_r forms with ρ the above-mentioned scroll Σ.

Since an arbitrary f_4 meets c_7 in one free point, it follows that c_7 meets each line l_r in four points, and that $\mathbf{c}_6$ has l_r for chords. The quadrics ρ and ρ' are self-corresponding; a space section of ρ corresponds to a conic and a space section of ρ' to a rational quintic. We return later to this representation*.

6. *Four planes.* Resuming the previous discussion we suppose now that V contains a fourth plane π_4 skew to π_r. Evidently this corresponds, by the former process, to a cubic surface f_3 passing through l_r and the

* More generally, we may consider the quartic form represented by quartic surfaces passing through a c_9^7, of which c_6 and l_r present a degenerate case; c_9 corresponds to a scroll of order 24; its twenty-one quadrisecants correspond to nodes, as may be verified by the fact that three quartic surfaces containing c_9 and a quadrisecant intersect elsewhere in two points. There are also three nodes corresponding to conics meeting c_9 in eight points.

section l_4 of π_4 by ω. The lines of π_4 correspond to twisted cubics having l_r for chords and meeting ψ in five points (§ 4).

Since any space section meets π_4 in a line, the corresponding f_6 meets f_3 in a twisted cubic; hence f_3 contains a part of the base-curve, which must be of order 9. In particular, the surface f_4 in ω meets π_4 in l_4; since it can have no free intersection with f_3, f_3 and f_4 must have in common a base-curve $\mathbf{c}_8$. Thus f_3 contains λ and a curve $\mathbf{c}_8$ forming part of ψ.

This may be shown otherwise. The lines meeting all four planes on V generate a cubic form which intersects V in a residual scroll F_8 corresponding to $\mathbf{c}_8$. Now these lines also meet a fifth associated plane; in particular, the generators of F_8 meet this plane in points of a plane section of V, so that F_8 is of genus 3. From the intersection of f_3 and f_4 we find that $\mathbf{c}_8$ meets each line l_r in five points, so that the remaining portion $\mathbf{c}_6$ of ψ meets each l_r in four points.

The plane π_4 gives rise to six more nodes on V, lying in π_4 but not in π_r. The scroll F_6 corresponding to $\mathbf{c}_6$ is generated by lines which meet π_r but not in general π_4; hence it meets π_4 in six points, which are the nodes just referred to; they correspond to intersections of $\mathbf{c}_6$ and $\mathbf{c}_8$ and through them pass common generators of F_6 and F_8. Since $\mathbf{c}_6$ and $\mathbf{c}_8$ together form a degenerate ψ, it follows that F_6 is elliptic. Hence

If V contains four skew planes $\pi_1, \ldots, \pi_4$, it contains a scroll F_8 of genus 3, whose generators meet all four planes, intersecting the planes in quintic curves with three nodes in each plane. The lines of V which meet three planes π_1, π_2, π_3 only generate an elliptic scroll F_6 meeting those planes in quartic curves with two nodes in each plane. F_6 meets π_4 in the six nodes of V other than those common to π_1, π_2, π_3 and intersects F_8 in the six generators through those nodes.

7. *Five planes.* Suppose that V contains a fifth plane π_5 skew to the other four. The lines meeting all five planes generate an elliptic scroll F_5 which must lie on V. Thus the lines of V which meet $\pi_1, \ldots, \pi_4$ break up into two scrolls: one, F_5, whose generators meet π_5 also; the other, F_3', whose generators meet $\pi_1, \ldots, \pi_4$ only. Similarly, the scroll F_6 breaks up into a scroll F_3'' of lines meeting $\pi_1, \pi_2, \pi_3, \pi_5$, and a scroll F_3 of lines meeting π_1, π_2, π_3 only.

These facts are at once verified by the representation, which readily yields other information concerning the incidence of the scrolls. Thus

If V contains five skew planes (and therefore thirty-five nodes), it contains the elliptic scroll F_5 of lines meeting all five planes, intersecting them in cubic

curves. It contains a cubic scroll F_3 of lines meeting each set of three planes and a scroll F_3' of lines meeting each set of four. F_3', F_5 have in common three generators, F_3', F_3 two generators, and F_3, F_5 one generator. Each directrix plane meets the corresponding cubic scrolls in conics.

8. *The polo-form containing five planes.* In a previous paper (already referred to) the writer has considered a representation of the polo-form Ω or determinantal quartic which in the present case yields a simpler representation, essentially different from that given in § 7 and from another which is to be given later (§ 10).

The general form Ω is represented upon a space ω by means of surfaces f_4 passing through a base-curve ψ of order 10 and genus 11, ψ_{10}^{11}, say. It is easily shown that, if ψ breaks up into a curve ψ_9 (necessarily of genus 9) and a trisecant t, then Ω contains a plane π, corresponding to the unique cubic surface which passes through ψ_9; ψ acquires three more nodes, which correspond to the intersections of t and ψ_9; of the remaining twenty, which in the general case are represented by the quadrisecants of ψ_{10}, six are found to correspond to the six quadrisecants of ψ_9, thereby verifying the remark made in § 1, that π contains nine nodes of Ω.

Similarly, ψ_{10} may break up into ψ_8^2 and two trisecants t_1, t_2; there are now two planes π_1, π_2 and three additional nodes, making twenty-six in all, and we may verify that the intersection of the planes is a node of Ω, as stated above. So we may consider the cases of ψ_7^5 and three trisecants, ψ_6^3 and four trisecants, and ψ_5^1 with five trisecants. The form corresponding to this last has $20+5.3=35$ nodes, precisely as in § 7; in the previous cases the number of nodes of Ω differed from that of the corresponding V with the same number of planes.

9. *Six planes.* If π_6 is a plane of V skew to the first five, this will correspond to a surface f_3''' passing through λ, l_r, and a base-curve of order 8; f_3''' must have a quintic in common with f_3' and f_3'' (corresponding to π_4 and π_5 respectively), besides λ and l_r, since no two of the surfaces have a free intersection. It follows that the curve $\mathbf{c}_5$ corresponding to F_5 of § 7 breaks up into a cubic c_3''' and a *conic* c_2, common to the three cubic surfaces, which meets each associated cubic in two points. Since $\mathbf{c}_5$ has l_r for trisecants and these are chords of the cubics, it follows that c_2 is secant to l_r; thus it represents a regulus which meets each of the six planes in a generator of the conjugate regulus. Hence

If V contains six skew planes, it contains a quadric meeting each of the planes in a line, i.e. there exists a space which meets the planes in six generators

of a regulus. There is a cubic scroll on V whose generators meet each set of three planes, and a cubic scroll whose generators meet each set of five planes. The lines of V meeting four of the planes must also meet a fifth plane.

10. *The second representation.* We shall show that six is the maximum number of skew planes which V can contain. It appears from the above result that, for the case of six planes, there is no loss of generality in using the representation by quartic surfaces (§ 5). For we may take the space ω to be identical with that of the above-mentioned regulus and thereby simplify the discussion. We arrive at the desired representation by considering first the cases of four and five planes.

Suppose that the form V mentioned in § 5 contains a plane π_4 skew to π_r; then *its section l_4 by ω must lie on the regulus defined by l_r.* For if not, we see, as in § 6, that π_4 corresponds to a surface f_3 passing through l_4, l_r, and a base quintic, and that a line of π_4 corresponds to a twisted cubic on f_3. Hence the free intersection of f_3 and the surface f_1, representing an arbitrary space section, should be a cubic: since, however, it is a quartic, the assumption is erroneous.

It follows at once that π_4 corresponds to a plane ϖ (which with ρ represents a degenerate f_3); the base-curve $\mathbf{c}_6$ breaks up into a conic $\mathbf{c}_2$ in ϖ—the section of ρ' by ϖ—and an elliptic quartic $\mathbf{c}_4$ on ρ', meeting $\mathbf{c}_2$ in four points and having l_r for chords. The curve c_7 corresponding to a plane section meets $\mathbf{c}_2$ in six points and ϖ in one free point, as it should.

The nodes in π_4 which do not lie in π_r correspond to the four intersections of $\mathbf{c}_4$ and $\mathbf{c}_2$, and the two generators of ρ which meet $\mathbf{c}_4$ at points not on l_r. Thus, with the acquisition of a fourth plane, the scroll F_{12} of lines meeting π_r breaks up into a scroll F_6 of lines meeting π_r only, and a scroll F_6' of lines meeting all four planes; as might have been expected from § 6.

11. If V contains a plane π_5 skew to the first four, its section l_5 by ω must, similarly, lie on the regulus conjugate to ρ. The scroll F_5 of § 7 consists of ρ and a scroll F_3, and the planes ϖ', ϖ'' representing π_4, π_5 meet in a base-line $\mathbf{c}_1$ which corresponds to F_3; each plane contains a second line ($\mathbf{c}_1'$, $\mathbf{c}_1''$, say) which forms with $\mathbf{c}_1$ a conic $\mathbf{c}_2$ of the previous case. An arbitrary c_7 has each of these lines for trisecants. The residual base-curve $\mathbf{c}_3$ is a twisted cubic with l_r and $\mathbf{c}_1$, etc., for chords; but, since $\mathbf{c}_3$ and the three lines $\mathbf{c}$ lie on a quadric ρ', it follows that $\mathbf{c}_3$, $\mathbf{c}_1$, and $\mathbf{c}_1'$ have a point A, and $\mathbf{c}_3$, $\mathbf{c}_1$, and $\mathbf{c}_1''$ a point B in common.

The line $\mathbf{c}_1$ represents a cubic scroll of lines meeting all five planes; $\mathbf{c}_3$ represents a cubic scroll of lines meeting π_r only; and $\mathbf{c}_1'$, $\mathbf{c}_1''$ represent

similar scrolls of lines meeting π_r and π_4, π_5 respectively. The points A and B correspond to common generators of three of the scrolls.

12. If there is a sixth plane π_6 skew to the first five, its section l_6 by ω must belong to the same regulus as l_r. The corresponding plane ϖ''' meets ϖ' and ϖ'' in base-lines $\mathbf{c}_1'$, $\mathbf{c}_1''$ concurrent with the intersection $\mathbf{c}_1'''$ of ϖ' and ϖ''; thus the points A and B of § 11 are coincident; $\mathbf{c}_3$ passes through A and meets the three lines elsewhere (evidently the quadric ρ' containing this configuration must be a cone). All four base-curves correspond to cubic scrolls; in an obvious notation, $\mathbf{c}_3$ represents the scroll (123), $\mathbf{c}_1'$ represents (12356), and so on; and the four scrolls have a common generator.

It is clear from the form of the base-curve that V cannot acquire another plane skew to the first six; for $\mathbf{c}_3$ would break up into a line through A and a conic meeting all four lines; and an arbitrary c_7, which is trisecant to those lines, would not intersect the conic. Hence the plane of the conic, which corresponds to part of a space section of V, would represent a surface of order seven. Thus, combining these results with § 9, we have in conclusion:

A quartic form containing six skew planes (*and therefore thirty-nine nodes*) *must contain a regulus having a generator in each plane*; *moreover, this is the maximum number of skew planes which the form can acquire. Each set of cubic scrolls of the type* (123), (12356), (12364), (12345) *has a generator in common.*

CONCERNING THE GENERALIZATION OF THE YOUNG-HAUSDORFF THEOREM AND THE HARDY-LITTLEWOOD THEOREMS ON FOURIER CONSTANTS

By H. P. MULHOLLAND.

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1. *Introduction.*

1.1. The problem with which this paper is concerned is that of generalizing certain theorems on Fourier constants, which may be stated as follows. Let $f(\theta)$ be an integrable function whose complex Fourier coefficients are c_m; thus

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_m e^{m\theta i}.$$

Then the theorem, proved in part by W. H. Young and completed by Hausdorff†, asserts that

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^q d\theta\right)^{1/q} \leqslant \left(\sum_{-\infty}^{\infty} |c_m|^{q'}\right)^{1/q'}, \tag{1.11}$$

$$\left(\sum_{-\infty}^{\infty} |c_m|^q\right)^{1/q} \leqslant \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^{q'}\right)^{1/q'}, \tag{1.12}$$

whenever $$q \geqslant 2, \quad \frac{1}{q}+\frac{1}{q'}=1.$$

Further, there are four theorems due to Hardy and Littlewood‡, of which

† W. H. Young, **18, 19, 20,** and Hausdorff, **10.** An improved version of Hausdorff's proof given by F. Riesz, **15,** and another proof by Hardy and Littlewood, **6,** Theorem 1.

‡ *Loc. cit.*, Theorems 2–7.

the following are two:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^q\,d\theta \leqslant A_q \sum_{-\infty}^{\infty}|c_m|^q\mu^{q-2}, \tag{1.13}$$

$$\sum_{-\infty}^{\infty}|c_m|^p\mu^{p-2} \leqslant A_p \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(\theta)|^p\,d\theta, \tag{1.14}$$

where $$\mu = |m|+1, \quad q \geqslant 2, \quad 1 < p \leqslant 2,$$

$A_q \leqslant (Aq)^q$, and $A_p = A_{p/(p-1)}^{p-1}$. The other two theorems are "reciprocals" of the above, obtainable from them by interchanging sums and integrals. These theorems have been stated in the form of inequalities. To each inequality there corresponds a "convergence" theorem asserting merely that the convergence of the right-hand side implies the existence and convergence of the left; for the theorems above, and others of similar character throughout the paper, we adopt the convention that, whenever an "inequality" theorem is stated, the corresponding "convergence" theorem is implied.

These theorems arose as generalizations of the Parseval and Riesz-Fischer theorems, to which they reduce when $q = q' = 2 = p = p'$; and it is natural to inquire whether they in their turn may be extended so as to have more general functions $\Phi(x)$, $\Psi(x)$ in place of the powers that appear in them†. For the sake of definiteness, let us take (1.11) and (1.12), or rather the corresponding convergence theorems, and state the kind of generalization of them that is sought. We are led to the following:

Let $\Phi(x)/x^2$ be increasing, and let $\Psi(x)$ be defined by the relation‡

$$\Psi^{-1}(x) = x\Phi^{-1}(1/x). \tag{1.15}$$

Then the convergence of $\Sigma\Psi(|c_m|)$ implies the existence and convergence of $\int\Phi|f(\theta)|\,d\theta$, and the convergence of $\int\Psi|f(\theta)|\,d\theta$ implies that of $\Sigma\Phi(|c_m|)$, provided that $\Phi(x)$ satisfies suitable conditions.

The first point to be discussed here is the relation (1.15) between $\Phi(x)$ and $\Psi(x)$, which replaces the relation between the q and q' of the Young-Hausdorff theorem. (1.15) is plainly satisfied by $\Phi(x) = x^q$, $\Psi(x) = x^{q'}$; but so also is

$$\Phi^{-1}(x)\,\Psi^{-1}(x) = x, \tag{1.16}$$

† Similar generalizations of the corresponding theorems on power series form the subject of another paper, **13**.

‡ We use $\Phi^{-1}(x)$ for the inverse of $\Phi(x)$; similarly for other functions.

which is the relation used by Cooper in his generalization of Hölder's inequality†. However, what we clearly require is a connection between the behaviour of $\Psi(x)$ when x is small and that of $\Phi(x)$ when x is large, and *vice versa*; hence (1.15) is appropriate rather than (1.16)‡. Moreover, taking (1.15), I am able to show in §4 (Theorems 1 and 2) that nothing stronger than the theorem stated above can be true; for instance, the convergence of $\Sigma\Psi(|c_m|)\epsilon(|c_m|)$, where $\epsilon(x)$ is increasing and tends to 0 with x, does *not* imply the existence and convergence of $\int \Phi|f(\theta)|\,d\theta$.

1.2. The negative result just mentioned is obtained by using an example belonging to a special type of function for which the coefficients c_m are real, non-negative, and decreasing as $|m|$ increases. Now such functions can be dealt with by comparatively elementary methods, and for them I have found the following generalization of Hausdorff's inequality (1.11):

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f(\theta)|\,d\theta \leqslant K\phi\Big(\sum_{-\infty}^{\infty}\Psi(c_m)\Big). \tag{1.21}$$

K here is a constant depending only on Φ, and $\Phi(x)=x\phi(x)$ is any function [with $\Phi(x)/x^2$ increasing] that is "sub-multiplicative"; *i.e.* that satisfies

$$\Phi(xy)\leqslant\Phi(x)\Phi(y) \quad (\text{all } x,\ y). \tag{1.22}$$

These conditions permit us to take, in particular,

$$\Phi(x)=x^q(1+\log^{+}x)^{\alpha} \quad (\alpha>0);$$

and the result is of interest as an indication of the form of inequality that we may expect to be true in the general case.

Returning to the case of functions $f(\theta)$ not restricted in any special way, we shall see, in §5, that generalizations of Hardy and Littlewood's theorems may be established very simply by the device of integrating them with respect to q (or p). For instance, if we integrate (1.13) with respect to q, we get§

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|f(\theta)|^q-|f(\theta)|^2}{\log|f(\theta)|}\,d\theta \leqslant K\sum_{-\infty}^{\infty}\frac{(\mu|c_m|)^q-(\mu|c_m|)^2}{\mu^2\log(\mu|c_m|)}, \tag{1.23}$$

† Quoted below as (2.37).

‡ Incidentally, we may notice that, with (1.15), convexity of $\Phi(x)$ implies convexity of $\Psi(x)$, and *vice versa*: this is easily proved by considering the concavity of their inverses.

§ I owe this remark to Prof. Littlewood. The integration argument has also been employed by Titchmarsh, in **16**, to get a similar generalization of M. Riesz's theorem on conjugate functions.

where $K = \operatorname{Max} A_t$ $(2 \leqslant t \leqslant q)$. Much more general results are obtainable if we multiply by a function $K(t)$ before integrating. Thus, instead of (1.23), we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H|f(\theta)|\,d\theta \leqslant K \sum_{-\infty}^{\infty} H(\mu|c_m|)\,\mu^{-2}, \tag{1.24}$$

where, by suitable choice of $K(t)$, we can get, for instance, with $\alpha > 0$,

$$H(x) \sim x^q(\log x)^{-\alpha} \quad (x \to \infty).$$

However, the integration process can only give us functions $H(x)$ with *negative* powers of $\log x$ in their asymptotic values; the question arises, therefore, whether there are similar results with positive powers of the logarithm. It is shown in § 7 that such results may be deduced from the theorems already obtained by the integration process. Moreover, these arguments can all be applied equally well to certain refinements on (1.13), (1.14), etc., that have been obtained recently†.

1.3. There remains the problem of generalizing the Young-Hausdorff theorem. Some results in this direction can be found by combining the arguments of § 5 with those of §§ 6 and 7; these are given in § 8. From the convergence point of view they are quite satisfactory; the proposition enunciated in (1.1) is established whenever either $\Phi(x)$ or $1/\Psi(1/x)$ is a function $H(x)$ of the type appearing in (1.24), Theorems 12 and 13 being devoted to this result. In particular, the following is true:

For $q > 2$ and any α, positive or negative, the convergence of

$$\sum_{-\infty}^{\infty} [|c_m|\{1+\overset{+}{\log}(1/|c_m|)\}^{\alpha}]^{q'} \tag{1.31}$$

implies the existence of a function $f(\theta)$, with Fourier coefficients c_m, such that

$$\int_{-\pi}^{\pi} [|f(\theta)|\{1+\overset{+}{\log}|f(\theta)|\}^{\alpha}]^q\,d\theta \tag{1.32}$$

converges.

However, the $\Phi(x)$ used for the more general results cannot be exponentially large as x becomes infinite. This restriction apparently is necessitated by the method and not by the result, since Hardy and

† Hardy and Littlewood, **9.**

Littlewood have proved† that the convergence of $\Sigma|c_m|/\log(1/|c_m|)$ implies that of $\int \exp[k|f(\theta)|]\,d\theta$.

When we come to obtaining a generalization of the *inequalities* (1.11), (1.12), the results are not so satisfying. The inequality (1.21) seems obtainable only when $\Sigma\Psi(|c_m|) \leqslant 1$ (Theorem 14). This restriction can be removed, but at the cost of introducing an added constant on the right-hand side (Theorem 15).

It might be thought that inequalities in a satisfactory form could be found by generalizing one of the known proofs of the Young-Hausdorff theorem, such as that due to F. Riesz or that given by Hardy and Littlewood. But my repeated attempts to do so have all failed. It may be that such proofs, which depend on very nicely adjusted "interpolation" arguments involving Hölder's inequality, are too delicate to be susceptible of generalization; for the available generalizations of Hölder's inequality are rather awkward, and the form for generalizations of Hausdorff's inequalities is somewhat narrowly prescribed.

Before proceeding to the details of the work, I should like to express my gratitude to Prof. Littlewood for suggesting to me the problems of this paper originally, and for much helpful advice since.

2. *Certain elementary results on inequalities.*

2.1. There are various results that will be needed later, and which it is convenient to deal with in the present paragraph.

We take first the generalization of the inequality‡

$$\sum_1^\infty (a_1+\ldots+a_n)^k n^{-c} \leqslant K \sum_1^\infty (na_n)^k n^{-c}, \quad (k \geqslant 1,\ c > 1), \tag{2.11}$$

where K depends only on k and c. The result may be stated as follows§:

LEMMA 1. *Let $\Phi(x)$ be a non-negative function such that $\Phi(x)/x$ is increasing, and $\Phi(x)/x^k$ decreasing for some $k > 1$. Then, if $c > 1$,*

$$\sum_1^\infty \Phi(a_1+a_2+\ldots+a_n)\, n^{-c} \leqslant K \sum_1^\infty \Phi(na_n)\, n^{-c}, \tag{2.12}$$

where K depends only on Φ and c.

---

† 9, Theorem 10.

‡ *Cf.* Hardy and Littlewood, 7, Theorem 1.

§ It remains valid when the two conditions on $\Phi(x)$ are slightly widened. The conditions are then necessary as well as sufficient for (2.12); these results, as also an alternative proof of (2.12) and various extensions of it, may possibly find a place in some future communication.

The proof to be given here proceeds on the same general lines as that of (2.11)†. We write

$$s_n = a_1 + a_2 + \ldots + a_n,$$

$$\sigma_n = n^{-c} + (n+1)^{-c} + \ldots < c(c-1)^{-1} n^{1-c} = c' n^{1-c}.$$

Then, applying Abel's identity, with $s_0 = 0$,

$$\sum_1^N \Phi(s_n) n^{-c} = \sum_1^N \Phi(s_n)(\sigma_n - \sigma_{n+1}) \leqslant \sum_1^N \{\Phi(s_n) - \Phi(s_{n-1})\} \sigma_n.$$

But, since $f(x) = \Phi(x)/x^k$ is decreasing, we have

$$\frac{\Phi(s_n) - \Phi(s_{n-1})}{s_n - s_{n-1}} = \frac{s_n{}^k f(s_n) - s_{n-1}^k f(s_{n-1})}{s_n - s_{n-1}}$$

$$\leqslant f(s_n) \frac{s_n{}^k - s_{n-1}^k}{s_n - s_{n-1}} \leqslant f(s_n)\, k s_n^{k-1};$$

and so, writing $\phi(x)$ for $\Phi(x)/x$,

$$(2.13) \qquad \sum_1^N \Phi(s_n) n^{-c} \leqslant k \sum_1^N \phi(s_n)(s_n - s_{n-1}) \sigma_n$$

$$\leqslant kc' \sum_1^N \phi(s_n)\, n a_n\, n^{-c}.$$

Now, since $\phi(x)$ is increasing, we have‡

$$(2.14) \qquad \phi(s_n)\, n a_n = t^{-1}\{t n a_n \phi(s_n)\} \quad (t \geqslant 1),$$

$$\leqslant t^{-1} \operatorname{Max}\{t n a_n \phi(n a_n),\ s_n \phi(s_n)\}$$

$$\leqslant t^{-1}\{\Phi(t n a_n) + \Phi(s_n)\} \leqslant t^{k-1} \Phi(n a_n) + t^{-1} \Phi(s_n).$$

Substituting this in (2.13), we get

$$\sum_1^N \Phi(s_n) n^{-c} \leqslant kc' \left\{ t^{k-1} \sum_1^N \Phi(n a_n) n^{-c} + t^{-1} \sum_1^N \Phi(s_n) n^{-c} \right\}.$$

Hence

$$(2.15) \qquad \sum_1^N \Phi(s_n) n^{-c} \leqslant \frac{kc'\, t^{k-1}}{1 - kc'\, t^{-1}} \sum_1^N \Phi(n a_n) n^{-c}.$$

In this we may give t any value greater than kc', and obtain a result differing

† Hardy and Littlewood, *loc. cit.*

‡ The inequality $x\phi(y) \leqslant \Phi(x) + \Phi(y)$, which may also be written $xz \leqslant x\phi(x) + z\phi^{-1}(z)$, is really a simpler, though cruder, version of Young's inequality; *cf.* **17**, § 2.

from (2.12) only in respect to the range of the series. The result for infinite series can then be deduced by letting the number of terms become infinite first on the right and then on the left of (2.15).

The integral analogue of (2.12) may be proved similarly, the result being

LEMMA 2. *With $\Phi(x)$ as in Lemma 1, and*

$$F(x) = \int_0^x f(t)\,dt \quad [f(x) \geqslant 0], \tag{2.16}$$

we have

$$\int_0^a \Phi[F(x)]\,x^{-c}\,dx \leqslant K \int_0^a \Phi[xf(x)]\,x^{-c}\,dx. \tag{2.17}$$

2.2. We consider next a certain class of functions obtained by integral transformation of the function x^t. Let $K(t)$ be a non-negative function, defined for $0 \leqslant a < t < b$, which is such that the integral

$$\int_a^b K(t)\,dt \tag{2.211}$$

is positive and finite. Define

$$H(x) = H_K(x) = \int_a^b K(t)\,x^t\,dt. \tag{2.212}$$

Then, in the first place, $H(x)$ increases from 0 to ∞ with x. Again, we have

$$H'(x) = \int_a^b K(t)\,tx^{t-1}\,dt, \tag{2.213}$$

since the latter integral clearly converges uniformly in x in any finite interval, and $K(t)$ is positive. Further, if $a \geqslant 1$, $H'(x)$ is increasing and hence $H(x)$ is convex. Next, it is clear that $H(x)/x^a$ is increasing, but $H(x)/x^b$ decreasing; and so we have

$$t^a H(x) \leqslant H(tx) \leqslant t^b H(x) \quad (t \geqslant 1), \tag{2.221}$$

$$\tau^b H(x) \leqslant H(\tau x) \leqslant \tau^a H(x) \quad (\tau \leqslant 1). \tag{2.222}$$

There is a further property of the functions $H(x)$ to establish which we require the following result, due to Tchebycheff†.

† A proof by Franklin is given in **4**; it is quoted by Goursat, **5**.

LEMMA 3. *If $A(t)$ and $B(t)$ are non-negative and monotonic, and if $u(t)$ is non-negative, in (a, b), then we have, whenever the integrals exist,*

$$\frac{\int_a^b A(t)\,B(t)\,u(t)\,dt}{\int_a^b B(t)\,u(t)\,dt} \geqslant \frac{\int_a^b A(t)\,u(t)\,dt,}{\int_a^b u(t)\,dt}, \tag{2.23}$$

when $A(t)$ and $B(t)$ are monotonic in the same sense, but the opposite is true when they are monotonic in different senses.

The lemma enables us to prove that the $H(x)$ of (2.212) possesses the property

$$H(x)/H(y) \leqslant H(tx)/H(ty) \quad (x \geqslant y > 0,\ t \geqslant 1). \tag{2.24}$$

It is clear that we require

$$\frac{\int_a^b x^s t^s K(s)\,ds}{\int_a^b y^s t^s K(s)\,ds} \geqslant \frac{\int_a^b x^s K(s)\,ds}{\int_a^b y^s K(s)\,ds}.$$

But this follows from the lemma by writing $A(s) = (x/y)^s$, $B(s) = t^s$, and $u(s) = y^s K(s)$; for, with $x > y$ and $t > 1$, $A(s)$ and $B(s)$ are both increasing. We may sum up the results so far obtained as follows:

LEMMA 4. *The function $H(x)$, defined in (2.212), is increasing, convex when $a \geqslant 1$, and concave when $b \leqslant 1$. Further, $H(x)$ satisfies the inqualities (2.221), (2.222), and (2.24).*

Coming now to the asymptotic behaviour of $H(x)$, we have, if $K(t) = 1$,

$$H(x) = \int_a^b x^t\,dt = \frac{x^b - x^a}{\log x}$$

$$\sim \frac{x^b}{\log x} \qquad (x \to \infty)$$

$$\sim \frac{x^a}{\log(1/x)} \quad (x \to 0).$$

More generally we find the following result.

LEMMA 5. *If $\alpha > 0$ and*

$$K(t) = (q-t)^{\alpha-1}/\Gamma(\alpha) \quad (a \leqslant t < q), \tag{2.251}$$

we have, as $x \to \infty$,

$$H_K(x) = \frac{1}{\Gamma(\alpha)} \int_a^q (q-t)^{\alpha-1} x^t \, dt \sim \frac{x^q}{(\log x)^\alpha}; \tag{2.252}$$

while, if

$$K(t) = (t-p)^{\alpha-1}/\Gamma(\alpha) \quad (p < t \leqslant b), \tag{2.261}$$

we have, as $x \to 0$,

$$H_K(x) = \frac{1}{\Gamma(\alpha)} \int_p^b (t-p)^{\alpha-1} x^t \, dt \sim \frac{x^p}{[\log(1/x)]^\alpha}. \tag{2.262}$$

For (2.252), we observe that

$$H_K(x) = \frac{x^q}{\Gamma(\alpha)(\log x)^\alpha} \int_0^{(q-a)\log x} u^{\alpha-1} e^{-u} \, du,$$

and the integral tends to $\Gamma(\alpha)$ as $x \to \infty$. The other result comes similarly.

When $\alpha = 1$, these results reduce to the case considered to start with, where $H(x)$ was obtained by integration of x^t with respect to t; and, in fact, the $H_K(x)$ defined in (2.252) is the repeated integral of order α (in the sense of Riemann and Liouville) of x^t. Clearly, therefore, if we wish to get an $H(x)$ with a power of $\log\log x$ in its asymptotic value, we must integrate the $H_K(x)$ of (2.252) *with respect to* α. This leads to the following lemma, which may be proved without difficulty.

LEMMA 6. *If we take*

$$K(t) = \frac{1}{\Gamma(\beta)} \int_\alpha^\infty \frac{(u-\alpha)^{\beta-1}(q-t)^{u-1}}{\Gamma(u)} \, du, \tag{2.271}$$

then, as $x \to \infty$,

$$H_K(x) = \int_a^q K(t) x^t \, dt \sim \frac{x^q}{(\log x)^\alpha (\log\log x)^\beta}. \tag{2.272}$$

There is, of course, a similar result when the range of integration with respect to t is (p, b) as in (2.262). Moreover, the process may clearly be carried further, so as to give powers of $\log\log\log x$, etc.

2.3. We now investigate functions with certain properties for which it is convenient to have names, and which may be defined as follows:

A function $f(x)$ *is said to be "sub-multiplicative" if*

$$f(xy) \leqslant f(x) f(y) \quad (\textit{all } x, y); \tag{2.311}$$

but to be "super-multiplicative" if

$$f(xy) \geqslant f(x)\,f(y) \quad (\textit{all } x, y). \tag{2.312}$$

It is well known that any continuous function satisfying both these inequalities must be a power of x; accordingly, it will be assumed that the functions considered satisfy only one of them. We state first the formal properties connected with (2.311) and (2.312); these are all obvious and will be left without proof. We suppose that both variables and functions are non-negative, and that $f^{-1}(x)$ is the inverse of $f(x)$.

(2.321) *If $f(x)$ is sub-multiplicative, then $1/f(x)$ and $f^{-1}(x)$ are super-multiplicative, and vice versa.*

(2.322) *If $f(x)$ and $g(x)$ are sub-multiplicative, so also is $f(x)\,g(x)$, and, when $f(x)$ is increasing, $f[g(x)]$. Similarly for super-multiplicative functions.*

(2.323) *If $f(x)$ is sub-multiplicative,*

$$f(x/y) \geqslant f(x)/f(y),$$

while, if $f(x)$ is super-multiplicative,

$$f(x/y) \leqslant f(x)/f(y).$$

It is of importance, in view of applications, to know what asymptotic properties sub-multiplicative functions can have. Now a sub-multiplicative function $f(x)$ satisfies $f(2x) \leqslant f(2)\,f(x)$, and it follows, by repeated application of this, that, if $f(x)$ is increasing,

$$f(x) \prec x^k \quad (x \to \infty).$$

This result and others similar have been obtained directly by Cooper; he proves further a theorem† which may be stated as follows:

If $f(x)$ is sub-multiplicative, and monotonic for large values of x, then $\log f(x)/\log x$ tends to a finite limit as $x \to \infty$; while, if $f(x)$ is monotonic for sufficiently small values of x, then $\log f(x)/\log x$ tends to a finite limit as $x \to 0$.

Though $f(x)$ is equal to a power of x when $\log f(x)/\log x$ is constant, this result does not mean that any sub-multiplicative $f(x)$ must be virtually a power of x as far as asymptotic properties are concerned; for it is clear that the first of Cooper's two relations is satisfied by $x^{\alpha}(\log x)^{\beta}$, and it will be found that sub-multiplicative functions of this type may

† 3, Theorem VIII.

readily be constructed. In the first place, we have

LEMMA 7. *The following functions are sub-multiplicative*:

$$1+x, \tag{2.331}$$

$$1+a\overset{+}{\log}x, \quad 1+a\overset{+}{\log}(1/x), \quad 1+a|\log x|, \tag{2.332}$$

$$1+a\log(1+x), \quad 1+a\log[1+(1/x)], \tag{2.333}$$

where $a>0$.

For the first part we have

$$(1+x)(1+y)=1+x+y+xy\geqslant 1+xy;$$

for the second

$$(1+a\overset{+}{\log}x)(1+a\overset{+}{\log}y)\geqslant 1+a\overset{+}{\log}x+a\overset{+}{\log}y\geqslant 1+a\overset{+}{\log}xy;$$

and the other two cases are similar. (2.333) comes by combining (2.331) and (2.332) in accordance with (2.322)†. From the above we get

LEMMA 8. *If* $c>0$, $a>0$, *the functions*

$$x^c[1+a\overset{+}{\log}x]^a, \quad x^c[1+a\overset{+}{\log}(1/x)]^a, \tag{2.341}$$

$$x^c[1+a\log(1+x)]^a, \quad x^c[1+a\log\{1+(1/x)\}]^a, \tag{2.342}$$

are sub-multiplicative when $\alpha>0$ *and super-multiplicative when* $\alpha<0$.

It is easy to construct more complicated examples with powers of $\log\log x$, etc.; for instance:

LEMMA 9. *If* $c>0$, $a, b>0$, *the function*

$$x^c[1+a\overset{+}{\log}x]^\alpha[1+b\log(1+a\overset{+}{\log}x)]^\beta \tag{2.35}$$

is sub-multiplicative when $\alpha, \beta>0$, *but super-multiplicative when* $\alpha, \beta<0$.

It is sufficient to prove this for the factor

$$[1+b\log(1+a\overset{+}{\log}x)]^\beta.$$

Now, when $\beta>0$, this is an increasing sub-multiplicative function of $1+a\overset{+}{\log}x$, which is itself sub-multiplicative; whence the result, by (2.322). Similarly when $\beta<0$.

Again, we have, for $0\leqslant\kappa\leqslant 1$,

$$(\overset{+}{\log}xy)^\kappa\leqslant(\overset{+}{\log}x+\overset{+}{\log}y)^\kappa\leqslant(\overset{+}{\log}x)^\kappa+(\overset{+}{\log}y)^\kappa,$$

† Or also directly; *cf.* 12, Lemma 1.

and it follows that the function $\exp[(\overset{+}{\log} x)^\kappa]$ is sub-multiplicative; hence, if $0 \leqslant \kappa \leqslant 1$, the function

$$x^c \exp[\beta(\overset{+}{\log} x)^\kappa] \tag{2.36}$$

is sub-multiplicative when $\beta \geqslant 0$.

Further examples may clearly be constructed to an unlimited extent; in each case functions like (2.35) will be sub-multiplicative when the indices of the factors with logarithms and repeated logarithms are all positive, but super-multiplicative when the indices are all negative.

Lastly, it is convenient in applications to assume that $f(x)$ is increasing, and sometimes also convex (though the latter restriction is not required for the results in this paper). As an instance, it may be mentioned that, when $f(x)$ is increasing, convex, and super-multiplicative, we have Cooper's generalization† of Hölder's inequality:

$$f[\Sigma ab] \leqslant \frac{f^{-1}[\Sigma f(a)]}{g^{-1}[1/\Sigma g(b)]}, \tag{2.37}$$

where $f^{-1}(x)g^{-1}(x) = x$. In the case of functions such as those considered in the lemmas above the conditions under which they are increasing (or convex) may be determined most simply by inspection of their first and second derivatives. Some typical results are given below. The conditions stated are sufficient for the properties in question, and, in so far as they restrict c and a, are also necessary; but the restrictions on a are given in the simplest available form and are not always necessary.

$$f(x) \equiv x^c(1+a\overset{+}{\log} x)^\alpha \quad (c > 0,\ a \geqslant 0) \tag{2.381}$$

is

(i) *sub-multiplicative when* $\alpha \geqslant 0$, *super-multiplicative when* $\alpha \leqslant 0$;

(ii) *increasing when* $\alpha \geqslant 0$, *or* $\alpha < 0$, $a < c/(-\alpha)$;

(iii) *convex when* $c \geqslant 1$; $\alpha \geqslant 1$, *or* $0 \leqslant \alpha < 1$, $a \leqslant 1$.

$$f(x) \equiv x^c[1+a\overset{+}{\log}(1/x)]^\alpha \quad (c,\ a > 0) \tag{2.382}$$

is

(i) *sub-multiplicative when* $\alpha \geqslant 0$, *super-multiplicative when* $\alpha \leqslant 0$;

(ii) *increasing when* $\alpha > 0$, $a < c/a$, *or* $\alpha \leqslant 0$;

(iii) *convex when* $c > 1$, $\alpha \geqslant 0$, $a \leqslant \mathrm{Min}\{1, (c-1)/2a\}$.

† 3, § 9.

For functions of more complicated form, such as the function

$$x^c[1+a\overset{+}{\log}x]^\alpha[1+b\log(1+a\overset{+}{\log}x)]^\beta$$

of (2.35), the calculations are much more involved, but it is clear that, if a and b are taken sufficiently small, the functions will have the required properties, subject to one or two obvious restrictions†.

3. *Elementary properties of functions of certain special types.*

3.1. We consider first functions

$$f(\theta)=\sum_0^\infty c_n e^{n\theta i}, \tag{3.11}$$

for which the coefficients c_n are non-negative, and decreasing (in the broad sense) to limit 0 as $|n|$ increases to ∞. The series (3.11) converges, uniformly in θ, for $|\theta|\geqslant\delta>0$, since it clearly satisfies Dirichlet's test. It follows‡ that, when $|f(\theta)|$ is integrable over $(-\pi,\pi)$, the series in (3.11) must be the Fourier series of $f(\theta)$.

LEMMA 10§. *If $f(\theta)$ is defined as above, and*

$$S_n=\sum_0^n c_m, \tag{3.12}$$

then

$$|f(\theta)|\leqslant S_n+2(n+1)c_{n+1}\quad(n\geqslant 1), \tag{3.13}$$

for $|\theta|$ in $\{\pi/(n+1),\pi/n\}$.

We have

$$|f(\theta)|=\left|\sum_0^\infty c_m e^{m\theta i}\right|\leqslant\sum_0^n c_m+\left|\sum_{n+1}^\infty c_m e^{m\theta i}\right|, \tag{3.14}$$

and, applying Abel's identity to the last sum, we have, since $c_m\to 0$ and

$$|S_m|\equiv\left|\sum_1^m e^{r\theta i}\right|=\left|\frac{1-e^{m\theta i}}{1-e^{\theta i}}\right|$$

† *E.g.*, the functions can only be convex if $c\geqslant 1$, and can only be concave if $c\leqslant 1$.

‡ In fact, we may obtain the n-th Fourier coefficient of $f(\theta)$ by multiplying the series in (3.11) by $e^{-n\theta i}$ and integrating term-by-term: for the legitimacy of this operation, *cf.* Hobson, **11**, §215.

§ This lemma is taken from Hardy and Littlewood's work; see **9**, Theorem 5.

is bounded,

$$\left|\sum_{n+1}^{\infty} c_m e^{m\theta i}\right| = \left|\sum_{n+1}^{\infty} S_m(c_m - c_{m+1}) - S_n c_{n+1}\right|$$

$$\leqslant \operatorname*{Max}_{m \geqslant n} |S_m| \left\{c_{n+1} + \sum_{n+1}^{\infty} (c_m - c_{m+1})\right\}$$

$$\leqslant 4c_{n+1}/|1-e^{\theta i}| \leqslant 2c_{n+1}(n+1),$$

provided that $\pi/(n+1) \leqslant |\theta| \leqslant \pi/n$. Combining this result with (3.14), we have (3.13).

LEMMA 11. *With $f(\theta)$ defined in the same way as in Lemma 1, we have*

$$|f(\theta)| \geqslant 2n(c_n - c_{3n})/\pi \quad (n \geqslant 1), \tag{3.15}$$

when

$$\pi/2(n+1) < |\theta| \leqslant \pi/2n; \tag{3.16}$$

and

$$|f(\theta)| \geqslant (c_0 - c_1)/\pi \quad (\tfrac{1}{2}\pi < |\theta| < \pi). \tag{3.17}$$

We write

$$\sigma_m \equiv \sum_0^m e^{(r+\frac{1}{2})\theta i} = \frac{i \sin \frac{1}{2}\theta(1-e^{(m+1)\theta i})}{1-\cos\theta},$$

$$\mathfrak{J}(\sigma_m) = \tfrac{1}{2} \operatorname{cosec} \tfrac{1}{2}\theta \{1-\cos(m+1)\theta\}.$$

Then, applying Abel's identity as before, we have

$$|f(\theta)| \geqslant |\mathfrak{J}\{f(\theta) e^{\frac{1}{2}\theta i}\}| = \left|\mathfrak{J}\left\{\sum_0^{\infty} c_m e^{(m+\frac{1}{2})\theta i}\right\}\right|$$

$$= \left|\mathfrak{J} \sum_1^{\infty} \sigma_m (c_m - c_{m+1})\right|$$

$$= \tfrac{1}{2} |\operatorname{cosec} \tfrac{1}{2}\theta| \sum_0^{\infty} (c_m - c_{m+1}) \{1-\cos(m+1)\theta\}$$

$$\geqslant \tfrac{1}{2} |\operatorname{cosec} \tfrac{1}{2}\theta| \sum_n^{3n-1} (c_m - c_{m+1}) \{1-\cos(m+1)\theta\}$$

$$\geqslant \tfrac{1}{2} |\operatorname{cosec} \tfrac{1}{2}\theta| \sum_n^{3n-1} (c_m - c_{m+1})$$

$$\geqslant (c_n - c_{3n})/|\theta| \geqslant 2n(c_n - c_{3n})/\pi,$$

where it is supposed that $n \geqslant 1$; the case $n = 0$ presents no difficulty.

3.2. Let us consider now non-negative integrable functions $f(\theta)$ which are decreasing in $(0, 2\pi)$; their Fourier coefficients are given by

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, e^{-m\theta i}\, d\theta. \tag{3.21}$$

For these functions we have

LEMMA 12. *If $f(\theta)$ and c_m are defined as above, then*

$$2\pi|c_n| \leqslant \int_0^{\theta} f(\phi)\, d\phi + \frac{4}{\pi}\,\theta f(\theta), \tag{3.22}$$

when

$$\begin{cases} \pi/|n| \leqslant \theta \leqslant \pi/(|n|-1) & (|n|>1), \\ \pi \leqslant \theta \leqslant \frac{3}{2}\pi \quad (|n|=1); & \frac{3}{2}\pi \leqslant \theta \leqslant 2\pi \quad (n=0). \end{cases} \tag{3.23}$$

LEMMA 13.

$$2\pi|c_n| \geqslant \theta[f(\theta)-f(3\theta)]/2\pi, \tag{3.24}$$

when

$$\begin{cases} \pi/2(|n|+1) \leqslant \theta \leqslant \pi/2|n| & (|n| \geqslant 1), \\ \frac{1}{2}\pi \leqslant \theta \leqslant 2\pi & (n=0). \end{cases} \tag{3.25}$$

The proofs of these two lemmas proceed on the same lines as those of Lemmas 10 and 11.

4. *Negative results concerning extensions of the Young-Hausdorff theorem.*

4.1. We consider continuous, strictly increasing functions $\Phi(x)$ and $\Psi(x)$ such that $\Phi(x)/x$ and $\Psi(x)/x$ are increasing, and

$$\Psi^{-1}(x) = x\Phi^{-1}(1/x). \tag{4.11}$$

These conditions are symmetrical as between $\Phi(x)$ and $\Psi(x)$, and correspond to the conditions

$$q>1, \quad q'>1, \quad \frac{1}{q}+\frac{1}{q'}=1,$$

satisfied by the powers appearing in Hausdorff's inequalities. Next, corresponding to the restriction $q \geqslant 2$, we suppose that

$$\Phi(x)/x^2 \text{ increases with } x \text{ to the limit } \infty. \tag{4.12}$$

It follows that $\Psi(x)/x^2$ is decreasing. (4.12) is not necessary to the truth of the theorems in this paragraph, but their results are of interest only when it is satisfied.

4.2. THEOREM 1. *Let $\Phi(x)$ and $\Psi(x)$ be defined as above, and let $\epsilon(x)$ be a continuous increasing function which tends to 0 with x. Then there is a series*

$$\sum_1^\infty c_n e^{n\theta i}$$

such that

$$\sum_1^\infty \Psi(|c_n|)\,\epsilon(|c_n|) < \infty, \tag{4.211}$$

but which is not the Fourier series of any function $f(\theta)$ for which†

$$\int_0^{2\pi} \Phi|f(\theta)|\,d\theta < \infty. \tag{4.212}$$

We take a function

$$s(\theta) = \sum_1^\infty c_n e^{n\theta i} \tag{4.213}$$

of the type considered in 3.1, *i.e.* with non-negative coefficients in order of descending magnitude. As was there remarked, the series (4.213) converges uniformly for $|\theta| \geqslant \delta > 0$. Hence, if there *are* any functions with the c_n's as their Fourier coefficients, $s(\theta)$ must be one of them. We shall show that the coefficients c_n of (4.213) can be chosen so as to satisfy (4.211) but to make

$$\int_0^{2\pi} \Phi|s(\theta)|\,d\theta$$

divergent. Since these c_n's are either the Fourier coefficients of $s(\theta)$, or are not the Fourier coefficients of any function at all‡, they provide the example whose existence the theorem asserts. We remark that $\Psi^{-1}(x)$ is increasing and tends to 0 with x; the same has been assumed of $\epsilon(x)$. Hence $\epsilon[\Psi^{-1}(1/x)]$ decreases to the limit 0 as x increases to ∞. Now§ let $m_0 = 0$, and m_1 be the smallest positive integer such that

$$\epsilon\,[\Psi^{-1}(2\pi/m_1)] \leqslant 1.$$

We than define $m_2, m_3, \ldots, m_r, \ldots$, successively as follows: let m_r be the

† Here and throughout the paper integrals with respect to θ are taken in the sense of Lebesgue, although functions of the special types dealt with in the last paragraph are also amenable to Riemann integration.

‡ In the case when $x^2/\Psi(x)\,\epsilon(x) \to 0$ with x, which is the most interesting case, the convergence of $\Sigma\Psi(c_n)\,\epsilon(c_n)$ implies that of Σc_n^2, and hence the existence of a function $f(\theta)$ with Fourier coefficients c_n.

§ This construction was suggested by the proof of a well-known theorem of Pringsheim on series of decreasing positive terms, *cf.* Pringsheim, **14**. The theorem is quoted by Borel, **1**.

smallest integer such that†

$$m_r > 2m_{r-1}, \tag{4.214}$$

$$\Psi^{-1}(2\pi/m_r) < \tfrac{1}{2}\Psi^{-1}(2\pi/m_{r-1}), \tag{4.215}$$

$$\epsilon\,[\Psi^{-1}(2\pi/m_r)] \leqslant 2^{-r}. \tag{4.216}$$

Now let

$$c_{m_r} = \Psi^{-1}(2\pi/m_r), \tag{4.217}$$

$$c_{m_{r-1}+1} = c_{m_{r-1}+2} = \ldots = c_{m_r}. \tag{4.218}$$

Then $c_{m_r} < \frac{1}{2}c_{m_{r-1}}$ and the c_m's form a decreasing sequence tending to the limit 0; further,

$$\sum_{m=1}^{\infty} \Psi(c_m)\,\epsilon(c_m) = \sum_{r=1}^{\infty} (m_r - m_{r-1})\,\Psi(c_{m_r})\,\epsilon(c_{m_r})$$

$$\leqslant \sum_{r=1}^{\infty} m_r \frac{2\pi}{m_r}\frac{1}{2^r} = \sum_{1}^{\infty} \frac{\pi}{2^{r-1}} = 2\pi,$$

so that (4.211) is satisfied. Again, with $s(\theta)$ defined by (4.213), Lemma 11 gives

$$|s(\theta)| \geqslant 2m(c_m - c_{3m})/\pi, \quad \{\pi/2(m+1) < |\theta| \leqslant \pi/2m\};$$

and, since the combined length of these two intervals is $\pi/(m+1)^2 \equiv \pi/\mu^2$, we have

$$\int_0^{2\pi} \Phi|s(\theta)|\,d\theta \geqslant \pi \sum_{1}^{\infty} \Phi\,[2m(c_m - c_{3m})/\pi]\,\mu^{-2}.$$

It is therefore sufficient to show that the series last written is divergent when the c_m's are defined by (4.217) and (4.218). To do this we consider the block of terms in the series for which $m_{r-1} < m \leqslant m_r$, and prove that their sum exceeds a positive constant. Writing n_r for the first integer greater than $\frac{1}{2}m_r$, we have

$$n_r \geqslant \tfrac{1}{2}m_r + \tfrac{1}{2} \geqslant m_{r-1} + 1,$$

$$3n_r \geqslant m_r + 1;$$

and thus, by (4.215),

$$c_{3n_r} \leqslant c_{m_r+1} = c_{m_{r+1}} \leqslant \tfrac{1}{2}c_{m_r}.$$

† If we use the property (4.12), (4.214) can be deduced from (4.215); but this property is not actually necessary to the truth of the theorem.

Hence

$$\sum_{m_{r-1}+1}^{m_r} \Phi[2m(c_m - c_{3m})/\pi]\mu^{-2} \geqslant \sum_{n_r}^{m_r} \Phi[2m(c_{m_r} - c_{3n_r})/\pi]\mu^{-2}$$

$$\geqslant (m_r - n_r)\,\Phi[2n_r c_{m_r}/2\pi](m_r+1)^{-2}$$

$$\geqslant \frac{m_r}{2(m_r+1)^2}\,\Phi\left[\frac{m_r c_{m_r}}{2\pi}\right] = \frac{m_r}{2(m_r+1)^2}\,\Phi\left[\frac{m_r}{2\pi}\,\Psi^{-1}\left(\frac{2\pi}{m_r}\right)\right]$$

$$= \frac{m_r}{2(m_r+1)^2}\,\Phi\left[\Phi^{-1}\left(\frac{m_r}{2\pi}\right)\right] \geqslant \frac{1}{2^2}\,\frac{1}{4\pi},$$

as required.

THEOREM 2. *Let $\Phi(x)$ and $\Psi(x)$ be defined as in* 4.1, *and let $\epsilon(x)$ be a continuous decreasing function which tends to* 0 *as $x \to \infty$. Then there is a function $f(\theta)$, with Fourier coefficients c_n, such that*

$$\int_0^{2\pi} \Psi\{|f(\theta)|\}\,\epsilon\{|f(\theta)|\}\,d\theta < \infty, \tag{4.221}$$

but

$$\sum_{-\infty}^{\infty} \Phi(|c_n|) = \infty. \tag{4.222}$$

The proof runs on the same lines as that of Theorem 1, and will be omitted.

5. *Results for functions of the special types.*

5.1. We proceed to show how generalizations of the Young-Hausdorff theorem and of the Hardy-Littlewood theorems can be obtained, which are valid for functions of one or other of the special types considered in § 2. For the sake of brevity, the results are given only for functions of the first special type; the results, and the proofs, for those of the second type are very similar. We give first the generalization of (1.13), and then deduce from it the generalization of (1.11).

5.2. THEOREM 3†. *Let $\Phi(x)$ be a function such that $\Phi(x)/x$ is increasing and $\Phi(x)/x^k$ decreasing for some $k > 1$. Then, if the coefficients c_m are non-negative and decreasing as $|m|$ increases, and if*

$$f(\theta) = \sum_{-\infty}^{\infty} c_m e^{m\theta i}, \tag{5.21}$$

† The special case when $\Phi(x) = x^k$ $(k > 1)$ is given by Hardy and Littlewood, **9**, Theorem 5; and the proof above is adapted from that of the special case.

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f(\theta)|\,d\theta \leqslant K\sum_{-\infty}^{\infty}\Phi(\mu c_m)\,\mu^{-2}, \tag{5.22}$$

where $\mu = |m|+1$, *K is a constant depending only on* Φ, *and the convergence of the right-hand side of* (5.22) *implies that of the left*; *it also implies that the* c_m*'s are the Fourier coefficients of* $f(\theta)$.

We remark that it is sufficient to prove the result for functions

$$f_1(\theta) = \sum_{0}^{\infty} c_m e^{m\theta i}.$$

For
$$f(\theta) = \sum_{-\infty}^{\infty} c_m e^{m\theta i} = \sum_{0}^{\infty} + \sum_{-\infty}^{-1} = f_1(\theta)+f_2(-\theta),$$

where $f_2(\theta)$ is another function equal to a series of positive powers of $e^{\theta i}$. But, since $\Phi(x)/x^k$ is decreasing,

$$\begin{aligned}\Phi|f(\theta)| &\leqslant \Phi[|f_1(\theta)|+|f_2(-\theta)|] \leqslant \Phi[2\,\mathrm{Max}\{|f_1(\theta)|,\ |f_2(-\theta)|\}]\\ &\leqslant 2^k\,\mathrm{Max}\{\Phi|f_1(\theta)|,\ \Phi|f_2(-\theta)|\} \leqslant 2^k\{\Phi|f_1(\theta)|+\Phi|f_2(-\theta)|\};\end{aligned}$$

and hence, if (5.22) has been established for functions such as f_1 and f_2,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f(\theta)|\,d\theta &\leqslant 2^k\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f_1(\theta)|\,d\theta+\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f_2(\theta)|\,d\theta\right)\\ &\leqslant 2^k K\left(\sum_{0}^{\infty}\Phi(\mu c_m)\,\mu^{-2}+\sum_{1}^{\infty}\Phi[\mu c_{-m}]\,\mu^{-2}\right) = K'\sum_{-\infty}^{\infty}\Phi(\mu c_m)\,\mu^{-2}.\end{aligned}$$

Since the precise value of K' is immaterial, this is sufficient. Now, for $f_1(\theta)$, we have, by Lemma 11,

$$\begin{aligned}|f_1(\theta)| &\leqslant \sum_{0}^{m} c_n + 2(m+1)\,c_{m+1} \quad [\pi/(m+1) \leqslant |\theta| \leqslant \pi/m]\\ &\leqslant 3\sum_{0}^{m} c_n = 3s_m \leqslant 4s_m,\end{aligned}$$

since c_m is decreasing. But the combined length of the two intervals is $2\pi/m(m+1)$; therefore

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f_1(\theta)|\,d\theta &\leqslant \sum_{1}^{\infty}\Phi(4s_m)/m(m+1)\\ &\leqslant 2^{2k+1}\sum_{0}^{\infty}\Phi(s_m)\,\mu^{-2} \leqslant 2^{2k+1}K\sum_{0}^{\infty}\Phi(\mu c_m)\,\mu^{-2},\end{aligned} \tag{5.23}$$

by Lemma 1. It is now plain that the integral in (5.22) must converge whenever $\Sigma\Phi(\mu c_m)\mu^{-2}$ converges. Lastly, since $\Phi(x)/x$ is increasing, $|f(\theta)|$ is integrable at the same time as $\Phi|f(\theta)|$; hence, as remarked in §3.1, the c_m's must be the Fourier coefficients of $f(\theta)$.

5.3. If, in addition to the restrictions already laid on $\Phi(x)$, we suppose that it is sub-multiplicative (or super-multiplicative), a generalization of the Young-Hausdorff theorem may be obtained which is valid for functions of the two special types.

THEOREM 4. *Let $\Phi(x) \equiv x\phi(x)$ be a function such that $\phi_1(x) \equiv \Phi(x)/x^2$ is increasing, but $\Phi(x)/x^k$ $(k > 2)$ is decreasing; and let $\Psi(x)$ be defined by*

$$\Psi^{-1}(x) = x\Phi^{-1}(1/x). \tag{5.31}$$

Suppose that

$$f(\theta) = \sum_{-\infty}^{\infty} c_m e^{m\theta i},$$

where the c_m's are non-negative and decreasing as $|m|$ increases; and that

$$S = \sum_{-\infty}^{\infty} \Psi(c_m). \tag{5.32}$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi|f(\theta)|\,d\theta \leqslant K \operatorname{Max}(S^{k-1}, S). \tag{5.33}$$

Further, if $\Phi(x)$ is sub-multiplicative,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi|f(\theta)|\,d\theta \leqslant K\phi(S), \tag{5.34}$$

and, if $\Phi(x)$ is super-multiplicative,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi|f(\theta)|\,d\theta \leqslant \frac{K}{\phi(1/S)}, \tag{5.35}$$

where K is a constant depending only on Φ, and the convergence of the right-hand side of each inequality implies that $\Phi|f(\theta)|$ is integrable; it also implies that the c_m's are the Fourier coefficients of $f(\theta)$.

In virtue of Theorem 3, it is sufficient to prove that

$$\sum_{-\infty}^{\infty} \Phi(\mu c_m)\mu^{-2}$$

is convergent and not greater than the respective expressions on the right whenever

$$S = \sum_{-\infty}^{\infty} \Psi(c_m) \equiv \sum_{-\infty}^{\infty} u_m$$

is convergent. Now, in the first place, the conditions imposed on $\Phi(x)$ clearly imply† that

$$\Phi(tx) \leqslant \Phi(x)\,\mathrm{Max}(t^2,\, t^k),$$

and hence

$$\begin{aligned}\Phi(\mu c_m) &= \Phi[\mu\Psi^{-1}(u_m)] = \Phi[\mu u_m \Phi^{-1}(1/u_m)] \\ &\leqslant (1/u_m)\,\mathrm{Max}\,[(\mu u_m)^2,\ (\mu u_m)^k] = \mu^2 u_m\,\mathrm{Max}\,[1,\ (\mu u_m)^{k-2}] \\ &\leqslant \mu^2 u_m\,\mathrm{Max}\,[1,\ S^{k-2}];\end{aligned}$$

since u_m decreases as $|m|$ increases, and so

$$\mu u_m = (|m|+1)\,u_m \leqslant u_m + u_{m-1} + \ldots + u_0 \leqslant S.$$

We now have

$$\begin{aligned}\sum_{-N}^{N} \Phi(\mu c_m)\,\mu^{-2} &\leqslant \sum_{-N}^{N} u_m\,\mathrm{Max}\,[1,\ S^{k-2}] \\ &\leqslant S\,\mathrm{Max}\,[1,\ S^{k-2}] = \mathrm{Max}\,[S,\ S^{k-1}],\end{aligned}$$

so that the series on the left converges to a value not greater than the expression last written. Secondly, supposing $\Phi(x)$ to be sub-multiplicative,

$$\begin{aligned}\Phi(\mu c_m) &= \Phi[\mu u_m \Phi^{-1}(1/u_m)] \\ &\leqslant \Phi(\mu u_m)(1/u_m) = \mu^2 u_m \phi_1(\mu u_m) \\ &\leqslant \mu^2 u_m \phi_1(S),\end{aligned}$$

since $\phi_1(x)$ is increasing and $\mu u_m \leqslant S$. Thus

$$\begin{aligned}\sum_{-N}^{N} \Phi(\mu c_m)\,\mu^{-2} &\leqslant \sum_{-N}^{N} u_m \phi_1(S) \\ &\leqslant S\phi_1(S) = \phi(S).\end{aligned}$$

When $\Phi(x)$ is super-multiplicative, the proof is similar.

6. *Theorems derived by the integration argument.*

6.1. We consider next the results that are obtainable by integrating known inequalities with respect to the index p or q appearing in them. Such results will be stated in terms of the functions $H_K(x)$ discussed in

† For, if $t \geqslant 1$, $\Phi(tx)/t^k x^k \leqslant \Phi(x)/x^k$, since $\Phi(x)/x^k$ is decreasing: thus $\Phi(tx) \leqslant t^k \Phi(x)$. Similarly for $t \leqslant 1$.

§2.2; more precisely, in terms of the functions

$$H(x) = \int_2^q K(t)\, x^t\, dt \quad (q > 2), \tag{6.11}$$

$$H_1(x) = \int_q^{2q} K_1(t)\, x^t\, dt; \tag{6.12}$$

$$G(x) = \int_{\frac{1}{2}(p+1)}^{p} K(t)\, x^t\, dt \quad (1 < p < 2), \tag{6.13}$$

$$G_1(x) = \int_p^2 K_1(t)\, x^t\, dt; \tag{6.14}$$

where the functions $K(t)$ and $K_1(t)$ are non-negative functions, whose integrals over their respective ranges are equal to 1. It will be recalled† that, by choosing $K(t)$ suitably, we can make $H(x)$ and $G(x)$ asymptotically equal to $x^q(\log x)^{-\alpha}$ and $x^p(\log x)^{-\alpha}$ $(\alpha > 0)$, respectively, for large x; and also to more complicated expressions with negative powers of $\log\log x$, etc. The corresponding results for $H_1(x)$ and $G_1(x)$ have $\log(1/x)$ in place of $\log x$.

Now, let us take the Hardy-Littlewood theorem (1.13), namely

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^q\, d\theta \leqslant A_q \sum_{-\infty}^{\infty} |c_m|^q\, \mu^{q-2} \quad (q \geqslant 2),$$

where $A_q = (Aq)^q$ and $\mu = |m| + 1$. It follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^t\, d\theta \leqslant A_q \sum_{-\infty}^{\infty} |c_m|^t\, \mu^{t-2},$$

when $2 \leqslant t \leqslant q$, if we suppose that the series converges for $t = q$. Hence, multiplying by $K(t)$ and integrating with respect to t from 2 to q, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H|f(\theta)|\, d\theta \leqslant A_q \sum_{-\infty}^{\infty} H(\mu|c_m|)\, \mu^{-2},$$

provided that the inversion of the order of integration on the left and the term-by-term integration on the right are legitimate. As regards the former, we observe first that the integrand is non-negative, and

$$\int_{-\pi}^{\pi} |f|^t\, d\theta = \int_{|f|\geqslant 1} |f|^t\, d\theta + \int_{|f|\leqslant 1} |f|^t\, d\theta.$$

The first integral on the right is an increasing, and the second a decreasing, function of t. Thus the integral on the left, which we know to be bounded, is integrable with respect to t; and, on multiplying it by $K(t)$ and integrating

† From §2.2, Lemma 5.

from 2 to q, we get an integral which is majorized by

$$\int_2^q K(t)\left\{\int_{|f|\geqslant 1}|f(\theta)|^q d\theta+\int_{|f|\leqslant 1}|f(\theta)|^2 d\theta\right\} dt.$$

Since this integral certainly exists, the repeated integral

$$\int_2^q K(t)\,dt\int_{-\pi}^{\pi}|f(\theta)|^t d\theta$$

must exist and be equal to the other repeated integral. The term-by-term integration is justifiable in a similar fashion†.

6.2. The result obtained above may be stated as follows.

THEOREM 5. *If $H(x)$ is defined as in* (6.11), *and*

$$f(\theta)\sim\sum_{-\infty}^{\infty}c_m e^{m\theta i}, \tag{6.21}$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}H|f(\theta)|\,d\theta\leqslant A_q\sum_{-\infty}^{\infty}H(\mu|c_m|)\,\mu^{-2}, \tag{6.22}$$

where $\mu=|m|+1$, $A_q=(Aq)^q$. Moreover, the convergence of the right-hand side implies that of the left.

The last assertion has still to be proved, for the inequality (6.22) was established under the assumption that $\Sigma|c_m|^q\mu^{q-2}$ converges. We proceed to show that it holds good whenever $\Sigma H(\mu|c_m|)\,\mu^{-2}$ is convergent. Consider the Fejér sum

$$\phi_n(\theta)=\sum_{-n}^{n}\left(1-\frac{|m|}{n}\right)c_m e^{m\theta i}\equiv\sum_{-n}^{n}\gamma_m e^{m\theta i},$$

say. Applying the inequality (6.22) to the polynomial $\phi_n(\theta)$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}H|\phi_n(\theta)|\,d\theta\leqslant A_q\sum_{-n}^{n}H(\mu|\gamma_m|)\,\mu^{-2}\leqslant A_q\sum_{-\infty}^{\infty}H(\mu|c_m|)\,\mu^{-2}.$$

Now $\phi_n(\theta)\to f(\theta)$ almost everywhere, and hence‡

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}H|f(\theta)|\,d\theta\leqslant\lim_{n\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}H|\phi_n(\theta)|\,d\theta\leqslant A_q\sum_{-\infty}^{\infty}H(\mu|c_m|)\,\mu^{-2}.$$

Thus the integral converges with the series.

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† *Cf.* Hobson, **11**, 304 (7).

‡ Hobson, **11**, 318.

If we apply a similar treatment to the other three theorems† of Hardy and Littlewood's in the same group with (1.13), we get the following, and also two other theorems involving $H_1(x)$ and $G_1(x)$.

THEOREM 6. *If* $G(x)$ *is defined by* (6.13), *then*

$$\sum_{-\infty}^{\infty} G(\mu|c_m|)\mu^{-2} \leqslant A_p' \frac{1}{2\pi}\int_{-\pi}^{\pi} G|f(\theta)|\,d\theta, \tag{6.23}$$

where $A_p' = [A(p+1)/(p-1)]^{\frac{1}{2}(p+1)}$.

This inequality, and the others whose statement has been omitted, have the same implications in respect to convergence as Theorem 5. The argument used above will establish this when the series appears on the right-hand side of the inequality; but for Theorem 6 we have to consider the function

$$f_n(\theta) = f(\theta) \quad (|f(\theta)| \leqslant n),$$
$$= 0 \qquad (|f(\theta)| > n).$$

Clearly, $f_n(\theta) \to f(\theta)$ as $n \to \infty$, and $|f_n(\theta)| \leqslant |f(\theta)|$; hence, if

$$f_n(\theta) \sim \Sigma \gamma_m e^{m\theta i},$$

since we have

$$\int_{-\pi}^{\pi} |f_n(\theta)|^p\,d\theta$$

finite, the integration argument which led to (6.23) will give

$$\sum_{-\infty}^{\infty} G(\mu|\gamma_m|)\mu^{-2} \leqslant A_p' \frac{1}{2\pi}\int_{-\pi}^{\pi} G|f_n(\theta)|\,d\theta \leqslant A_p' \frac{1}{2\pi}\int_{-\pi}^{\pi} G|f(\theta)|\,d\theta, \tag{6.24}$$

the series being convergent, of necessity. Now the existence of the integral last written implies that $|f(\theta)|$ is integrable in $(-\pi, \pi)$: $f(\theta)$ therefore possesses Fourier coefficients c_m, for which we have

$$|c_m - \gamma_m| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \{f(\theta) - f_n(\theta)\} e^{-m\theta i}\,d\theta\right|$$
$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta) - f_n(\theta)|\,d\theta = \frac{1}{2\pi}\int_{|f|>n} |f(\theta)|\,d\theta,$$

in virtue of the definition of $f_n(\theta)$. Hence $\gamma_m \to c_m$, and so, using (6.24),

$$\sum_{-N}^{N} G(\mu|c_m|)\mu^{-2} = \lim_{n\to\infty} \sum_{-N}^{N} G(\mu|\gamma_m|)\mu^{-2} \leqslant A_p' \frac{1}{2\pi}\int_{-\pi}^{\pi} G|f(\theta)|\,d\theta.$$

† 6, Theorems 2, 3, and 5.

Consequently, $\Sigma G(\mu|c_m|)\mu^{-2}$ is convergent and its sum does not exceed the expression last written.

6.3. Next, suppose that the numbers $|c_m|$ are rearranged so as to form a sequence $\{c_m^*\}$ of non-negative numbers such that

$$c_m^* \leqslant c_n^* \quad (|m| \geqslant |n|). \tag{6.311}$$

We may assume, for the sake of definiteness, that, when m is positive, $c_m^* \geqslant c_{-m}^*$. In these circumstances the numbers c_m^* are said to be "in star order", and we define a function

$$f^*(\theta) \sim \sum_{-\infty}^{\infty} c_m^* e^{m\theta i}, \tag{6.312}$$

whenever such a function exists. We can now enunciate the following refinements on theorems (1.13), (1.14), etc., which have been obtained by Hardy and Littlewood†:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^q d\theta \leqslant A_q \sum_{-\infty}^{\infty} (\mu c_m^*)^q \mu^{-2}, \tag{6.32}$$

$$\sum_{-\infty}^{\infty} (\mu c_m^*)^p \mu^{-2} \leqslant A_p \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^p d\theta. \tag{6.33}$$

Applying the integration argument to these, we get

THEOREM 7.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H|f(\theta)|\, d\theta \leqslant A_q \sum_{-\infty}^{\infty} H(\mu c_m^*)\mu^{-2}, \tag{6.34}$$

where $H(x)$ and A_q are as in Theorem 5.

THEOREM 8.

$$\sum_{-\infty}^{\infty} G(\mu c_m^*)\mu^{-2} \leqslant A_p' \frac{1}{2\pi}\int_{-\pi}^{\pi} G|f(\theta)|\, d\theta, \tag{6.35}$$

where $G(x)$ and A_p' are as in Theorem 6.

As before, there are two other theorems involving $H_1(x)$ and $G_1(x)$. Further, as a deduction from the theorems of type (6.32), Hardy and

† 9, Theorems 1 and 2.

Littlewood have established the inequalities:

$$\int_{-\pi}^{\pi} |f(\theta)|^q \, d\theta \leqslant A_q \int_{-\pi}^{\pi} |f^*(\theta)|^q \, d\theta \quad (q \geqslant 2), \tag{6.36}$$

$$\int_{-\pi}^{\pi} |f^*(\theta)|^p \, d\theta \leqslant A_p \int_{-\pi}^{\pi} |f(\theta)|^p \, d\theta \quad (1 < p \leqslant 2), \tag{6.37}$$

and from these we can get, by the integration argument,

THEOREM 9.

$$\int_{-\pi}^{\pi} H|f(\theta)| \, d\theta \leqslant A_q \int_{-\pi}^{\pi} H|f^*(\theta)| \, d\theta, \tag{6.38}$$

$$\int_{-\pi}^{\pi} G|f^*(\theta)| \, d\theta \leqslant A_p' \int_{-\pi}^{\pi} G|f(\theta)| \, d\theta, \tag{6.39}$$

where $H(x)$, A_q, and $G(x)$, A_p' are as in Theorems 5 and 6 respectively.

7. *Deduction of results "conjugate" to those of the preceding paragraph.*

7.1. The theorems of the last paragraph were stated in terms of the functions $H(x)$, etc.; and it appeared that, by appropriate choice of $K(t)$, we could get

$$H(x) \sim Ax^q (\log x)^{-\alpha},$$

for any $\alpha > 0$, A being a constant. However, positive powers of the logarithm were not obtained; this was inevitable, since

$$H(x) = \int_2^q K(t) x^t \, dt \leqslant H(1) x^q,$$

for $x \geqslant 1$, which is plainly incompatible with

$$H(x) \sim Ax^q (\log x)^{\alpha} \quad (\alpha > 0).$$

Similarly for $H_1(x)$, $G(x)$, and $G_1(x)$. However, theorems in terms of functions whose asymptotic values contain positive powers of $\log x$, or $\log(1/x)$, may be deduced from the theorems of the last paragraph†.

† In the analogous problem of the generalization of M. Riesz's theorem on conjugate functions, the same procedure is adopted. See Titchmarsh, **16**; it was this paper which suggested to me the possibility of the deduction given in the present paragraph, though the method which I follow, being an adaptation of the argument employed by Hardy and Littlewood, in **6**, is quite different from Titchmarsh's.

We suppose that

$$U(x) \equiv xu(x), \quad V(x) \equiv xv(x), \tag{7.11}$$

where $u(x)$ and $v(x)$ are increasing and

$$u(x) = v^{-1}(x). \tag{7.12}$$

In these circumstances we have†

$$U^{-1}(x)\, V^{-1}(x) = x. \tag{7.13}$$

We consider first the case where the two functions

$$u_1(x) \equiv u(x)/x, \quad v_1(x) \equiv v(x)/x \tag{7.14}$$

are respectively increasing and decreasing; a condition that is satisfied, in particular, when

$$U(x) = \int_r^s K(t)\, x^t\, dt \quad (s > r \geqslant 2).$$

Further, we shall require the property‡

$$t^r U(x) \leqslant U(tx) \leqslant t^s U(x) \quad (t \geqslant 1). \tag{7.15}$$

7.2. We shall make use of the following definitions:

$$f(\theta) \sim \sum_{-\infty}^{\infty} c_m e^{m\theta i}, \tag{7.211}$$

$$f_n(\theta) = \sum_{-n}^{n} c_m e^{m\theta i}. \tag{7.212}$$

The coefficients c_m* are defined as before, and we write

$$T_U(f) = \sum_{-\infty}^{\infty} U(\mu c_m^*)\, \mu^{-2} \quad (\mu = |m| + 1), \tag{7.221}$$

$$T_U^{(n)}(f) = \sum_{-n}^{n} U(\mu c_m^*)\, \mu^{-2}, \tag{7.222}$$

$$J_U(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U|f(\theta)|\, d\theta. \tag{7.223}$$

† *Cf.* Cooper, 2, Lemma.

‡ This states that $U(x)/x^r$ is increasing, while $U(x)/x^s$ is decreasing.

Similarly for $T_V(f)$ and $J_V(f)$. Next, let

$$\lambda_V = \overline{\text{bound}}\, T_V(f)/J_V(f), \tag{7.231}$$

$$\lambda_U' = \overline{\text{bound}}\, J_U(f)/T_U(f), \tag{7.232}$$

where f ranges over all functions that make the denominators finite. We may notice that $\lambda_V \geqslant 1$, $\lambda_U' \geqslant 1$; for the corresponding ratios are equal to 1 when f is a constant.

7.3. It remains to be shown that λ_V and λ_U' are both finite†. If $U(x)$ [or $V(x)$] is one of the functions $H(x)$, etc., that were employed in the last paragraph, the results there obtained show that λ_U' (or λ_V) is finite. It will be sufficient therefore to prove that λ_V and λ_U' are finite or infinite together. We prove, in fact, that

$$\lambda_V \leqslant r'(r\lambda_U')^{r'-1}, \tag{7.311}$$

$$\lambda_U' \leqslant s(s'\lambda_V)^{s-1}, \tag{7.312}$$

where $r' = r/(r-1)$, $s' = s/(s-1)$. Suppose that, in the star rearrangement of the coefficients c_m of $f(\theta)$, the star coefficient with suffix m_k is derived from the original coefficient with suffix k, so that

$$c^*_{m_k} = |c_k|. \tag{7.32}$$

Now write $\mu_k = |m_k|+1$, and define

$$d_k = v(\mu_k|c_k|)\,\overline{\text{sgn}}\, c_k . \mu_k^{-1}, \tag{7.331}$$

$$\phi_{(n)}(-\theta) = \sum_{|m_k|\leqslant n} d_k e^{k\theta i}. \tag{7.332}$$

Then, for the star rearrangement of the d's, we have

$$d^*_{m_k} = |d_k|. \tag{7.34}$$

For, suppose that $c^*_{m_k} = |c_k| > |c_j| = c^*_{m_j}$,

so that $\mu_k \leqslant \mu_j$. Then

$$\begin{aligned} |d_k| &= v(\mu_k|c_k|)\,\mu_k^{-1} > v(\mu_k|c_j|)\,\mu_k^{-1} \\ &= \mu_k|c_j|\,v_1(\mu_k|c_j|)\,\mu_k^{-1} = |c_j|\,v_1(\mu_k|c_j|) \\ &\geqslant |c_j|\,v_1(\mu_j|c_j|) = v(\mu_j|c_j|)\,\mu_j^{-1} = |d_j|, \end{aligned}$$

† In Hardy and Littlewood, **6**, the bounds λ and λ' corresponding to the problems of that paper are first defined by using $f_n(\theta)$, when it can be shown that they are finite for each n; the proof is then carried through and shows that λ and λ' have upper bounds independent of n. However, for the purposes of the present argument, a more direct course is possible.

since $\mu_k \leqslant \mu_j$ and $v_1(x)$ is decreasing. Thus $|d_k| > |d_j|$ whenever $|c_k| > |c_j|$, and it is clear that the d's follow the c's in the star rearrangement; hence (7.34).

Now, to prove (7.311), we have

$$(7.35) \qquad T_V^{(n)}(f) = \sum_{-n}^{n} V(\mu c_m^*)\,\mu^{-2}$$

$$= \sum_{-n}^{n} c_m^* d_m^* = \sum_{|m_k| \leqslant n} c_k d_k$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,\phi_{(n)}(\theta)\,d\theta \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\,|\phi_{(n)}(\theta)|\,d\theta$$

$$= \frac{t}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|\,t^{-1}|\phi_{(n)}(\theta)|\,d\theta$$

$$\leqslant \frac{t}{2\pi}\int_{-\pi}^{\pi} V|f(\theta)|\,d\theta + \frac{t}{2\pi}\int_{-\pi}^{\pi} U|t^{-1}\phi_n(\theta)|\,d\theta,$$

using the modification of Young's inequality†; but, by (7.15),

$$t^r\,U\,[t^{-1}|\phi_{(n)}(\theta)|] \leqslant U|\phi_n(\theta)| \quad (t \geqslant 1).$$

Hence (7.35) gives

$$T_V^{(n)}(f) \leqslant tJ_V(f) + t^{1-r}J_U(\phi_{(n)}) \leqslant tJ_V(f) + t^{1-r}\lambda_U'\,T_U(\phi_{(n)}).$$

Now
$$T_U(\phi_{(n)}) = \sum_{m=-n}^{n} U(\mu d_m^*)\,\mu^{-2},$$

and
$$\mu_k d_{m_k}^* = \mu_k |d_k| = \mu_k v(\mu_k |c_k|)\,\mu_k^{-1} = v(\mu_k c_{m_k}^*).$$

Thus

$$U(\mu d_m^*) = U\,[v(\mu c_m^*)]$$

$$= v(\mu c_m^*)\,u\,[v(\mu c_m^*)] = v(\mu c_m^*)\,\mu c_m^* = V(\mu c_m^*),$$

so that
$$T_U(\phi_{(n)}) = T_V^{(n)}(f).$$

We now have
$$T_V^{(n)}(f)(1 - t^{1-r}\lambda_U') \leqslant tJ_V(f),$$

and, therefore, whenever $t^{1-r}\lambda_U' < 1$,

$$T_V^{(n)}(f)/J_V(f) \leqslant t(1 - t^{1-r}\lambda_U')^{-1}.$$

The same is clearly true of $T_V(f)$; hence

$$\lambda_V \leqslant t(1 - t^{1-r}\lambda_U')^{-1} \quad (t \geqslant 1),$$

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† See foot-note to (2.14).

and the right-hand side is least when we choose $t^{1-r}\lambda_U' = 1/r$, as we may, since $\lambda_U' \geqslant 1$. This gives

$$\lambda_V \leqslant r'(r\lambda_U')^{r'-1}.$$

7.4. Next, we suppose that λ_V is finite and introduce the Fejér mean

$$\phi_n(\theta) = \sum_{-n}^{n} \left(1 - \frac{|m|}{n}\right) c_m e^{m\theta i}, \tag{7.41}$$

which tends to $f(\theta)$ almost everywhere as $n \to \infty$. Define

$$\psi(-\theta) = u|\phi_n(\theta)| \overline{\operatorname{sgn}}\, \phi_n(\theta), \tag{7.421}$$

$$\psi(\theta) \sim \sum_{-\infty}^{\infty} \gamma_m e^{m\theta i}. \tag{7.422}$$

We then have

$$J_U(\phi_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_n(\theta)\psi(-\theta)\,d\theta = \sum_{-n}^{n}\left(1-\frac{|m|}{n}\right) c_m\gamma_m$$

$$\leqslant \sum_{-n}^{n} |c_m||\gamma_m| \leqslant \sum_{-\infty}^{\infty} |c_m||\gamma_m| \leqslant \sum_{-\infty}^{\infty} c_m^*\gamma_m^*,$$

for the inequality $\Sigma ab \leqslant \Sigma a^* b^*$ can be proved very simply†. The remainder of the argument may be carried through as before, and we get

$$J_U(\phi_n) \leqslant s(s'\lambda_V)^{s-1} T_U(f).$$

But, since $\phi_n \to f$ almost everywhere, we have‡

$$\int_{-\pi}^{\pi} U|f(\theta)|\,d\theta \leqslant \varliminf_{n\to\infty} \int_{-\pi}^{\pi} U|\phi_n(\theta)|\,d\theta.$$

Thus
$$J_U(f) \leqslant \varliminf_{n\to\infty} J_U(\phi_n) \leqslant s(s'\lambda_V)^{s-1} T_U(f).$$

Hence
$$\lambda_U' \leqslant s(s'\lambda_V)^{s-1}.$$

7.5. We can now establish

THEOREM 10. *Let $K(t)$ be positive and let*

$$H(x) = \int_2^q K(t)x^t\,dt. \tag{7.511}$$

† See Hardy and Littlewood, 8.

‡ See Hobson, 11, 318.

Then, if

$$V^{-1}(x)\,H^{-1}(x) = x,$$

so that $V(x)/x^2$ *is decreasing, we have*

$$\sum_{-\infty}^{\infty} V(\mu c_m^*)\,\mu^{-2} \leqslant 4A_q \frac{1}{2\pi}\int_{-\pi}^{\pi} V|f(\theta)|\,d\theta, \tag{7.512}$$

where $A_q = (Cq)^q$.

For we have, by Theorem 7,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H|f(\theta)|\,d\theta \leqslant A_q \sum_{-\infty}^{\infty} H(\mu c_m^*)\,\mu^{-2}. \tag{7.513}$$

Now apply the results of the present paragraph with $U(x) = H(x)$, $r = 2$, $s = q$. We have, from (7.513),

$$\lambda_U' \leqslant A_q;$$

and, therefore, using (7.311),

$$\lambda_V \leqslant 2(2A_q) = 4A_q,$$

which is equivalent to (7.512).

Similarly we have

THEOREM 11. *Let* $K(t)$ *be positive and let*

$$G(x) = \int_{\frac{1}{2}(p+1)}^{p} K(t)\,x^t\,dt. \tag{7.521}$$

Then, if

$$U^{-1}(x)\,G^{-1}(x) = x,$$

so that $U(x)/x^2$ *is increasing, we have*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U|f(\theta)|\,d\theta \leqslant s(s'A_s')^{s-1}\sum_{-\infty}^{\infty} U(\mu c_m^*)\,\mu^{-2}, \tag{7.522}$$

where $A_s' = (As)^s$, *and* $s = (p+1)/(p-1)$.

This is deduced from Theorem 8 by taking

$$V(x) = G(x), \quad s = (p+1)/(p-1), \quad r = p' = p/(p-1),$$

and applying (7.312).

Lastly, there are similar results for the theorems involving the $H_1(x)$ and $G_1(x)$ of (6.1).

8. *Applications of the foregoing to generalize the Young-Hausdorff theorem.*

8.1. In §5 we obtained generalizations of the Young-Hausdorff theorem that were valid for functions of type $f^*(\theta)$, while in §6 in-

equalities were given connecting a quite general function $f(\theta)$ with the $f^*(\theta)$ corresponding to it. It is thus possible to get results of the required type, without restriction on $f(\theta)$, by combining the arguments of §§ 5 and 6.

We take first $\Phi(x)$ equal to the $H(x)$ of Theorem 7, so that we have, for any $f(\theta)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f(\theta)|\,d\theta \leqslant A_q \sum_{-\infty}^{\infty}\Phi(\mu c_m^*)\,\mu^{-2}, \tag{8.11}$$

where $A_q=(Aq)^q$. Also, from the definition of $\Phi(x)$, we have, by (2.22),

$$\Phi(tx) \leqslant \mathrm{Max}(t^2, t^q)\,\Phi(x). \tag{8.12}$$

Now let $\Psi(x)$ be defined by

$$\Psi^{-1}(x) = x\Phi^{-1}(1/x),$$

and write

$$S = \Sigma u_m \equiv \Sigma\,\Psi(c_m^*) = \Sigma\,\Psi|c_n|.$$

Arguing as in the proof of Theorem 4, we remark that

$$\mu u_m \leqslant \Sigma u_m = S,$$

since u_m decreases as $|m|$ increases. Then

$$\begin{aligned}\Phi(\mu c_m^*) &= \Phi\,[\mu\Psi^{-1}(u_m)] = \Phi\,[\mu u_m \Phi^{-1}(1/u_m)]\\ &\leqslant \mathrm{Max}\,[(\mu u_m)^2,\ (\mu u_m)^q](1/u_m) = \mu^2 u_m\,\mathrm{Max}\,[1,\ (\mu u_m)^{q-2}],\end{aligned}$$

in virtue of (8.12). Since $q>2$, we have $(\mu u_m)^{q-2} \leqslant S^{q-2}$, and so

$$\Sigma\,\Phi(\mu c_m^*)\,\mu^{-2} \leqslant \Sigma u_m\,\mathrm{Max}\,[1,\ S^{q-2}] = \mathrm{Max}\,[S,\ S^{q-1}].$$

Combining this with (8.11), we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Phi|f(\theta)|\,d\theta \leqslant A_q\,\mathrm{Max}\,[\Sigma\,\Psi|c_n|,\ (\Sigma\,\Psi|c_n|)^{q-1}]. \tag{8.13}$$

As far as convergence is concerned, the results may be summed up as follows.

THEOREM 12. *Let $K(t)$ be non-negative and integrable over the ranges involved. Suppose further that* EITHER

$$\Phi(x) = \int_2^q K(t)\,x^t\,dt \quad (q \geqslant 2), \tag{8.141}$$

OR

$$\frac{1}{\Psi(1/x)} = \int_{\frac{1}{2}(q'+1)}^{q'} K(t)\,x^t\,dt \quad (1 < q' \leqslant 2); \tag{8.142}$$

and that $\Phi(x)$ *and* $\Psi(x)$ *are connected by*

$$\Psi^{-1}(x) = x\Phi^{-1}(1/x). \tag{8.15}$$

Then, if

$$\sum_{-\infty}^{\infty} \Psi|c_n| < \infty, \tag{8.161}$$

there is a function $f(\theta)$, *with Fourier coefficients* c_n, *such that*

$$\int_0^{2\pi} \Phi|f(\theta)|\,d\theta < \infty. \tag{8.162}$$

When $\Phi(x)$ is defined by (8.141) the result has been proved. When $\Psi(x)$ is defined by (8.142) we may use an argument similar to that above, but based on Theorem 11 instead of Theorem 7; it leads to an inequality resembling (8.13).

Recalling the results of Lemma 5 for the asymptotic values of the integrals in (8.141) and (8.142), we observe that a suitable choice of $K(t)$ in (8.141) gives

$$\Phi(x) \sim x^q(\log x)^{-\alpha} \quad (x \to \infty), \tag{8.171}$$

where $\alpha > 0$; we then have, with $q' = q/(q-1)$ as usual,

$$\left\{\begin{aligned} &\Phi^{-1}(x) \sim x^{1/q} q^{-\alpha/q}(\log x)^{\alpha/q}, \\ &\Psi^{-1}(x) \equiv x\Phi^{-1}(1/x) \sim x^{1/q'} q^{-\alpha/q} [\log(1/x)]^{\alpha/q} \quad (x \to 0), \\ &\Psi(x) \sim x^{q'} q^{\alpha(q-1)} [q' \log(1/x)]^{-\alpha(q'-1)}. \end{aligned}\right. \tag{8.172}$$

On the other hand, a suitable choice of $K(t)$ in (8.142) gives

$$\Psi(x) \sim x^{q'} [\log(1/x)]^{\beta} \quad (x \to 0),$$

with β positive. Choosing β and multiplying by a constant in an appropriate way, we can thus get a $\Psi(x)$ that satisfies (8.172) for a *negative* α. The restriction on the sign of α in (8.171) and (8.172) is therefore removed; and, on taking the special case of Theorem 12 corresponding to functions $\Phi(x)$ and $\Psi(x)$ of this type, we get the proposition enunciated in § 1.3.

It remains for us to deal with the second half of the Young-Hausdorff theorem. For this we take $H(x)$ to be equal to the integral in (8.141) and we define $\Psi(x)$ by

$$\Psi^{-1}(x)\, H^{-1}(x) = x.$$

Then $H(x)$ has the property (8.12), from which we may deduce that

$$\Psi(tx) \geqslant \mathrm{Min}(t^2, t^{q'})\, \Psi(x). \tag{8.181}$$

Supposing that $\Phi(x)$ is defined by (8.15), and writing

$$S = \Sigma u_m \equiv \Sigma \Phi(c_m^*) = \Sigma \Phi |c_n|,$$

we have, as before,

$$\begin{aligned}\Sigma \Psi(\mu c_m^*)\mu^{-2} &\geqslant \Sigma \Psi[\mu u_m \Psi^{-1}(1/u_m)]\mu^{-2} \\ &\geqslant \Sigma \operatorname{Min}[(\mu u_m)^2, (\mu u_m)^{q'}](1/u_m)\mu^{-2} \\ &\geqslant \Sigma u_m \operatorname{Min}[1, (\mu u_m)^{q'-2}] \geqslant S \operatorname{Min}[1, S^{q'-2}],\end{aligned}$$

since $q' < 2$, $\mu u_m \leqslant S$. The assumption that Σu_m converges can be avoided if we take the series finite, and then let the number of terms tend to ∞ at the end. We now have, applying Theorem 10,

$$\begin{aligned}(8.19) \qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} \Psi|f(\theta)|\,d\theta &\geqslant A_{q'} \Sigma \Psi(\mu c_m^*)\mu^{-2} \\ &\geqslant A_{q'} \operatorname{Min}[S, S^{q'-1}].\end{aligned}$$

On combining this with the similar result based on Theorem 8, we have

THEOREM 13. *Let $K(t)$ be positive and let* EITHER

$$(8.191) \qquad \frac{1}{\Phi(1/x)} = \int_2^q K(t)\,x^t\,dt$$

OR

$$(8.192) \qquad \Psi(x) = \int_{\frac{1}{2}(q'+1)}^{q'} K(t)\,x^t\,dt;$$

further, let $\Phi(x)$ and $\Psi(x)$ be related by (8.15). *Then*

$$(8.193) \qquad \int_{-\pi}^{\pi} \Psi|f(\theta)|\,d\theta < \infty$$

implies that

$$(8.194) \qquad \sum_{-\infty}^{\infty} \Phi|c_n| < \infty.$$

8.2. The theorems above settle questions of convergence, but as a generalization of Hausdorff's inequalities nothing beyond (8.13) and (8.19) is provided. It remains to be seen what more refined inequalities may be obtainable. For the sake of brevity discussion will be confined to the inequalities (8.13). If the $\Phi(x)$ appearing in this was sub- (or super-) multiplicative, we could clearly argue as for Theorem 4. Now, by (2.24), $\Phi(x)$ has the property

$$(8.211) \qquad \Phi(x)/\Phi(y) \leqslant \Phi(tx)/\Phi(ty) \quad (x \geqslant y > 0,\ t \geqslant 1),$$

and from this we have†, in particular,

$$\Phi(ab) \leqslant \Phi(a)\Phi(b)/\Phi(1) \quad (a \leqslant 1 \leqslant b), \tag{8.212}$$

$$\Phi(ab) \geqslant \Phi(a)\Phi(b)/\Phi(1) \quad (a, b \leqslant 1;\ a, b \geqslant 1). \tag{8.213}$$

Accordingly we can use (8.212) in place of the sub-multiplicative property, or, alternatively, we can so modify the definition of $\Phi(x)$ as to make it super-multiplicative.

Let us consider first what may be established with $\Phi(x)$ still defined as in (8.141). It is necessary to assume that $\Phi(1) = 1$, and that

$$\Sigma \Psi(c_n) = \Sigma \Psi(c_m{}^*) = \Sigma u_m \leqslant 1.$$

Then, if we observe as before that $\mu u_m \leqslant \Sigma u_m \leqslant 1$, we have, pursuing the line of argument that led to Theorem 12,

$$\Phi(\mu c_m{}^*) = \Phi[\mu u_m \Phi^{-1}(1/u_m)] \leqslant \Phi(\mu u_m)(1/u_m);$$

for $1/u_m \geqslant 1/\Sigma u_m \geqslant 1$, and $\Phi(\mu u_m) \leqslant \Phi(1) = 1$. Thus, writing

$$\Phi(x) \equiv x\phi(x) \equiv x^2 \phi_1(x),$$

we get

$$\begin{aligned}\Sigma \Phi(\mu c_m{}^*)\mu^{-2} &\leqslant \Sigma \Phi(\mu u_m)/\mu^2 u_m \leqslant \Sigma u_m \phi_1(\mu u_m) \\ &\leqslant \Sigma u_m \phi_1(S) = \phi(S).\end{aligned}$$

We now have

THEOREM 14. *If $\Phi(x)$ is defined by* (8.141), *with $\Phi(1) = 1$, then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi|f(\theta)|\,d\theta \leqslant A_q \phi\left(\sum_{-\infty}^{\infty} \Psi|c_n|\right), \tag{8.221}$$

where $\phi(x) \equiv \Phi(x)/x$, provided that

$$\sum_{-\infty}^{\infty} \Psi|c_n| \leqslant 1. \tag{8.222}$$

Next, suppose that

$$H(x) = \int_2^q K(t)\, x^t\, dt,$$

with $H(1) = 1$, and

$$\Phi(x) = [x/(x+1)]^q H(x+1).$$

Then $\Phi(x)$ is super-multiplicative. For it is clear that $H(x)x^{-q}$ is de-

† For, on putting $x = 1$, $y = a$, $t = b$ in (8.211), we have (8.212); and (8.213) comes in a similar way.

creasing, and that $H(x)$ has the property (8.213); hence, remembering that $xy+1 \leqslant (x+1)(y+1)$, we have

$$\begin{aligned}\Phi(xy) &= x^q y^q H(xy+1)(xy+1)^{-q} \\ &\geqslant x^q y^q H[(x+1)(y+1)](x+1)^{-q}(y+1)^{-q} \\ &\geqslant x^q y^q H(x+1)\,H(y+1)(x+1)^{-q}(y+1)^{-q} = \Phi(x)\,\Phi(y).\end{aligned}$$

Secondly, $\Phi(x)$ satisfies

$$2^{2-q}H(x)-1 \leqslant \Phi(x) \leqslant H(x), \tag{8.231}$$

for the right-hand inequality follows from the fact that $H(x)\,x^{-q}$ is decreasing, and the left-hand inequality, for $x \geqslant 1$, from the fact that $H(x)x^{-2}$ is increasing; while, when $x \leqslant 1$, $2^{2-q}H(x)-1 \leqslant 0$.

Now, by Theorem 7,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H|f(\theta)|\,d\theta \leqslant A_q \sum_{-\infty}^{\infty} H(\mu c_m{}^*)\,\mu^{-2},$$

and this, with (8.231), gives

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi|f(\theta)|\,d\theta &\leqslant A_q \sum_{-\infty}^{\infty} 2^{q-2}\{\Phi(\mu c_m{}^*)+1\}\,\mu^{-2} \\ &= 2^{q-2}A_q\left\{\sum_{-\infty}^{\infty}\mu^{-2}+\sum_{-\infty}^{\infty}\Phi(\mu c_m{}^*)\,\mu^{-2}\right\} \leqslant 2^q A_q\left\{1+\sum_{-\infty}^{\infty}\Phi(\mu c_m{}^*)\,\mu^{-2}\right\}.\end{aligned}$$

From this and the super-multiplicative property of $\Phi(x)$ we may argue as for (5.315) in Theorem 4, and obtain the following result.

THEOREM 15. *Let*

$$H(x) = \int_2^q K(t)\,x^t\,dt \quad [H(1)=1], \tag{8.241}$$

and

$$\Phi(x) = [x/(x+1)]^q H(x). \tag{8.242}$$

Then, with the usual relation between $\Phi(x)$ *and* $\Psi(x)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi|f(\theta)|\,d\theta \leqslant 2^q A_q\left\{1+\frac{1}{\phi(1/\Sigma\Psi|c_n|)}\right\}. \tag{8.243}$$

There are similar inequalities obtainable also when $\Phi(x)$ is sub-multiplicative, and further there are inequalities which correspond to Theorem 13. But, since all these are somewhat awkward in form and without the appearance of finality, I have contented myself with the indications above.

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ON REPRESENTATIONS OF LINE-CONGRUENCES OF THE SECOND AND THIRD ORDERS

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The most natural method of dealing with congruences of lines in ordinary space is that which uses the representation of the lines of space by the points of a quadric Ω in S_5. In this way, the lines of a congruence of order m and class n are represented on Ω by the points of an (m, n) surface meeting the planes of the two systems on Ω in m and n points respectively, and the geometry of line-congruences is reduced at once to the geometry of surfaces on a quadric in S_5.

It is rather unfortunate, however, that these surfaces, in the large majority of cases, are very special of their type*, this being due simply to the fact that they have to lie on a quadric; and conversely it is usually very difficult to find directly what degree and character of specialization must be imposed on a surface of a given type in order that it may lie on a quadric, the more so as there are often several distinct ways, leading to different congruences, in which a surface may be so specialized. Furthermore, the specialization required is of a kind very difficult to specify exactly, either geometrically or algebraically, with a view to a possible parametric representation of the congruence; for it consists usually of the condition that the surface must contain isolated plane curves and improper nodes which it would not normally have. These difficulties have had the effect of seriously restricting the application of the representational method to congruences; and, in fact, Fano, in his great memoir on congruences of the third order†, restricts himself very largely to direct arguments in ordinary space.

The object of the present paper is to show that, although the surfaces which represent, on Ω, the congruences of ordinary space are rather

* A surface may be said to be *special of its type* if it is not a general projection of the corresponding normal surface in higher space.

† Fano, *Mem. della R. Acc. delle Scienze di Torino* (2), 51 (1902), 1–79.

special and complicated, yet representations of a much simpler kind of the same congruences can usually be found on surfaces in a four-dimensional space S_4. In fact, we use representations of the lines of S_3 on the points of S_4 instead of on the points of Ω in S_5, and we obtain representations, on very simple well-known surfaces of S_4, of nearly all the congruences of the second and third orders without singular lines.

It seems that these representations furnish a much better description of the congruences than do the special methods usually given; for it will be seen that each such representation that we shall give sets clearly in evidence the various singular points and double or multiple rays of the congruence, and since the representing surface is usually general of its type and of low order, it makes possible a convenient parametric representation when such is required. A mere plane representation of a congruence is usually quite useless for such requirements.

As an example of the kind of result which we propose to obtain, we may perhaps mention one of two irrational (3, 9) congruences whose existence was first established by Fano*. The congruence in question has, in addition to some quadratic singular points, two singular points of order three and six of order five; and it would be represented on Ω by a very special surface of order twelve. By our method it is represented on a perfectly general elliptic sextic scroll of S_4, ${}^1F^6$ say, sections of it by linear complexes being represented by sections of ${}^1F^6$ by quadrics through three of its generators, l_1, l_2, l_3 say. Now the surface ${}^1F_2{}^6$ is known to possess exactly two plane cubics and three plane elliptic quartics†; and it is just those two plane cubics which represent the two singular cones of order three, while the three plane quartics represent three of the six singular cones of order five, the remaining three such cones being represented by the twisted quartic curves in which the surface is met by solids through pairs of l_1, l_2, l_3. Furthermore, in § 13.3, we use this representation to arrive at a new definition of the congruence as the residual intersection of two cubic complexes which have six point-stars in common. All the relations of the singular points to each other and the various double rays can be deduced immediately from the representation.

Representations of the quadric Ω on S_4.

1.1. We use for the purposes of this paper representations of the quadric Ω of [5] on a four-dimensional space S_4, such representations

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* *Loc. cit.*, 49.

† *Cf.* Semple, *Proc. London Math. Soc.* (2), 32 (1931), 369-401 (390).

being evidently equivalent* to representations of the lines of S_3 on the points of S_4. The simplest such is evidently that obtained by projecting Ω from a point of itself; but this (the Schumacher) representation is practically useless for our problem. The representations which we shall use here are those in which prime sections of Ω become quadrics of S_4 which all pass through a rational normal C^4, complete or degenerate. The most general case of this, in which C^4 is complete, has already been set forth by the author†; but, for purposes of reference, we review here some details of the correspondence and investigate in addition the two very useful particular cases, in which C^4 degenerates into a cubic curve and a line, or into three skew lines and their common transversal.

1.2. The fact that a general quadric Ω of [5] can be represented on a [4] S_4 by quadrics through a rational normal quartic curve C^4 is very easily proved analytically as follows. Let C^4 be taken to have equations

$$\left\| \begin{matrix} x_1 & x_2 & x_3 & x_4 \\ x_1' & x_2' & x_3' & x_4' \end{matrix} \right\| = 0,$$

where the elements of the matrix are linear functions of the five homogeneous coordinates of a point of S_4. There are evidently six linearly independent quadrics through the curve, namely those given by the vanishing of the minors of the matrix, say X_{23}, X_{31}, X_{12}, X_{14}, X_{24}, X_{34}. If we take these, then, as proportional to the coordinates of a point of S_5, there will correspond to any point of S_4 a definite point of S_5; and, since the six minors, like the six coordinates of a line, are connected by the relation

$$X_{23}X_{14}+X_{31}X_{24}+X_{12}X_{34}\equiv 0,$$

it follows that the points of S_4 are represented on the points of a general quadric of S_5. We shall denote by Φ the system of quadrics through C^4 representing prime sections of Ω.

1.3. Of the details of the above representation we note the following, referring the reader for fuller discussion to the above-mentioned account by the author.

Sections of Ω by primes are represented, as remarked already, by quadrics Φ through C^4; sections by solids are represented by Φ-surfaces,

* We refer, of course, to the usual representation of the lines of S_3 by points of a quadric in [5].

† *Phil. Trans. Royal Soc.*, 228 (1929), 331–376 (350).

i.e. by surfaces of intersection of Φ-quadrics; and plane sections are represented by Φ-curves, which are rational quartics meeting C^4 in six points.

To neighbourhoods of points of C^4 correspond, on Ω, planes of an ∞^1 system generating a V_3^6; to chords of C^4—the exceptional lines of the correspondence—correspond *points* of Ω forming a surface ψ, the double surface evidently of V_3^6. To trisecant planes of C^4 correspond the planes of one system, say α-planes, of Ω; and to cubic scrolls* containing C^4 there correspond the β-planes of Ω. The planes of V_3^6 are a special system of β-planes. The surface ψ is easily seen to be a non-ruled (3, 1) surface on Ω, *i.e.* a Veronese surface, and V_3^6 is just the locus of conic planes of ψ which lie on Ω.

To primes of S_4, finally, there correspond sections of Ω by quadrics through the surface ψ, and conversely; and each such section is a V_3^4 containing a [4, 4] of planes of Ω corresponding to the vertices and faces of the tetrahedron defined by the four points in which the prime meets C^4. The system of quadrics through ψ, which system we shall denote by Ψ, evidently includes Ω, and, being defined by its base Veronese surface, it is a homaloidal system† of quadrics in S_5.

1.4. *Special cases of the representation.* There are, as we shall now show, two ways in which the base quartic C^4 can degenerate without affecting the generality of the resulting Ω in S_5. We consider each of these in turn.

1.41. *First degeneration of* C^4. Consider the system of quadrics Φ in S_4 through a rational cubic curve C^3 and a line l meeting C^3. We can take the equations of the cubic to be

$$\frac{x}{y} = \frac{y}{z} = \frac{z}{t}; \quad u = 0,$$

and those of the line to be

$$y = z = t = 0.$$

The six independent quadrics of Φ can then be taken to be

$$X = yt - z^2; \quad Y = yz - xt; \quad Z = xz - y^2;$$

$$X' = yu; \quad Y' = zu; \quad T' = tu;$$

* The cubic scrolls referred to are not those generated by an involution of pairs of points of C^4. They are those which have each a chord of C^4 as directrix line and C^4 itself as a simple directrix curve.

† *Cf.* Semple, *Phil. Trans. Royal Soc.*, 228 (1929), 351.

and these are again connected by the identical relation

$$XX'+YY'+ZZ'\equiv 0,$$

the equation of a general quadric primal Ω in S_5.

The Veronese surface ψ, the base of the system of quadrics Ψ in S_5, breaks up into an α-plane α_0, whose points correspond to chords of C_3 (*i.e.* α_0 corresponds to the whole solid Σ containing C_3) and into a cubic scroll R, meeting α_0 in a line, whose points correspond to lines of S_4 which meet both l and C^3. The Φ-curves now meet l twice and C^3 four times; and the planes of S_4 which correspond to α-planes are those which meet l once and C^3 twice, there being, as we remarked, *one special* α*-plane* α_0, *which represents the whole solid* Σ.

1.42. *Second degeneration of* C^4. We now make C^4 degenerate into three skew lines l_1, l_2, l_3 and their unique transversal line λ, noting that the line λ imposes no additional conditions on quadrics through l_1, l_2, l_3 in order that they should contain it. Taking λ, then, as the edge XY of the simplex of reference, l_1 as XZ, l_3 as YU, and l_2 as the join of T to the point $(1, -1, 0, 0, 0)$, we find, as our six independent quadrics through the four lines,

$$X_1=tu; \quad X_2=uz; \quad X_3=zt;$$

$$X_1'=yz; \quad X_2'=-t(x+y); \quad X_3'=xu,$$

and these again are connected by the identity

$$X_1X_1'+X_2X_2'+X_3X_3'\equiv 0.$$

The Veronese surface ψ now breaks up into three α-planes α_1, α_2, α_3, corresponding to the three solids Σ_1, Σ_2, Σ_3 containing pairs of the lines l_1, l_2, l_3, and into a fourth plane β_0 whose points do not correspond to any system of chords of the degenerate C^4. This fourth plane is the unique β-plane of Ω which meets α_1, α_2, α_3 in lines and it represents, as is easy to see, the neighbourhood of every point* of the line λ. It is implied, as a base plane of Ψ, by α_1, α_2, α_3.

The quartic Φ-curves in S_4 now have l_1, l_2, l_3 as chords; and the *effective* trisecant planes of C^4 (corresponding to α-planes on Ω) are simply the planes which meet the same three lines. It should again be remem-

* The neighbourhood of every point of λ is represented homographically on the same β-plane β_0 of Ω. The section of the neighbourhood of λ by a cubic scroll through λ, l_1, l_2, l_3 is represented by a single point of β_0.

bered, however, that the three special α-planes α_1, α_2, α_3 are represented exceptionally by solids.

1.5. The interpretation of the above representations of Ω as representations of the lines of ordinary space on the points of S_4 is now perfectly straightforward. The linear complexes of S_3, since they are represented by prime sections of Ω, are represented by Φ-quadrics in S_4; and similarly the linear congruences and the reguli of S_3 are represented by Φ-surfaces and Φ-curves respectively. Trisecant planes of C^4 represent the point-stars of S_3 and cubic scrolls through C^4 represent the ruled planes.

Chords of C^4 represent the lines of a (3, 1) congruence Π in S_3, *i.e.*, in general, the congruence of axes of a certain cubic developable R, and points of C^4 represent exceptionally the planes of R; while to all the quadratic complexes which contain Π, these being all tetrahedral, there correspond the complete system of primes of S_4. We note, however, that when C^4 is degenerate the congruence Π also breaks up into components; and, in fact, when C^4 breaks up into three lines and their transversal, Π becomes the degenerate (3, 1) congruence consisting of three point stars (P_1), (P_2), (P_3) together with the plane $P_1P_2P_3$ joining their vertices.

It is this last form of the representation in reality that we shall find most useful, since in making use of it the three points P_1, P_2, P_3 can evidently be chosen to be any three points we please.

Representation of a given congruence.

2.1. Coming now to the problem of representing a given line-congruence K by the simplest possible surface in S_4, we think first of the surface, F say, which represents the given congruence on Ω. This is an (m, n) surface on Ω, m and n being the order and class of K; and if K has a singular point of order k, through which there pass ∞^1 lines of the congruence forming a cone of order k, there will correspond to it on Ω an α-plane meeting F in a curve of order k, When, therefore, Ω is represented on S_4 in some such way as we have outlined, F will be represented in S_4 by a surface F' in such a way that prime sections of F' correspond to sections of F by quadrics through the surface ψ on Ω; and it is evident that, in order to obtain F' as a surface of as low order as possible, the surface ψ must be chosen to contain a suitable curve or curves of F. This is effected most easily and directly by using the special representation (§ 1.42) in which ψ breaks up into planes α_1, α_2, α_3, and β_0, and by choosing α_1, α_2, α_3 to be the three α-planes which represent the three singular points of highest order of the congruence. This we shall call the *direct method* of representing a given congruence.

Now although this method is nearly always successful, yet we shall prefer sometimes, for reasons of symmetry, to use a more *indirect method.* For this we start with a surface F' in S_4, specially chosen perhaps in regard to C^4, and we find what surface it represents on Ω, and hence what congruence in S_3; and we trust to an investigation of the singular points to indicate to us whether the congruence so arrived at is a general one of its kind.

2.2. *Representation in S_4 of singular points and double rays.* We note finally that any singular point of our congruence K will be represented in S_4 either by

(a) a trisecant plane of C^4 which meets F' in a curve,

(b) an exceptional solid such as Σ, Σ_1, Σ_2, Σ_3 of §§ 1.41, 1.42,

(c) a point of intersection of F' with C^4,

observing that in the last case the singular point represented is necessarily of order one; for the neighbourhood on F' of the point of C^4 represents a line of Ω and a plane pencil therefore in S_3.

Similarly a double ray of K will be represented either by an *improper* double point of F' or by an exceptional line—a chord of C^4—which meets F' in two points not on C^4, or possibly even by an exceptional plane, such as (l_1, λ) in § 1.42, which meets the surface in two free points.

2.3. *Notation.* We shall use the symbol ${}^pK^{(m,n)}[\alpha^i; \beta^j; \ldots]$ to denote a congruence of order m, class n, section genus p, having i singular points of order α, j singular points of order β, and so on.

Congruences of the second order.

3.1. All the congruences $K^{(2,n)}$ $(2 \leqslant n \leqslant 7)$ without singular lines have section genus one and they are all represented by normal or projected Del Pezzo surfaces on Ω. These surfaces, except for $n = 2, 3$, are all special of their kind. The congruences which they represent are seven in number, there being two distinct cases for $n = 6$, say $K_1^{(2,6)}$ and $K_2^{(2,6)}$.

We propose to show now that the five congruences $K^{(2,2)}$, $K^{(2,3)}$, $K^{(2,4)}$, $K^{(2,5)}$, $K_2^{(2,6)}$ can all be represented in S_4 by ordinary quadric surfaces meeting C^4 in 4, 3, 2, 1, 0 points respectively.

3.2. *Congruences representable on a quadric surface.* The direct method (§ 2.1) could be applied here; but it is simpler to begin by taking a general

quadric surface Q in S_4 and then to find what congruence it represents. We assume for the present that Q does not meet C^4; it lies in a solid, Π say, which meets C^4 in four points P, Q, R, S.

Q represents a surface F, say, on Ω; and F is represented on Q by means of its sections by quadrics through C^4. Thus F is a projected octavic Del Pezzo surface of the second kind; and, since trisecant planes of C^4 meet Q in two points, F is a (2, 6) surface on Ω. It represents, therefore, a (2, 6) congruence $K_2^{(2,6)}$ in S_3.

The only plane curves on Q are its plane sections and its generators. Of the former, four lie in trisecant planes of C^4, namely the sections of Q by the faces of the tetrahedron $PQRS$; and, since a general conic of S_4 represents a quartic curve on Ω, the faces of $PQRS$ represent singular points of order four of K.

To find how many generators of each system of Q lie in trisecant planes of C^4 we observe that all the lines which lie in all such planes define a quadratic complex* of lines of S_4. The section of this by Π is a tetrahedral complex in Π having P, Q, R, S as singular points, and this has evidently four lines in common with each regulus of Q. These lines represent conics on Ω (lying in α-planes), and singular cones of K, therefore, of order two. *Thus K has four singular points of order four and eight of order two; and it is, therefore, a congruence $K_2^{(2,6)}[4^4; 2^8]$, and evidently a general one of this kind*†.

The six edges of the tetrahedron $PQRS$ represent evidently the *six double rays of K*; and each of the latter is seen to be a join of two of the four singular points of order four, any singular cone at one of these points having three of the double rays as double generators.

The eight points 2^8 evidently divide into two tetrads A_i, B_j, whose sixteen cross-joins and thirty-two joins to P, Q, R, S all belong to K.

3.2. By taking the surface Q now to contain r of the points P, Q, R, S ($r = 1, 2, 3, 4$), we obtain the remaining four congruences of the set. We still obtain twelve singular points as above; only some of them, and in the last case all, have their orders reduced because the conics and lines representing their singular cones pass through some of the points P, Q, R, S of C^4; and we obtain in addition r new singular points *of order one* representing the neighbourhoods on Q of the r points in which Q meets C^4.

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* The trisecant planes of C^4 which pass through an arbitrary point generate, in fact, a threefold quadric cone.

† For a general résumé of these congruences, see, for example, Jessop, *Treatise on the line complex*, 280. They were first investigated by Kummer, *Berlin Abh.* (1866), 1–120.

The congruences so obtained are easily identified as the general congruences

$K^{(2,2)}[1^{16}]$ for $r = 4$,

$K^{(2,3)}[2^5;\ 1^{10}]$,, $r = 3$,

$K^{(2,4)}[3^2;\ 2^6;\ 1^6]$,, $r = 2$ (one double ray),

$K^{(2,5)}[4^1;\ 3^3;\ 2^6;\ 1^3]$,, $r = 1$ (three double rays),

and the relation of the singular points and double rays is in each case obvious from the figure.

3.4. Finally, since Q lies in a solid, each of the above congruences belongs to at least one tetrahedral complex.

4.1. *Representation of $K_1^{(2,6)}$ on a cubic scroll.* The other congruence of class six not included in the group just discussed is that which has a symbolical representation $K_1^{(2,6)}[5^1;\ 3^6;\ 2^4;\ 1^1]$. Here again we could use the direct method to prove that any such congruence can be represented in S_4 by a general cubic scroll; but for reasons of symmetry—there being only one singular point of order five—we look for a slightly more general representation than that given by the most special form of the representation of Ω on S_4.

4.2. Consider then the representation of Ω on S_4 in which C^4 breaks up into a line l and a cubic curve C^3, and let us investigate the congruence which is represented in S_4 by a general cubic scroll Γ having l as a generator.

We note, in the first place, that Γ represents a surface F on Ω which is in fact an octavic (projected) Del Pezzo surface of the first kind; for prime sections of F are represented on Γ by its intersections with quadrics through l and C^3; and in the usual plane representation of Γ such sections are evidently represented by cubic curves with a single base point*. The surface F, moreover, is clearly a (2, 6) surface on Ω; for any plane in S_4 representing an α-plane meets l once and C^3 twice, and such a plane therefore meets Γ in just two free points. *Thus Γ represents a congruence of the type $K_1^{(2,6)}$ in S_3.*

4.3. We note secondly that the principal singular point of the congruence just obtained is that which corresponds to the solid Σ containing C^3, its singular cone being represented, of course, by the curve E^3 in which

* Γ is represented by curves $C^2[1]$; quadric sections by curves $C^4[2]$. Removing the fixed generator, represented by $C^1[1]$, we are left with $C^3[1]$ as required.

Γ meets Σ. This curve passes necessarily through the intersection, O say, of l and C^3; and being met, therefore, by a variable quadric surface through C^3 in five variable points, it represents a plane quintic curve on Ω. *Thus the solid Σ represents a singular point of order five of the congruence.*

4.4. The other singular points must be represented by planes which, in addition to meeting l once and C^3 twice, meet Γ either in a conic or a line. We classify them as follows:

(a) The cubic curves C^3 and E^3 have ten common chords of which six do *not* pass through O. Through the pair of points of E^3 on any one of the latter there passes a unique conic lying on Γ; and this conic, since it necessarily meets l, represents a plane cubic curve on Ω. *Thus the six conic-planes of Γ which meet Σ in the six* proper *common chords of C^3 and E^3 represent six singular points of order three of $K_1^{(2,6)}$.*

(b) Let the directrix line λ of Γ meet Σ in the point Λ of E^3, and draw the four chords of C^3 which meet the line $O\Lambda$ and the curve E^3 each in one point. Through any one of these chords there can be drawn a plane containing a generator of Γ and this plane will also meet l*. *Thus these four planes represent quadratic singular points of $K_1^{(2,6)}$.*

(c) Finally, the plane containing the directrix λ and the chord of C^3 through Λ evidently represents the remaining *singular point of order one.*

The six double rays of the congruence are, of course, represented by the six *proper* common chords of C^3 and E^3.

Thus the congruence $K_1^{(2,6)}$ represented by Γ is evidently a quite general sextic congruence of the first kind.

5.1. *Representation of $K^{(2,7)}$ on a Veronese surface.* We consider now the last congruence of order two, namely that of class seven whose full symbolical representation is $K^{(2,7)}[6^1;\ 3^{10}]$.

The procedure is exactly the same as in the case immediately preceding; only we now begin with a projected Veronese surface F' having l as a double line instead of, as before, with a cubic scroll through l. Sections of F' by quadrics through l and C^3 are evidently represented, in the plane representation of F', by cubics in the plane with no base points†; and

* For the plane in question meets both $O\Lambda$ and the directrix λ, and l lies in the plane determined by these two lines.

† F' is represented by all conics which have a given pair of points P, Q as conjugate points; and the double line is represented simply by the join PQ.

F' represents, therefore, a projected Del Pezzo surface of order nine on Ω. This again meets α-planes in two points and gives a (2, 7) congruence $K^{(2,7)}$.

F' meets Σ in a curve E^4 having a double-point at O and representing therefore a plane sextic curve on Ω; so that Σ represents a *sextic singular point* of our congruence $K^{(2,7)}$.

Also E^4 and C^3 are easily seen to have ten proper common chords. These represent the *ten double rays* of the congruence; and, since planes can be drawn through each of them [just as in § 4.4 (a)] to contain conics of F', it follows that $K^{(2,7)}$ has *ten cubic singular points* whose joins to the sextic singular point are the ten double rays. The relations of the singular points and double rays indicate that the congruence is quite general of its type.

Thus the general (2, 7)*congruence can be represented simply in S_4 on a projected Veronese surface with a double line.*

6.1. We could also obtain, evidently, similar simple representations of the congruences $(2, n)$ which have singular lines. For example, using the degenerate $C^4 \equiv (l_1, l_2, l_3, \lambda)$, we see that an elliptic quintic scroll through l_1, l_2, l_3 represents a (2, 6) congruence which is, in fact, that formed by all the chords of an elliptic C^4, the ∞^1 directrix planes which meet the surface in cubic curves evidently representing the points of C^4 (*cf.* § 13.1).

6.2. We pass over these, however, for the present, and consider next the congruences of the third order $K^{(3,n)}$ without singular lines. In regard to the singular points of these we recall the following general remarks of Fano in his paper on congruences of order three*:

(a) Any singular cone of any of the congruences $K^{(3,n)}$ is either rational or elliptic.

(b) Through any singular point there may pass one, but not more than one, isolated ray of the congruence not belonging to the singular cone.

(c) The singular cone at any singular point through which there passes an isolated ray is always rational. The converse is usually, but not always, true.

6.3. Since we shall be using largely henceforth the representation of Ω by quadrics through (l_1, l_2, l_3, λ), we may note here conveniently also a property of the special quadratic complex of all lines which lie in transversal planes of l_1, l_2, l_3. This property is, namely, that the *complex in question has λ as triple line*, meaning by this that, of the $2k$ lines which

* Fano, *loc. cit.*, 8.

the complex has in common with any scroll of order k which contains λ, three coincide with λ. This can be shown geometrically; or we can use the simple representation* of the lines of S_4 on the points of S_6, in which case the theorem follows from the fact that the complex can be transformed to a cubic cone† in S_6 whose vertex is the point corresponding to λ. In a similar way it can be shown that the *complex has* l_1, l_2, l_3 *double lines*.

We shall denote this complex in future by the symbol Θ.

6.4. We remark, lastly, that, except in cases of special difficulty, we shall, in passing from a known surface on Ω or in S_4 to the corresponding surface in S_4 or on Ω, omit the details of the process of identifying the new surface. We shall, however, take pains to specify exactly in both cases the special relation of the surface in S_4 to the base lines l_1, l_2, l_3, λ, and of the surface on Ω to the base planes a_1, a_2, a_3, β_0.

Congruences of order three and section genus one.

7.1. The whole set of congruences ${}^1K^{(3,n)}(n = 2, \ldots, 6)$ was obtained by Segre‡, and independently by Castelnuovo§, by projecting various systems of lines on certain determinantal cubic primals in S_4; and it was left to Fano‖ to show that the congruences so obtained are the most general third order congruences of section genus one. Fano also remarks that all singular cones of these congruences are necessarily rational and have each an *isolated* ray of the congruence through their vertices. We consider each of them in turn, using suitable symbols to indicate their known properties. They are represented on Ω, of course, by various projected Del Pezzo surfaces so specialized that they are $(3, n)$ surfaces on a quadric in S_5.

7.2. *The congruence* ${}^1K^{(3,6)}$ [3^6]. This congruence is represented on Ω by a novenic surface ${}^1F^9$, and we can take a_1, a_2, a_3 to be three of the six

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* *Cf.* Semple, *Proc. London Math. Soc.* (2), 32 (1930), §§7.1, 10.1, with $n = 4$, $k = 1$.

† In fact, if l_1, l_2, l_3, λ are taken as XU, YW, ZV, UV, where $XYZUV$ is the pentad of reference and W is the point (0, 0, 0, 1, 1), then, in the representation in S_6 referred to, the complex in question is represented by the cubic cone whose equation is

$$x_3(x_4x_2 - x_1x_5) = x_4(x_3x_5 - x_2x_6),$$

where the equations of the base V_3^3 of planes in S_6 are

$$x_0 = 0; \quad \frac{x_1}{x_4} = \frac{x_2}{x_5} = \frac{x_3}{x_6}.$$

‡ Segre, *Mem. della R. Acc. di Torino* (2), 39 (1888), 3–50.

§ Castelnuovo, *Atti del R. Ist. Veneto* (6), 5 (1887), 1249–1281; (6), 6 (1888), 525–579.

‖ Fano, *loc. cit.*, 58.

α-planes which meet the latter surface in rational cubic curves. Each of these planes, by §7.1, meets the surface in one further isolated point; and, using the plane representation, it is easy to see that the corresponding surface in S_4 is a *general projected sextic Del Pezzo surface* ${}^1F^6$ through l_1, l_2, l_3. Conversely, any such surface as ${}^1F^6$ through l_1, l_2, l_3 is easily seen to represent a (3, 6) Del Pezzo surface on Ω.

Each of the solids $\Sigma_1, \Sigma_2, \Sigma_3$ evidently contains one of the three remaining lines of ${}^1F^6$, and it meets the surface residually in a cubic curve meeting l_1, l_2, l_3 in three points of which one lies on λ.

Now a general projected sextic Del Pezzo surface in S_4 contains six *plane* cubic curves, of which three belong to the system of rational cubic curves of the surface which meet l_1, l_2, l_3, and the remaining three to the opposite system associated with the three remaining lines of the surface; for the corresponding normal surface ${}^1N^6$ in [6] is the double surface of a determinantal cubic primal ${\Omega_5}^3$, its two systems of cubic curves lying in the two systems of generating solids of ${\Omega_5}^3$; and on projecting then from a general line, which meets ${V_5}^3$ in three points, three curves of each system project evidently into plane cubics of the surface in S_4.

Thus, of the six cubic singular points of the congruence, three are evidently represented by the solids $\Sigma_1, \Sigma_2, \Sigma_3$, *and the remaining three by the triad of planes which contain the three plane cubics of* ${}^1F^6$ *meeting* l_1, l_2, l_3.

The isolated rays associated with the first three are represented by the three remaining lines of ${}^1F^6$, and those of the second three by isolated points of ${}^1F^6$ in the planes. The six double-rays are represented, partly by the three improper double points of ${}^1F^6$ and partly by those three chords of the cubic curves in $\Sigma_1, \Sigma_2, \Sigma_3$ which meet a pair of the lines l_1, l_2, l_3.

7.31. *The congruence* ${}^1K_1^{(3,5)}[3^3; 2^3; 1]$. This congruence, like some of those to follow, presented a certain amount of difficulty on account of the fact that none of the authorities already quoted explains precisely the very special relation of the seven singular points and their associated singular cones and isolated rays to each other. Both Segre and Castelnuovo explain that all seven singular points lie in two planes, the three cubic points P_3, P_3', P_3'' lying in one plane ϖ, the three quadratic points P_2, P_2', P_2'' lying in another plane ϖ', and the remaining one P_1 on the line of intersection of ϖ and ϖ'. From their method of generation, also, it is easy to deduce that ϖ and ϖ' are singular planes, ϖ' being, in fact, the plane of the pencil of rays through P_1, while ϖ contains a conic envelope E of rays of the congruence in addition to the three joins of P_3, P_3', P_3'', which also belong to the congruence. This arrangement, however, gives

in the plane ϖ *four* rays through each of the *cubic* points P_3, P_3', P_3'' and two through the point P_1; and we must therefore assume that the isolated rays associated with each of these four points all lie in the plane ϖ and belong in fact to E.

Interpreting this now on Ω and taking the planes α_1, α_2, α_3 to be those corresponding to the points P_3, P_3', P_3'', so that β_0 then corresponds to the plane ϖ, and denoting the vertices of the triangle in which α_1, α_2, α_3 meet β_0 by A_1, A_2, A_3, we see that the Del Pezzo surface ${}^1F^8$ (of the first kind) on Ω, which represents the congruence ${}^1K_1^{(3,5)}$, meets α_1, α_2, α_3 in cubic curves and β_0 in a conic; and that the cubic curves in question meet the sides of the triangle $A_1A_2A_3$ in pairs of the points A_1, A_2, A_3 and in three further points B_1, B_2, B_3 say; and that the conic referred to meets the same sides in B_1, B_2, B_3 and in three further points, which are indeed the necessary isolated points of ${}^1F_2{}^8$ which lie in α_1, α_2, α_3 and which correspond namely to the isolated rays through the singular points P_3, P_3', P_3''.

It is easy to see now that the surface in S_4 which corresponds to ${}^1F^8$ is simply a cubic scroll Γ having λ as a generator*; and we verify this immediately by noting that to any such scroll there does, in fact, correspond on Ω a (3, 5) projected Del Pezzo surface of the first kind.

7.32. The relations between Γ and ${}^1K_1^{(3,5)}$ are now fairly obvious. The three cubic singular points are represented, of course, by the solids Σ_1, Σ_2, Σ_3. The quadratic complex Θ of lines lying in transversal planes of l_1, l_2, l_3 (*cf.* § 6.3) contains besides λ three other generators of Γ, and these represent the three quadratic singular points. The directrix line of Γ represents the unique plane pencil of ${}^1K_1^{(3,5)}$. And, finally, the three double rays are represented by the three lines which meet two of l_1, l_2, l_3 and lie in the planes of the three conics in which Γ meets Σ_1, Σ_2, Σ_3. The two singular planes of the congruence are represented one by the neighbourhood of λ and the other by a cubic scroll containing l_1, l_2, l_3, λ, and the directrix of Γ.

Thus the congruence ${}^1K_1^{(3,5)}$ *can be represented simply on a cubic scroll of* S_4 *which has* λ *as a generator.*

7.41. *The congruence* ${}^1K_2^{(3,5)}[4^1, 2^6]$. In this congruence, as in the previous one, the seven singular points lie by fours in two planes ϖ, ϖ',

* ${}^1F_2{}^8$ is represented by cubics $C^3(1)$ with one base point, and general quadric sections by curves $C^6(2)$; and, removing the three general lines corresponding to the three cubics and a line through the base point corresponding to the conic, we get curves $C^3(1)$ representing a cubic ruled surface Γ. The earlier discussion was necessary in order to show that there are no additional base points arising from isolated points of ${}^1F_2{}^8$ lying in α_1, α_2, α_3.

the quartic singular point P_4 being on the intersection of ϖ, ϖ'. Both the planes ϖ, ϖ' now contain conic envelopes, E, E' say, of lines of the congruence, and from Castelnuovo's work in S_4 it is easy to see that the six lines joining P_4 to the triads (P_2, P_2', P_2'') and (Q_2, Q_2', Q_2'') of quadratic singular points also belong to K. This implies, as in the previous case, that the isolated rays through P_4, P_2, P_2', P_2'' all lie in the plane ϖ and belong to E, and that through P_4, in fact, must be simply the line of intersection of ϖ and ϖ'.

The congruence is represented on Ω by an octavic Del Pezzo surface of the second kind; and taking α_1, α_2, α_3 to be the α-planes which correspond to P_4, P_2, P_2', so that β_0 corresponds to ϖ, we find that ${}^1K_2^{(3,5)}$ is represented in S_4 by a quartic scroll ${}^0F^4$ having λ and l_1 as generators, the intersection, D_1 say, of these two lines being the unique improper node of the surface. The reverse process, beginning with ${}^0F^4$, gives an immediate verification of this result.

The points P_4, P_2, P_2' are represented by the solids Σ_1, Σ_2, Σ_3 respectively; the triad of quadratic points Q_2, Q_2', Q_2'' is represented by the three generators of ${}^0F^4$, other than l_1 and λ, which belong to the complex Θ; and the remaining point P_2'' is represented* by the unique conic-plane of ${}^0F^4$ which passes through λ.

Thus ${}^1K_2^{(3,5)}$ can be represented in S_4 by a rational quartic scroll through λ and l_1.

The congruence contains two ∞^1 systems of quadric reguli represented on ${}^0F^4$ by its generators and conics.

7.5. *The congruence* ${}^1K^{(3,4)}[3^1; 2^4; 1^3]$. The procedure in this case is very like that in the two previous cases; and the result is that ${}^1K^{(3,4)}$ *can be represented in S_4 by a cubic scroll Γ having λ as a generator and meeting l_1 in one further point R.* Σ_1, Σ_2, Σ_3 give the one cubic and two quadratic singular points; the three generators of Γ which belong to Θ give two further quadratic and one simple singular point; while the two remaining simple singular points arise from the neighbourhood of P and the directrix line of Γ respectively.

The two ∞^1 systems of reguli of the congruence are represented by the generators of Γ and the conics of Γ which pass through P.

7.6. *The congruence* ${}^1K^{(3,3)}[2^3; 1^6]$. This congruence admits of two simple representations on S_4. The one similar to those just outlined is

* Two conics of ${}^0F^4$ pass through the double point D_1, and one of these meets l_1 again while the other meets λ again; *cf.* the plane representation.

by a cubic scroll Γ having λ as a generator and meeting each of l_1, l_2 in one further point. The other, however, is a better representation than this owing to the fact that it is more symmetrical and also because it emphasises the fact that the congruence in question belongs to a tetrahedral complex. *It represents the lines of K on the points of a 3-nodal cubic surface in* S_3, the nodes being the points where the containing solid meets l_1, l_2, l_3.

We note, first, the fact that any cubic surface F^3 having nodes at three points A, B, C of l_1, l_2, l_3 respectively does represent a general (3, 3) projected sextic Del Pezzo surface* on Ω, and therefore a congruence ${}^1K^{(3,3)}$ in S_3. Moreover, the latter has three quadratic singular cones P_2, P_2', P_2'' corresponding to the conics in which F^3 meets Σ_1, Σ_2, Σ_3 respectively, the lines BC, CA, AB of the surface giving the associated isolated rays; also it has six simple singular points whose plane pencils are represented by the three pairs of lines of the surface, other than joins of A, B, C, which pass through the nodes. The same congruence finally has three *quadratic* singular planes as has Castelnuovo's congruence of this type (*cf.* Castelnuovo, *loc. cit.*, 540), namely those represented by the three base points A, B, C.

The fact that the congruence so obtained is the *general* one can now be verified by taking as α_1, α_2, α_3 the three planes of Ω which correspond to the quadratic singular points of the general congruence, this giving immediately a cubic surface in S_4. The other representation would be derived by taking one of the quadratic singular planes as the fundamental plane in S_3 and two quadratic and one simple singular point in it as the three fundamental points.

The fact that the representative 3-nodal cubic surface lies in a solid is equivalent to saying that the general ${}^1K^{(3,3)}$ belongs to one tetrahedral complex of lines. The three systems of reguli of the congruence are represented by the systems of conics lying in planes through BC, CA, AB respectively.

Thus the general congruence ${}^1K^{(3,3)}$ *can be represented symmetrically by a 3-nodal cubic surface in* S_4, *the nodes being base points of* Φ.

We note, in conclusion, that a *general* projected sextic Del Pezzo surface in S_5 lies on a pencil of quadrics and is a (3, 3) surface on any one

* A general 3-nodal cubic surface is represented on the plane by cubic curves through the vertices P, Q, R of a triangle and through three further points P', Q', R' of the sides. The three sides represent the nodes; and sections of the surface by quadrics through the nodes are evidently represented by cubic curves through P', Q', R' and so represent prime sections of a sextic Del Pezzo surface ${}^1F^6$. The two lines, beside node-joins, of the cubic surface which pass through the node represented by $QP'R$ are represented by the point P' and the line $Q'R'$ respectively.

of them. The projected Del Pezzo surface which represents a (2, 4) congruence is special in that it has an improper double point. It lies on ∞^2 quadrics.

7.7. *The congruence* ${}^1K^{(3,2)}[1^{10}]$. The best way to study this congruence is by means of the dual congruence ${}^1K^{(2,3)}$ (*cf.* § 3.3). The simplest direct representation in S_4 seems to be by a cubic scroll Γ having λ as a generator and meeting each of l_1, l_2, l_3 in one further point.

Congruences of order three and section genus two.

8.1. We are evidently concerned in this section with congruence surfaces on Ω of order $n+3$ and section genus two; and we begin, therefore, by recalling some of the facts about such surfaces. In the first place, they all contain a pencil of conics and they are all rational surfaces of order m, where $4 \leqslant m \leqslant 12$. Castelnuovo derives them all from four supernormal* surfaces of order twelve in [11], which we may denote by F_3, F_2, F_1, F_0, having as minimum directrix curves (*i.e.* curves meeting all the conics) a pencil of cubic curves, a conic, a line, and a point respectively. These surfaces can be defined, after Castelnuovo, by their plane representations; or we can say simply that F_3 and F_1 are the surfaces represented on the two types of rational normal quartic scroll† in [5] by means of their intersections with quadrics through a fixed generator; while F_2 and F_0 are the surfaces represented on a general cubic scroll and a rational cubic cone in [4] by means of all their quadric sections. Evidently F_1 and F_0 can be considered as limiting cases of F_3 and F_2 respectively‡.

Congruences ${}^2K^{(3,n)}$, then, will evidently divide themselves, likewise, with repetitions of course, into four families according as they are represented on Ω by surfaces of the four families governed by F_3, F_2, F_1, F_0, and we note that the first two families coincide except for the highest members. Each of the congruences will contain, as remarked by Fano, $18-2n$ simple singular points and $n-3$ quadratic singular points corresponding respectively to lines of the associated congruence surface on Ω

* By a *supernormal* surface of order n we mean one which cannot be derived by projection from a surface of order $n+1$ in higher space.

† There are, of course, two types of normal quartic scroll—the first having conics as minimum directrix curves, and the second having a directrix line. They are represented by curves $C^3(2, 1)$ and $C^4(3; 1^3)$ respectively, the three simple base points in the latter case being collinear.

‡ F_2 is represented by curves $C^4(2)$, equivalent birationally to $C^8(6; 2^4)$, while F_0 is also representable by curves $C^8(6; 2^4)$ provided that the four double base points are collinear. Similarly F_3 is representable by curves $C^5(3; 2)$, equivalent birationally to $C^7(5; 2^3)$, and F_1 is also representable by $C^7(5; 2^3)$ provided that the three double base points are collinear.

and to conics of the surface which lie in α-planes. Fano also shows that there are $\frac{1}{2}(n-2)(n-3)$ double rays belonging to each congruence.

Fano has shown moreover, by inventing congruences associated with an ∞^1 system of quadrics of order three, that there is at least one congruence of every class n $(3 \leqslant n \leqslant 9)$ with no other singular points than those just mentioned. We shall extend this result to show that, with the exception of the highest members of the first and third families, congruences of all families exist, for the same range of values of n, having only the singular points named. For the two exceptional cases, we shall obtain congruences ${}^2K^{(3,9)}$, which have, in addition to the necessary six quadratic singular points, a singular point of order seven.

8.21. *Three families of congruences representable on cubic scrolls.* We use here the general representation of Ω on S_4 by means of quadrics through an undegenerate C^4, and, as before, we denote by Θ the quadratic complex of lines of S_4 which lie in trisecant planes of C^4.

Consider then the congruence which is represented in S_4 by a general cubic scroll Γ. The surface on Ω which corresponds to Γ is evidently an outside projection of the surface F_2 referred to in the preceding section, its directrix conic being derived, in fact, from the directrix line of Γ; and our congruence is therefore the leading member of the second family of congruences. No conics of Γ lie in general in trisecant planes of C^4; and thus the only singular points of the congruence are the *six quadratic singular points* derived from the six generators of Γ which belong to Θ.

Double rays of the congruence must be represented by common chords of C^4 and Γ, and it is not difficult to see, in fact, that there are twenty-one such chords. For Γ meets the cubic locus of chords of C^4 in a C^9 of genus seven which is represented on Ω by a ${}^7C^{18}$ having double points at all the points corresponding to exceptional chords of Γ. This ${}^7C^{18}$, however, lies on the base Veronese surface ψ on Ω, and the number of its double points is therefore twenty-one, as is obvious from the plane representative of ψ.

Thus the general cubic scroll in S_4 represents the leading member ${}^2K^{(3,9)}$ $[2^6]$ of the second family of congruences of section genus two, and it has twenty-one common chords with C^4, representing the twenty-one double rays of the congruence.

8.22. Consider now the effect of making Γ contain one or more points P of C^4. In the first place, the class of the congruence is evidently reduced by one for each point P; and in addition one of the six generators of Γ which belong to Θ is now the generator of Γ which passes through P and gives rise consequently to a simple instead of a quadratic singular

point of K. There is, however, a new simple singular point introduced, derived from the neighbourhood of P on Γ.

Thus, corresponding to cubic scrolls Γ *which meet* C^4 *in* r *points* $(r = 0, \ldots, 6)$, *we get all the congruences* ${}^2K^{(3,\,9-r)}[2^{6-r};\ 1^{2r}]$ *of the second family of congruences of order three and section genus two.*

These coincide, of course, with the single exception of the leading member, with various congruences of the first family; and they include the well-known congruence of *Roccella* ${}^2K^{(3,\,3)}[1^{12}]$, which has twelve first order singular points.

The number of *double rays* is easily seen in general to be $\frac{1}{2}(7-r)(6-r)$, which, since $n = 9-r$, is equal to $\frac{1}{2}(n-2)(n-3)$, agreeing with Fano's result.

8.23. If now we replace the general scroll Γ by a rational cubic cone Γ_1 whose vertex is V, we evidently get a *complete series of congruences* ${}^2K^{(3,\,n)}$ *of the fourth family* $(3 \leqslant n \leqslant 9)$, with exactly the same array of singular points and double rays as before. These congruences, however, have all a certain line v, namely that corresponding to the vertex V, as a *triple ray*; and, moreover, since V is a *proper* multiple point of Γ_1, v is a triple ray *of the second kind* of each of the congruences. All the quadratic singular points evidently lie on this line v; and the ∞^1 reguli which generate the congruence all contain v.

8.24. Lastly, if we choose Γ so that its directrix λ meets C^4, we obtain another series of congruences ${}^2K^{(3,\,n)}$ $(3 \leqslant n \leqslant 8)$, likewise having the same disposition of singular points, but belonging evidently to the third family of congruences of section genus two; for the congruence surfaces on Ω have all a directrix line corresponding to λ. *We thus obtain all but the leading member* ${}^2K^{(3,\,9)}$ *of the third family of congruences of section genus two.*

8.3. The two remaining congruences ${}^2K^{(3,\,9)}$, of the first and third families respectively, are not easily representable, if indeed they exist at all, unless they have, in addition to the necessary six quadratic singular points, a singular point of order seven. We can, however, obtain congruences, of both types, with this extra singular point as follows.

We use the degenerate form (l, C^3) of the base curve C^4 in S_4, and we observe that a rational quartic scroll F^4 containing l represents a congruence ${}^2K^{(3,\,9)}$ of the *first* family if its minimum directrix curve is a conic, and a congruence ${}^2K^{(3,\,9)}$ of the *third* family if its minimum directrix curve is a line. The solid Σ containing C^3 meets F in a rational C^4 meeting

l; this C^4 represents therefore a singular cone of order seven of the congruence, and there are as before six generators of F^4 belonging to Θ. These give the usual six quadratic singular points.

Thus to the two types of rational quartic scroll which can be taken to contain l, there correspond congruences ${}^2K^{(3,9)}\,[7^1;\ 2^6]$ which belong to the first and third families respectively.

8.4. We can, of course, obtain other special varieties of the various congruences which we have described, and we mention in particular that to a rational quintic scroll having l as double generator there corresponds a congruence ${}^2K^{(3,9)}$, of the second family, having a singular point of order eight in addition to its six quadratic singular points. And, in fact, by making the same quintic scroll contain in addition a number $9-n$ of points of C^3, we see that *special congruences ${}^2K^{(3,n)}$ exist, for all values of n $(3 \leqslant n \leqslant 9)$, which have, in addition to the usual simple and quadratic singular points, a singular point of order $n-1$.*

In this case the $\frac{1}{2}(n-2)(n-3)$ double rays of the congruence are the double rays of the singular cone of order $n-1$. Fano obtains these special congruences by an application of Caporali's method of generating congruences by means of a birational representation of a surface on a plane.

Congruences of section genus three.

9.1. Congruences ${}^3K^{(3,n)}$ of section genus three are not of very great interest, and in view of the fact that they are all *monoidal*, *i.e.* they all contain a singular point* of order $n-1$, they present no difficulties to an elementary treatment. We mention, for the sake of completeness, Fano's remark, by means of which he arrives at his result, namely that there cannot be third order congruences of hyperelliptic section genus greater than two.

The general types of congruence ${}^3K^{(3,n)}$ are, therefore, all representable on Ω by rational surfaces ${}^3F^{n+3}$ of the type which are representable on the plane by means of quartic curves with a number of simple base points. They form a single well-defined series for values of n satisfying $4 \leqslant n \leqslant 13$; and they have each an elliptic singular point of order $n-1$, whose singular cone is represented on the plane by a cubic curve through the base points, and $13-n$ plane pencils associated with the $13-n$ lines of ${}^3F^{n+3}$.

9.2. There is no very simple way of representing all these congruences in S_4; but, by using the special representation in which C^4 consists of a

* Fano, *loc. cit.*, 63.

line l and a C^3, we can obtain, at any rate, sufficiently general representations as follows.

For $4 \leqslant n \leqslant 8$ we obtain congruences ${}^3K^{(3,n)}[n-1;\ 1^{13-n}]$ by taking a Segre quartic surface through l and through r further points of C^3 $(r = 0, \ldots, 4)$.

For $9 \leqslant n \leqslant 13$ we have to use projected Del Pezzo surfaces of orders five, six, seven, eight, nine respectively having l as a multiple line of order two, three, four, five, or six. These surfaces certainly represent congruences ${}^3K^{(3,n)}$ of the required type in S_3.

The solid Σ containing C^3 represents in all cases the singular point of order $n-1$, and the $13-n$ plane pencils are represented either by lines of the representing surface which meet l or by base points on the surface. The double rays of any one of the congruence are, of course, the double generators of the singular cone. There are various special types of the congruences, but we do not consider these here.

Irrational congruences of order three and section genus four.

10.1. Of all third order congruences without singular lines the only irrational ones are:

(*a*) The congruence ${}^4K^{(3,3)}$, the intersection of a linear and a cubic complex.

(*b*) Two congruences ${}^4K^{(3,9)}$, discovered by Fano in an investigation of Montesano's cubic complex (the complex of all generators of a net of quadrics).

(*c*) Certain congruences ${}^4K^{(3,n)}$, contained in a tetrahedral complex and obtained by Fano from a net of quadrics with a common base line.

(*d*) A congruence ${}^6K^{(3,7)}$.

In the present section we propose to give very simple representations of the two congruences (*b*) and of the series of congruences (*c*).

10.21. *The two congruences* ${}^4K^{(3,9)}[5^6;\ 3^2;\ 2^6]$ *and* ${}^4K^{(3,9)}[6^2;\ 4^6;\ 2^6]$ *generated by elliptic pencils of reguli.*

These two very interesting congruences of the third order suggest themselves immediately when the linear complexes of space are represented in S_4 by quadrics through three skew lines l_1, l_2, l_3 and their transversal λ. For in this representation an elliptic sextic scroll through l_1, l_2, l_3, and an elliptic quintic scroll through l_1, l_2, evidently both represent third

order congruences of some kind in S_3, each generated by an elliptic infinity of reguli; and these can be none other than the congruences in question. We consider each case in turn.

10.22. We take then an elliptic sextic scroll ${}^1F^6$ through l_1, l_2, l_3; and we remark, to begin with, that it certainly does represent a congruence ${}^4K^{(3,9)}$; for any quadric through l_1, l_2, l_3 meets it in l_1, l_2, l_3 and in a residual C^9 of genus* four meeting each of l_1, l_2, l_3 in two points; and since such a curve represents a scroll of order twelve and genus four in S_3 our congruence, being of order three, is evidently of class nine and section genus four.

We call to mind certain properties of the general elliptic sextic scroll in S_4, namely that it possesses three plane directrix quartic curves and two plane directrix cubic curves, each of the former having nodes at two of the three improper nodes of the surface†. Now the planes of all these five curves meet l_1, l_2, l_3, and they represent therefore *three singular points of order five P_5 and two singular points of order three P_3*. From the exceptional solids $\Sigma_1, \Sigma_2, \Sigma_3$, moreover, we get *three further singular points of order five*; and lastly, from the six free generators of ${}^1F^6$ which belong to Θ (two are absorbed by each of l_1, l_2, l_3) we get *six quadratic singular points P_2*.

Thus the general elliptic sextic scroll ${}^1F^6$ through l_1, l_2, l_3 represents the general congruence ${}^4K^{(3,9)}[5^6; 3^2; 2^6]$.

It is easy now to see that every join of two of the six points P_5 is a double ray of the congruence, there being fifteen such rays in all; while all the other joins of singular points, except the join of the two points P_3 and the inter-joins of the six points P_2, are simple rays of the congruence.

10.23. Consider next then an elliptic quintic scroll ${}^1F^5$ through l_1, l_2 and note immediately, as in the preceding case, that it too represents a congruence ${}^4K^{(3,9)}$ in S_3. We identify the singular points of the latter as follows:

(i) The solids Σ_1 and Σ_2, since they meet ${}^1F^5$ residually in quartic curves each of which meets both l_1 and l_2, represent *two elliptic singular points of order six P_6*.

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* A complete quadric section of ${}^1F^6$ has genus seven, as is seen by applying Segre's formula for the genus of a curve on a ruled surface; and when the section breaks up into three lines and a curve meeting each of the lines twice, the genus p of the curve is given by the decomposition formula

$$7 = 0+0+0+p+6-3,$$

i.e.

$$p = 4.$$

† *Cf.* Semple, *Proc. London Math. Soc.* (2), 32 (1931), 390.

(ii) The solid Σ_3 meets ${}^1F^5$ in a plane cubic meeting l_1, l_2 and it represents therefore *a singular point of order four.*

(iii) There is a simple infinity of planes which meet ${}^1F^5$ in plane directrix cubics; and since these generate a $V_3{}^5$ meeting l_3 in five points, there are five transversal planes of l_1, l_2, l_3 which meet the surface in (elliptic) cubic curves. *These give five further elliptic singular points of order four.*

(iv) Lastly, there are $10-2.2=6$ generators of ${}^1F^5$ which belong to Θ and these give *six singular points of order two* P_2.

Thus the congruence represented by an elliptic quintic scroll ${}^1F^5$ *through* l_1, l_2 *is the second of the two congruences required, namely* ${}^4K^{(3,9)}[6^2;\ 4^6;\ 2^6]$.

The plane (λ, l_3) meets F in three further points and represents evidently a *triple ray* of the congruence, joining the two points P_6; and it is easy to see also that the join of any of the six quartic points P_4 to either point P_6 is a *double ray*, giving twelve such double rays in all. All the remaining joins of the fourteen singular points, except the cross-joins of the quadratic points, are simple rays of the congruence.

We shall refer to these two congruences again in § 12.3; we now pass on to consider the irrational congruences of order three which belong to a tetrahedral complex.

10.3. *Irrational congruences belonging to a tetrahedral complex.* We have already remarked (§ 1.31) that, in any one of our representations of the lines of S_3 on the points of S_4, any solid of S_4 represents a tetrahedral complex of S_3, the vertices and faces of the fundamental tetrahedron of the latter being derived from the faces and vertices of the tetrahedron formed by the four points in which the solid meets C^4.

Any congruence of order three then in S_3 which belongs to a tetrahedral complex can be represented by a *cubic* surface of some kind, lying in a solid, Π say, of S_4; for being of the third order its representing surface in S_4 must meet an arbitrary trisecant plane of C^4 in three points.

We see, moreover, conversely, that any cubic surface lying in a solid Π of S_4 represents a congruence of order three belonging to at least one tetrahedral complex. Such a surface, provided that it has not a *multiple* point at any of the four base points P, Q, R, S in Π, is met by quadrics through C^4 in curves of genus four; and it is met by cubic curves through P, Q, R, S (*i.e.* by the sections by Π of cubic scrolls through C^4) in $9-r$ points, where r is the number of the points P, Q, R, S which the surface contains simply.

Thus any cubic surface which contains $r\ (0 \leqslant r \leqslant 4)$ *of the four points* P, Q, R, S *represents a congruence* ${}^4K^{(3,\,9-r)}$ *of* S_3 *belonging to a tetrahedral complex.*

In a subsequent section (§ 10), we shall discuss the family of congruences represented by *rational* cubic surfaces in Π; but, since we are now looking for irrational congruences, we consider only irrational cubic surfaces in Π, *i.e.* cubic cones; and in particular we take an *elliptic cubic cone* R in Π through r of the points P, Q, R, S and examine the corresponding congruence in S_3.

If $r = 0$, we get a congruence ${}^4K^{(3,\,9)}$ generated by an elliptic pencil of quadric reguli having all a fundamental generator v in common, corresponding to the vertex V of R. This congruence has evidently four elliptic singular points P_6 of order six, corresponding to the four planes of the tetrahedron P, Q, R, S, and six quadratic singular points P_2, all lying on the fundamental line v and derived from the six generators of R which belong to Θ. The six edges of the tetrahedron P, Q, R, S represent, evidently, *six triple rays* of ${}^4K^{(3,\,9)}$ of the first kind; and the line v, since it represents a proper multiple point of R, represents *a triple ray of the second kind**.

In the cases $r = 1, \ldots, 4$ there are the usual modifications. Each base point on the cone R reduces the order of three of the points P_6 by one; it replaces one of the points P_2 by a point P_1, and it adds a further simple singular point corresponding to its own neighbourhood on the surface. We easily construct the following table which includes the case $r = 0$:

$r = 0$ ${}^4K^{(3,\,9)}\,[6^4;\ 2^6]$: joins of points 6^4 are triple rays and all points 2^6 lie on a further triple ray v of the second kind.

$r = 1$ ${}^4K^{(3,\,8)}\,[6^1;\ 5^3;\ 2^5;\ 1^2]$: joins of 6^1 to 5^3 are triple rays; joins of 5^3 to each other are double rays; the plane of 5^3 contains one of the plane pencils; v is a triple ray of second kind.

$r = 2$ ${}^4K^{(3,\,7)}\,[5^2;\ 4^2;\ 2^4;\ 1^4]$: one triple ray besides v and four double rays.

$r = 3$ ${}^4K^{(3,\,6)}\,[4^3;\ 3^1;\ 2^3;\ 1^6]$: only three double rays besides the triple ray v.

$r = 4$ ${}^4K^{(3,\,5)}\,[3^4;\ 2^2;\ 1^8]$: no multiple rays except v.

* Fano makes the distinction between a *multiple ray of the first kind*, represented by an improper multiple point of the congruence surface on Ω, and a *multiple ray of the second kind* represented by a proper multiple point of that surface, at which there is a proper tangent cone to the surface.

It seems hardly necessary to enter into further detail concerning the mutual relations of the singular points, since these can easily be deduced from the representations. In the last case, for example, the fundamental triple ray v contains the two quadratic and four of the simple singular points. while the plane pencils of the remaining four lie in the faces of the tetrahedron formed by the four cubic singular points.

The rational congruences of order three and section genus four.

11. We have already indicated in the preceding section that there exist in S_3 rational congruences ${}^4K^{(3,\,9-r)}$ $(0 \leqslant r \leqslant 4)$, these being defined simply as the *congruences which correspond in S_3 to ordinary cubic surfaces in S_4 through r points $(0 \leqslant r \leqslant 4)$ of the base curve C^4.*

These congruences, as it happens, are the only rational congruences of order three and section genus four which exist in S_3; for they are all contained, as we have seen, in a tetrahedral complex, and Fano* has, in fact, shown that every rational third order congruence of section genus four lies in a tetrahedral complex. The theory of such congruences is, therefore, very simple indeed, and presents marked analogies to the theory of $(2, n)$ congruences representable on a quadric surface. In fact, if Π is the solid containing the cubic surface in S_4 and if P, Q, R, S are the points in which Π meets C^4, then the congruence ${}^4K^{(3,\,9-r)}$ can only contain four elliptic singular points corresponding to the faces of the tetrahedron P, Q, R, S, and r plane pencils corresponding to the neighbourhoods of those of the points P, Q, R, S which lie on the cubic surface. The edges of the tetrahedron $PQRS$ will evidently represent triple, double, or simple rays according as neither, one, or both of their end points lie on the surface.

We easily construct the following table:

$r = 0$ ${}^4K^{(3,\,9)}[6^4]$: six triple rays joining the four points 6^4.

$r = 1$ ${}^4K^{(3,\,8)}[6^1;\ 5^3;\ 1]$: three triple and three double rays.

$r = 2$ ${}^4K^{(3,\,7)}[5^2;\ 4^2;\ 1^2]$: one triple and four double rays.

$r = 3$ ${}^4K^{(3,\,6)}[4^3;\ 3^1;\ 1^3]$: three double rays.

$r = 4$ ${}^4K^{(3,\,5)}[3^4;\ 1^4]$: no multiple rays.

The plane pencils belonging to the last four congruences evidently lie in the faces of the tetrahedron formed by the four elliptic singular points.

* Fano, *loc. cit.*, 69.

The rational congruence ${}^5K^{(3,6)}$ of order three and section genus five.

12.1. In order to obtain a congruence of order three and section genus five we have to find a surface F in S_4 which meets trisecant planes of C^4 (simple or degenerate) in three variable points and which is met by quadrics through C^4 in variable curves of genus five. The simplest such surface is easily seen to be a projectively generated sextic surface ${}^3F^6$ through C^4, or, when C^4 breaks up into (l_1, l_2, l_3, λ), a similar surface through l_1, l_2, l_3; and we shall show now that the congruence so obtained is the unique congruence obtained by Fano*, of order three and section genus five in S_3.

12.2. We take the surface ${}^3F^6$ to pass through the non-degenerate C^4; this gives exactly the same result as if C^4 was degenerate, but the work is more symmetrical. Nevertheless, the degenerate curve affords a more convenient way of passing from the *general* congruence surface ${}^5F^9$ on Ω to the sextic surface ${}^3F^6$ in S_4, since for this purpose it is only necessary to take as fundamental planes a_1, a_2, a_3 on Ω any three of the ten planes which meet ${}^5F^9$ in cubic curves.

The surface ${}^3F^6$ is represented on a plane by curves $C^4(1^{10})$ and the curve C^4, which lies on it, is represented by a line of the plane. Thus the corresponding surface on Ω is represented on the plane by curves $C^7(2^{10})$, *i.e.* it is a (3, 6) surface ${}^5F^9$ of section genus five, *normal in* S_5. It is easily seen to contain ten plane elliptic cubic curves represented on ${}^3F^6$ by the ten plane cubics of the latter surface and on the plane by cubic curves through all but one of the ten base points. These plane cubics, which meet two by two in points, lie evidently in a-planes of Ω and represent therefore the ten elliptic singular points of order three of the associated congruence ${}^5K^{(3,6)}$. There are no other singular points of any kind and no multiple rays.

Thus the general congruence ${}^5K^{(3,6)}[3^{10}]$ can be represented in S_4 by a general projectively generated sextic surface ${}^3F^6$ through the base curve C^4.

12.3. It is easy to see directly that the normal surface ${}^5F^9$ lies on a quadric in S_5; for exactly one quadric can evidently be taken to contain the planes of five of the cubic curves of the surface, the number of conditions so imposed being $6+5+4+3+2=20$, and, since such a quadric must meet the surface residually in one of the five remaining cubic curves, it must evidently contain them all and pass through the whole surface ${}^5F^9$.

* Actually Fano starts from the rational congruence surface ${}^5F^9$ on Ω and, by means of the system of curves adjoint to the prime sections, arrives at the surface ${}^3F^6$ which we use for representing the congruence in S_4.

It is easily calculated, moreover, from the plane representation that ${}^5F^9$ lies on ∞^{12} cubic primals of S_5, and the congruence ${}^5K^{(3,6)}$ belongs therefore to ∞^{12} cubic complexes of which any two evidently intersect residually in a congruence ${}^5K^{(6,3)}$, the dual of the former. We proceed now, by breaking up the latter congruence, to obtain a simple definition of ${}^5K^{(3,6)}$.

Let ${}^5K^{(3,6)}$ be represented in S_4 by a surface ${}^3F^6$ through l_1, l_2, l_3, so that its ten singular points, of which three are the three fundamental points P_1, P_2, P_3 of the representation, are represented partly by the three exceptional solids Σ_1, Σ_2, Σ_3 in S_4, and partly by the planes, $\gamma_1, \ldots, \gamma_7$ say, of the seven elliptic cubic curves of ${}^3F^6$ which meet l_1, l_2, l_3. A cubic primal through l_1, l_2, l_3 represents a cubic complex of S_3 which contains the three point-stars (P_1), (P_2), (P_3); and, since ${}^3F^6$ lies on ∞^3 cubic primals, ${}^5K^{(3,6)}$ belongs to ∞^3 cubic complexes containing (P_1), (P_2), (P_3). Two independent cubic primals through ${}^3F^6$ can be made to contain any two of the seven planes $\gamma_1, \ldots, \gamma_r$, say γ_1, γ_2, and they intersect residually in one further plane δ which meets γ_1, γ_2 in lines but does not meet l_1, l_2, l_3. The two planes γ_1, γ_2 represent two singular points, P_4, P_5 say, of ${}^5K^{(3,6)}$ and δ represents evidently a (1, 3) congruence which is easily seen to have all five points $P_1, \ldots, P_5$ as singular points. We deduce, therefore, the following definition of ${}^5K^{(3,6)}$:

If two cubic complexes in S_2 have five point-stars $(P_1), \ldots, (P_5)$ in common and also the congruence of chords of a cubic curve through $P_1, \ldots, P_5$, then their residual intersection is a general congruence ${}^5K^{(3,6)}$.

Five of the ten singular points of ${}^5K^{(3,6)}$ are at $P_1, \ldots, P_5$.

12.4. We can also obtain another definition of ${}^5K^{(3,6)}$ directly by interpreting in S_3 the projective definition of ${}^3F^6$ by means of four related line-stars in S_4. In fact, if we use the general base curve C^4 in S_4, we find that our congruence is generated by the unique variable line common to related complexes of four collinear nets of quadratic (tetrahedral) complexes which have all a (3, 1) congruence in common. Or by making the latter degenerate (as we may) we find that

${}^5K^{(3,6)}$ is the congruence generated by the unique variable line common to corresponding complexes of four related nets of (tetrahedral) quadratic complexes which have three fixed singular points in common.

12.5. We remark in conclusion on the rather curious interpretation in S_3 of the well-known *double-ten* configuration of lines of ${}^3F^6$ and planes which meet it in cubic curves. For, when C^4 is undegenerate, the

ten planes in question are represented by ten points $P_1, \ldots, P_{10}$ in S_3, the cubic curves are represented by the ten cubic cones which contain the joins of any one of the points to the remaining nine, and the ten lines of ${}^3F^6$ are represented by the ten quadrics through sets of nine of the points P_i, each quadric being considered as a single regulus. It is obvious that the configuration in S_3 is special in regard to the base (3, 1) congruence; for each of the ten reguli must contain three lines of this congruence in order that it may represent a line of S_4; but I do not know how special the configuration must be, regarded simply as a configuration in S_3.

Line systems associated with quadrics.

13.1. *Pencil of quadrics.* In this last section of our paper we use exclusively the representation of the lines of S_3 on the points of S_4 in which the fundamental elements in S_3 are three point-stars (P_1), (P_2), (P_3), and the plane containing them; and we note to begin with that any quadric regulus through P_1, P_2, P_3 represents a line of S_4 and conversely. Two *conjugate* reguli through P_1, P_2, P_3 (belonging to the same quadric) represent, as we have shown in a previous paper*, *two lines of S_4 which form with l_1, l_2, l_3 a set of five associated lines.*

We consider first, then, the (2, 6) congruence ${}^2K^{(2,6)}$ of all the generators of quadrics of a pencil of quadrics L, this being the same, of course, as the congruence of chords of an elliptic ${}^1C^4$, the base curve of L. We take P_1, P_2, P_3 to lie on ${}^1C^4$, and this ensures that every regulus contained in K is represented by a line of S_4. Every point of ${}^1C^4$ is an elliptic cubic singular point of the congruence, and the congruence is evidently represented in S_4, therefore, by an elliptic quintic scroll through l_1, l_2, l_3 (*cf.* § 6.1); for such a surface is known to contain an elliptic series of ∞^1 plane directrix cubics, and it therefore certainly represents a congruence of order two with an elliptic singular curve, *i.e.* the congruence of chords of a ${}^1C^4$. Two conjugate reguli of the congruence represent a pair of generators which belong to an involution of such pairs of the quintic scroll.

Thus the congruence ${}^2K^{(2,6)}$ of generators of quadrics of a pencil can be represented by an elliptic quintic scroll through l_1, l_2, l_3.

13.2. *Net of quadrics.* We consider next how to represent in S_4 the cubic complex, Ψ say, of all generators of quadrics of a net G. Quadrics of G have eight base points, which we can take to be P_1, P_2, P_3, and five others, say $P_4, \ldots, P_8$; and all the reguli will again be represented by

* *Journal London Math. Soc.*, 7 (1932), 267.

lines in S_4. In fact, all reguli through P_1, P_2, P_3 are represented by all lines in S_4, and adding any further number r of base points in S_3 is exactly equivalent to imposing on the lines of S_4 the condition that they should meet the r corresponding transversal planes of l_1, l_2, l_3. Now, if we add four base points P_4, ..., P_7, the fifth point, being an eighth associated point, is thereby implied; and it follows then that all lines which meet four transversal planes of l_1, l_2, l_3 meet a fifth such plane.

Thus in the present transformation the theorem of eight associated points in S_3 becomes the theorem of five associated planes in S_4.

To continue then, we note that all the lines which meet four planes (and therefore a fifth) generate a Segre ten-nodal cubic primal V_3^3. The five points P_4, ..., P_8 represent one set of five associated planes of V_3^3, and the ten joins of the same five points represent, therefore, the ten nodes of the primal. Two lines representing conjugate reguli of Ψ meet any one of the five planes in a pair of points which move in a Geiser involution* in that plane, the base points being the seven points in which the plane is met by the other four planes and by the lines l_1, l_2, l_3. The locus of united points of this involution is a sextic (of genus three) with nodes at the seven base points; and the points of this curve represent the cones of G (which are known to form an ∞^1 family of genus three)†.

Thus the cubic complex of generators of a net of quadrics can be represented by a Segre cubic primal through l_1, l_2, l_3.

13.3. It is easy now to see how the two irrational congruences ${}^4K^{(3,9)}$ arise as belonging to the cubic complex Ψ (*cf.* § 10.21). For we have only to pick out an elliptic scroll from among the ∞^2 generators of V_3^3 and choose it in reference to l_1, l_2, l_3 so that it meets transversal planes of l_1, l_2, l_3 in three free points. The quintic scroll through l_1, l_2 and the sextic scroll through l_1, l_2, l_3 are the two obvious cases.

In regard to the sextic scroll through l_1, l_2, l_3, we note that it is the partial intersection of two cubic primals whose fixed intersection is three skew planes‡ (the three planes of the directrix quartic curves of the surface). It follows then that

${}^4K^{(3,9)}$ is the residual intersection of two cubic complexes which have six singular points in common.

Similarly as regards the quintic scroll through l_1, l_2, we know that it belongs to ∞^4 cubic primals of which exactly one will contain l_1, l_2, l_3;

* The Geiser involution of pairs of points of the plane is that defined by cubic curves which pass through seven fixed points.

† I am indebted for this last remark to the Referee.

‡ *Cf.* author, *loc. cit.*, 390.

but it cannot, as is obvious for other reasons, be defined as a partial intersection of two cubic complexes. A definition involving a cubic and a quartic complex could easily be given.

13.4. The only congruence without singular lines and of the third order, not so far dealt with, is the congruence ${}^6K^{(3,7)}$ of section genus six. This congruence* has twenty elliptic cubic singular points arranged in ten pairs, so that every join of two singular points belongs to the congruence unless the two points form a pair.

The dual congruence ${}^6K^{(7,3)}$ has been known a long time as the congruence of exceptional lines associated with a linear ∞^3 system of quadrics, W say, an exceptional line being a line which lies on an infinity of quadrics W or, which is the same thing, a line which joins a pair of base points of any net of the quadrics W. The system W contains ten plane pairs which are the ten pairs of singular planes of the congruence.

The congruence ${}^6K^{(7,3)}$ is chiefly interesting on account of the fact that it is irrational but representable on a regular surface of genus zero and bigenus unity. Fano obtains this result by projecting the congruence surface ${}^6F^{10}$ on Ω from a trisecant line into three dimensions. All we propose to do here is to obtain a rather different representation of it on an interesting septimic surface which is obviously of genus zero and bigenus unity.

13.5. We remark to begin with that the congruence may be considered as the residual intersection of two cubic complexes which have the congruence of chords of an elliptic C^4 in common. This follows immediately from the property of exceptional lines, the two cubic complexes being those defined by the generators of any two nets of quadrics contained in W.

Now one of the two cubic complexes can be represented by a cubic primal through l_1, l_2, l_3, the congruence of chords of ${}^1C^4$ being represented, at the same time, by an elliptic quintic scroll ${}^1F^5$ through l_1, l_2, l_3 (*cf.* § 12.1). The other cubic complex will then be represented by a sextic primal having l_1, l_2, l_3, and λ as triple lines and passing also through ${}^1F^5$; and the congruence ${}^6K^{(3,7)}$ will be represented, therefore, by the residual intersection of the two primals, which is a surface of order thirteen, F^{13}.

Now cubic primals through ${}^1F^5$ give a Cremona transformation† of S_4 into S_4', of which the reverse transformation is by means of quadrics through an elliptic C^5. And in this transformation the generators of ${}^1F^5$

* *Cf.* Fano, *loc. cit.*, 72.

† *Cf.* Semple, *Phil. Trans. Royal Soc.* (A), 228 (1929), 334.

represent cubic scrolls of S_4' generated by chords of C^5, while to lines of S_4' there correspond conics of S_4 which meet ${}^1F^5$ in five points. Applying this transformation then, we see that the cubic primal is transformed to a solid Π of S_4' and that the sextic primal, since it contains ${}^1F^5$, is transformed to a septimic primal $V_3{}^7$ having C^5 as a triple curve; and, finally, interpreting the fact that the sextic primal has l_1, l_2, l_3 as triple lines, we see that $V_3{}^7$ has the three cubic scrolls corresponding to l_1, l_2, l_3 as double surfaces*. Taking the section of $V_3{}^7$ by Π, we get as the surface corresponding to F^{13} a septimic surface having three double twisted cubics meeting in five common triple points, namely the points where Π meets C^5.

Thus the congruence ${}^6K^{7,3}$ (*or* ${}^6K^{(3,7)}$) *can be represented on a septimic surface* ${}^6F^7$ *which has three double twisted cubic curves meeting in five triple points of the surface.*

This surface is necessarily regular and, since there are no adjoints of order three (there is evidently no cubic surface through the three double cubics), its genus is zero. There is one sextic biadjoint surface, namely the sextic surface composed of *the three quadrics through pairs of the cubics*; and the surface has, therefore, bigenus unity. Its genus P_3 is also zero, and by a theorem of Enriques†, therefore, the surface (and the congruence) is birationally equivalent to the sextic surface which has the edges of a tetrahedron as double lines.

* The $V_3{}^7$ will have a special point at the *point* corresponding to λ, but as this point does not lie in Π, it need not be considered.

† "Sopra le superficie algebriche di bigenere uno", *Mem. di mat. e fis. Soc. It. delle Scienze* (3), 14 (1907), 327–352.

CONCOMITANTS OF POLYNOMIALS IN NON-COMMUTATIVE ALGEBRA

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PART I.

The fact that the theory of four linear or of four quadratic forms in four real variables is equivalent to that of a single linear or of a single quadratic form in one real quaternion variable suggests that the solution of such a real problem may be investigated most simply by solving a corresponding problem in quaternions. Indeed, it implies that a preliminary step in the solution of a real problem, involving several polynomials in n-variables, is the determination of a suitable algebra in which the investigation may be made to depend upon a smaller number of variables and polynomials†. So far two difficulties have stood in the way of the immediate application of this method to the theory of numbers, to the theory of simultaneous equations, and to general algebra. The first is the existence of identical relations, *i.e.*, expressions which are zero for all values of the variables. The determination of all such identical relations, and, as a corollary, the definition of a suitable canonical form for the linear and quadratic polynomials, is necessary, and is solved in this paper by methods that have been used to find the identical relations in quaternions‡. The second difficulty is that the necessary technique for working in non-commutative algebra has not been developed. Thus, apart from the theory of determinants and of systems of linear equations, nothing is known about the algebra of non-commutative systems. It is the object of this paper to begin the study of the invariants and covariants§ of linear and quadratic forms, in

† The theta functions of Jacobi may be found from a single expression in quaternions.

‡ D. E. Littlewood, *Proc. London Math. Soc.* (2), 32 (1931), 312–320.

§ In matrix form.

which the variables and coefficients are numbers in a non-commutative algebra. The algebra is carried sufficiently far to show that an entirely new field lies open to mathematical investigation.

Conventions and abbreviations.

The non-commutative algebra is taken with basis $e_1, e_2, \ldots, e_n$, and it is defined by n^3 constants γ_{pqr} $(1 \leqslant p, q, r \leqslant n)$ and the multiplication table

$$e_p e_q = \Sigma \gamma_{pqr} e_r.$$

The algebra is associative, and hence

$$\sum_r \gamma_{pqr}\gamma_{rst} = \sum_r \gamma_{qsr}\gamma_{prt}.$$

To avoid an ambiguity of expression, γ's will be used with any number of suffixes to represent constants defined to satisfy the following equations:

$$e_a e_b \ldots e_f = \Sigma \gamma_{ab\ldots fg} e_g.$$

Hence the two expressions $\sum_r \gamma_{pqr}\gamma_{rst}$ and $\sum_r \gamma_{qsr}\gamma_{prt}$ will be represented simply as γ_{pqst}. The convention may be used for two suffixes only, so that γ_{pq} represents unity if $p = q$, and zero otherwise.

In writing matrices and determinants, great simplification of expression can often be obtained. In theoretical work, since the matrices are usually defined by rules, it is often sufficient to give the element in the s-th row and t-th column as the specification of the complete matrix. The letters s and t will in future be reserved to represent respectively the row and the column of the matrix from which the element given is taken.

For example, $[(a_{st})]$ is used as an abbreviation for the matrix

$$\begin{bmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ \ldots & \ldots & & \ldots \\ a_{n1}, & & \ldots, & a_{nn} \end{bmatrix}.$$

The law of matrix multiplication may be expressed shortly in the form

$$[(a_{st})][(b_{st})] = [(\sum_u a_{su} b_{ut})].$$

The corresponding determinant will be written $|(a_{st})|$. The abbreviation for bordered matrices needs no explanation. For matrices of matrices, or for matrices with n^2 or nm rows and columns, s and s' may be used to indicate rows, and t and t' to indicate columns, *e.g.* for $n = 2$

$[(a_{st}b_{s't'})]$ represents the matrix

$$\begin{bmatrix} \begin{pmatrix} a_{11}b_{11}, & a_{12}b_{11} \\ a_{21}b_{11}, & a_{22}b_{11} \end{pmatrix} \begin{pmatrix} a_{11}b_{12}, & a_{12}b_{12} \\ a_{21}b_{12}, & a_{22}b_{12} \end{pmatrix} \\ \begin{pmatrix} a_{11}b_{21}, & a_{12}b_{21} \\ a_{21}b_{21}, & a_{22}b_{21} \end{pmatrix} \begin{pmatrix} a_{11}b_{22}, & a_{12}b_{22} \\ a_{21}b_{22}, & a_{22}b_{22} \end{pmatrix} \end{bmatrix}.$$

This example may be regarded as a four-rowed matrix, or as a two-rowed matrix in which the elements are matrices.

The matrix $$T = [(e_t e_s)]$$

is used continually in this paper. For quaternions,

$$T = \begin{bmatrix} 1, & i, & j, & k \\ i, & -1, & -k, & j \\ j, & k, & -1, & -i \\ k, & -j, & i, & -1 \end{bmatrix}$$

is not a singular matrix, in fact, its *left-hand determinant* is equal to 4. The transposed matrix $[(e_s e_t)]$ *is singular*. It is, in fact, the product of the column $[(e_s)]$ with the row $[(e_t)]$, and hence is of rank unity. Thus for complex numbers, in fact for any commutative algebra of order greater than one, T is singular. However, for any algebra which by an extension of the field to complex numbers becomes simply isomorphic with a simple matrix algebra, T is non-singular.

The word *singular*, in connection with matrices with elements in the algebra, is used in the following sense. Let A represent any matrix with elements in the algebra. The set of matrices of the same order as A with elements in the algebra forms an algebra with a finite basis. Hence, if A is a factor of zero in this algebra, we can find another non-zero matrix B such that

$$AB = 0.$$

If A is not a factor of zero, there is a unique matrix B such that

$$AB = BA = 1.$$

In the first case, A is said to be singular, in the second case non-singular.

In a division algebra A is singular or non-singular according as its left-hand determinant is or is not zero.

THEOREM I. *T is invariant for a change of basis of the algebra.*

Suppose the basis of the algebra to be changed to $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, where

$$\epsilon_p = \Sigma\, a_{pq}\, e_q.$$

Then
$$\begin{aligned} T_1 &= [(\epsilon_t\, \epsilon_s)] \\ &= [(\Sigma\, a_{sp}\, a_{lq}\, e_q\, e_p)] \\ &= [(a_{st})][(e_t\, e_s)][(a_{ts})]. \end{aligned}$$

Hence T is invariant.

THEOREM II. *T is non-singular for a simple matrix algebra.*

Let the algebra be of order n^2, and take for its basis

$$e_{11}, e_{12}, \ldots, e_{1n}, e_{21}, e_{22}, \ldots, e_{nn},$$

where e_{pq} represents the matrix with unity in the p-th row and the q-th column and with zero elsewhere. The multiplication table for the algebra is thus

$$e_{pq}\, e_{rs} = e_{ps}\, \gamma_{qr}.$$

The matrix T is of order n^4, and may be represented by

$$[(e_{tt'}\, e_{ss'})] = [(e_{ts'}\, \gamma_{st'})].$$

By a rearrangement of rows, this becomes equal to a matrix in which the matrix $[(e_{ts})]$ is repeated n times down the leading diagonal, there being zeros in other positions. This matrix is singular or non-singular with T.

But $[(e_{ts})]\,[(e_{ts})] = [(\Sigma\, e_{us}\, e_{tu})] = [(\Sigma\, e_{uu}\, \gamma_{st})] = 1$. Hence T is non-singular.

COROLLARY. *In any algebra which, by suitable extension of the field, becomes simply isomorphic with a simple matrix algebra T is non-singular.*

Since it is known† that every normal division algebra over a non-modular field becomes simply isomorphic with a simple matrix algebra, if its field is extended suitably, then T must be non-singular for any normal division algebra over a non-modular field.

THEOREM III. *If T is non-singular for each of two algebras, it is non-singular for the direct product of these algebras.*

Consider the direct product of two algebras with bases $e_1, e_2, \ldots, e_n$, and $\epsilon_1, \epsilon_2, \ldots, \epsilon_m$ respectively. This algebra has nm basal elements $e_p\, \epsilon_q$, each of the symbols e_p commuting with each of the symbols ϵ_q.

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† Dickson, *Algebras and their arithmetics* (1923), 227, Theorem II.

For the direct product the matrix T is $[(e_t \epsilon_{t'} e_s \epsilon_{s'})]$, which may be factorised in the form

$$[(e_t e_s)\gamma_{t's'}][(\epsilon_{t'} \epsilon_{s'})\gamma_{ts}].$$

The theorem follows.

THEOREM IV. *The matrix T is non-singular for the following algebras:*

(*a*) *A simple matrix algebra.*

(*b*) *A normal division algebra over a non-modular field.*

(*c*) *The direct product of a simple matrix algebra and a normal division algebra over a non-modular field.*

Throughout this paper it will be assumed that we are dealing with an algebra which has a modulus, and for which T is non-singular. The majority of the results, however, are quite formal and may be justified by direct multiplication. Hence they will be true in a perfectly general algebra. Many of them are not invalidated even if the field is modular. Our primary interest is, however, with algebras for which T is non-singular. We shall show later that, for such an algebra, the general equation of degree r in the algebra is equivalent to n perfectly general equations of degree r in the n variables.

For some theorems we need the special assumption that the algebra is a division algebra. This will be indicated.

Linear substitutions in a division algebra.

The most general linear substitution in *one* variable may be written

$$x = \sum_{p,\,q=1}^{n} l_{pq} e_p X e_q = \sum_{q=1}^{n} l_q X e_q = \sum_{p=1}^{n} e_p X \tilde{l}_p, \tag{1}$$

where l_q and $\tilde{l}_p$ are numbers of the algebra and l_{pq} is scalar. It is sometimes more convenient to use the first symmetric form with scalar coefficients, while again it may be more convenient to absorb either the e_p or the e_q with the scalar coefficient, so as to form a coefficient in the algebra.

The substitution is non-singular, that is (1) may be solved uniquely for X in terms of x if the left-hand determinant (2) is non-zero†.

$$\Delta_1 = |(\Sigma \gamma_{qst} l_q)|. \tag{2}$$

† A. R. Richardson, *Proc. London Math. Soc.* (2), 28 (1928), 395–420.

The inverse substitution may be expressed in the form

$$X = \sum_{q=1}^{n} L_q x e_q.$$

In *two* variables the general linear substitution is expressed by the two equations

$$\left.\begin{aligned} x &= \Sigma(l_q X e_q + m_q Y e_q), \\ y &= \Sigma(l_q' X e_q + m_q' Y e_q), \end{aligned}\right\} \tag{3}$$

and the condition for singularity is that $\Delta_2 = 0$, where

$$\Delta_2 = \left\| \begin{matrix} \left(\sum_q \gamma_{qst} l_q\right)\left(\sum_q \gamma_{qst} m_q\right) \\ \left(\sum_q \gamma_{qst} l_q'\right)\left(\sum_q \gamma_{qst} m_q'\right) \end{matrix} \right|, \tag{4}$$

a determinant of order $4n^2$.

In the special case of real quaternions these become

$$\left.\begin{aligned} \Delta_1 &= \begin{vmatrix} l_1, & l_2, & l_3, & l_4 \\ -l_2, & l_1, & l_4, & -l_3 \\ -l_3, & -l_4, & l_1, & l_2 \\ -l_4, & l_3, & -l_2, & l_1 \end{vmatrix}, \\ \Delta_2 &= \begin{vmatrix} l_1, & l_2, & l_3, & l_4, & m_1, & m_2, & m_3, & m_4 \\ -l_2, & l_1, & l_4, & -l_3, & -m_2, & m_1, & m_4, & -m_3 \\ -l_3, & -l_4, & l_1, & l_2, & -m_3, & -m_4, & m_1, & m_2 \\ -l_4, & l_3, & -l_2, & l_1, & -m_4, & m_3, & -m_2, & m_1 \\ l_1', & l_2', & l_3', & l_4', & m_1', & m_2', & m_3', & m_4' \\ -l_2', & l_1', & l_4', & -l_3', & -m_2', & m_1', & m_4', & -m_3' \\ -l_3', & -l_4', & l_1', & l_2', & -m_3', & -m_4', & m_1', & m_2' \\ -l_4', & l_3', & -l_2', & l_1', & -m_4', & m_3', & -m_2', & m_1' \end{vmatrix}, \end{aligned}\right\} \tag{5}$$

The extension to n variables is obvious.

The set of non-singular linear substitutions in a given algebra and in a fixed number of variables is a group, termed the *group of linear substitutions*. In the present state of knowledge of the structure of algebras a complete

analysis of this group in a general algebra is impossible. It will be shown later that the group may be generated by two sub-groups, which will be termed the group of *orthogonal* and the group of *diagonal* substitutions. In the case of quaternions it will be shown that the group of linear substitutions may be generated by the three sub-groups

$$(1) \quad x = \lambda X,$$

$$(2) \quad x = X\mu,$$

$$(3) \quad x = X - \nu(X - iXi - jXj - kXk),$$

where λ and μ are quaternions of unit norm, and ν is scalar.

Another set of generators of the linear group in quaternions is given by the following:

$$\left.\begin{aligned} R: &\quad x = \rho X, \\ S: &\quad x = X + \sigma Xi, \\ T: &\quad x = X + \mu Xj + \nu Xk. \end{aligned}\right\} \tag{6}$$

The proof may be readily constructed.

Generators of the linear group in two variables.

Consider
$$x = \sum_{q=1}^{n} (l_q Xe_q + m_q Ye_q),$$

$$y = \sum_{q=1}^{n} (l_q' Xe_q + m_q' Ye_q).$$

The matrix of order $4n^2$, of this substitution, will be represented symbolically as

$$\left[\begin{array}{c|c} L & M \\ \cdots & \cdots \\ L' & M' \end{array}\right], \tag{1}$$

where L denotes the matrix $[(\Sigma \gamma_{qst} l_q)]$, and L', M, M' denote the corresponding expressions in l_q', m_q, m_q'.

It is always possible to choose a non-singular substitution

$$X = X' + \lambda Y'$$

which will make M' non-singular. Suppose then that M' is non-singular, and therefore that it is also non-zero.

In order that (1) may be the product of matrices, written symbolically as

$$\left[\begin{array}{c|c} L & M \\ \cdots & \cdots \\ L' & M' \end{array}\right] = \left[\begin{array}{c|c} \nu & 0 \\ \cdots & \cdots \\ 0 & \nu \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \cdots & \cdots \\ 0 & \rho \end{array}\right]\left[\begin{array}{c|c} 1 & \lambda \\ \cdots & \cdots \\ 0 & 1 \end{array}\right]\left[\begin{array}{c|c} 1 & 0 \\ \cdots & \cdots \\ \mu & 1 \end{array}\right],$$

we must have

$$L = \nu(1+\lambda\mu), \quad L' = \nu\rho\mu,$$

$$M = \nu\lambda, \qquad M' = \nu\rho.$$

Hence

$$\mu = M'^{-1}L',$$

$$\nu = L - MM'^{-1}L' = M'^{-1}\Delta,$$

$$\lambda = \Delta^{-1}M'M,$$

$$\rho = \Delta^{-1}M'^{2},$$

where

$$\Delta = \left\| \begin{array}{c|c} L & M \\ \cdots & \cdots \\ L' & M' \end{array} \right|.$$

Further, the order of the factors is immaterial. Hence we have

THEOREM V. *The linear group in two variables is generated by*

$$\begin{aligned} R: \quad & x = \nu X, \\ & y = \nu Y; \\ S: \quad & x = X, \\ & y = \rho Y; \\ T: \quad & x = X + \lambda Y, \\ & y = Y; \\ U: \quad & x = X, \\ & y = \mu X + Y; \end{aligned}$$

where ν, ρ, λ, μ are linear operators on X and Y, each of which may be expressed as a product of generators as in the last paragraph.

Reducibility of the determinant of transformation.

It is now necessary to notice the fact that the development of a determinant does not consist solely of polynomials in the elements, but

of these multiplied by a permutator, *e.g.* the operator $d(\ )d^{-1}$ in

$$\left\|\begin{matrix} a, & b \\ c, & d \end{matrix}\right| = da - dbd^{-1}c \quad \text{if} \quad d \neq 0.$$

The degree in the coefficients of all terms in the development is however the same, so that, for convenience of statement, the presence of permutators in a term will be disregarded and the word polynomial used in a wider sense. Thus $dbd^{-1}c$ will be termed a polynomial of the second degree in d, b, c.

With this convention a slight extension of the proof usually given in text-books† suffices to show that the general development (right- or left-handed) is irresoluble into factors which are polynomials in the elements of the determinant.

Invariants of a linear form in one variable.

Let $$a(x) = \sum_{q=1}^{n} a_q x e_q = \sum_{p,q=1}^{n} a_{pq} e_p x e_q$$

be a linear form, where a_q is a number of the algebra, but a_{pq} is scalar.

When subjected to the general linear transformation

$$x = \sum_{r=1}^{n} l_r X e_r = \sum_{m,r=1}^{n} l_{mr} e_m X e_r,$$

the linear form becomes

$$a'(X) = \sum_{q,r} a_q l_r X e_r e_q = \sum_{p,m,r,q} a_{pq} l_{mr} e_p e_m X e_r e_q.$$

Hence $$a'_{p'q'} = \Sigma a_{pq} l_{mr} \gamma_{pmp'} \gamma_{rqq'}.$$

THEOREM VI. *The two matrices*

$$A = [(\Sigma a_{pq} e_p e_l e_q e_s)], \quad \tilde{A} = [(\Sigma a_{pq} e_l e_p e_s e_q)]$$

are invariants of the linear form $a(x) = \Sigma a_{pq} e_p x e_q$.

The summation, unless otherwise indicated, is taken with respect to the repeated suffixes.

As above, the transform of

$$a(x) = \Sigma a_{pq} e_p x e_q$$

is $$a'(X) = \Sigma a_{pq} l_{mr} e_p e_m X e_r e_q.$$

† Elliott, *Algebras of quantics* (1895), 19.

Hence the transformed matrix is given by

$$\begin{aligned} A_1 &= [(\Sigma a_{pq} l_{mr} e_p e_m e_t e_r e_q e_s)] = [(\Sigma a_{pq} l_{mr} \gamma_{mtru} e_p e_u e_q e_s)] \\ &= [(\Sigma a_{pq} e_p e_t e_q e_s)]\,[(\Sigma l_{mr} \gamma_{mtrs})] \\ &= A\,[(\Sigma l_{mr} \gamma_{mtrs})]. \end{aligned}$$

Similarly $$\tilde{A}_1 = [(\Sigma l_{mr} \gamma_{msrt})]\,\tilde{A}.$$

The factor $T = [(e_t e_s)]$ may be taken out of A or $\tilde{A}$ on the left or on the right.

Thus $$A = A'\,T = T A'',$$

$$\tilde{A} = T\tilde{A}' = \tilde{A}''\,T,$$

where $$A' = [(\Sigma a_{pq} \gamma_{qst} e_p)], \quad \tilde{A}' = [(\Sigma a_{pq} \gamma_{tps} e_q)],$$

$$A'' = [(\Sigma a_{pq} \gamma_{ptqs})], \quad \tilde{A}'' = [(\Sigma a_{pq} \gamma_{psqt})].$$

The determinant of A' is the determinant of transformation, if the linear form is used as a substitution.

COROLLARY. *The matrices A', $\tilde{A}'$, A'', $\tilde{A}''$ are invariants.*

Representing the invariants of the transposed linear form by the addition of a suffix 1, and representing the corresponding invariants of the linear substitution by L, $\tilde{L}$, L', $\tilde{L}'$, L'', $\tilde{L}''$, we see that

$$\begin{aligned} A_1' &= [(\Sigma a'_{p'q'} \gamma_{q'st} e_{p'})] = [(\Sigma a_{pq} l_{mr} \gamma_{pmp'} \gamma_{rqq'} \gamma_{q'st} e_{p'})] \\ &= [(\Sigma a_{pq} l_{mr} \gamma_{rqst} e_p e_m)] \\ &= [(\Sigma a_{pq} \gamma_{qst} e_p)]\,[(\Sigma l_{mr} \gamma_{mst} e_r)], \\ A_1'' &= [(\Sigma a'_{p'q'} \gamma_{p'tq's})] = [(\Sigma a_{pq} l_{mr} \gamma_{pmp'} \gamma_{rqq'} \gamma_{p'tq's})] \\ &= [(\Sigma a_{pq} l_{mr} \gamma_{pmtrqs})] \\ &= [(\Sigma a_{pq} l_{mr} \gamma_{mtru} \gamma_{puqs})] \\ &= [(\Sigma a_{pq} \gamma_{ptqs})]\,[(\Sigma l_{mr} \gamma_{mtrs})]. \end{aligned}$$

Hence the behaviour of these invariants may be expressed by the following equations:

$$A_1 = A L'', \qquad \tilde{A}_1 = \tilde{L}''\,\tilde{A},$$

$$A_1' = A' L', \qquad \tilde{A}_1' = \tilde{L}'\,\tilde{A}',$$

$$A_1'' = A'' L'', \quad \tilde{A}_1'' = \tilde{L}''\,\tilde{A}''.$$

It is clear that, since A'' and $\tilde{A}''$ are transposed scalar matrices, all these six forms may be regarded as equivalent.

These, however, do not exhaust the invariants of the linear form; there is another set of invariants that is especially important in dealing with conjugate and contragredient substitutions.

The second invariant of the linear form.

THEOREM VII. $\mathcal{A} = [(\Sigma a_{pq} e_q e_l e_p e_s)]$ *is an invariant matrix.*

There is a corresponding form

$$\tilde{\mathcal{A}} = [(\Sigma a_{pq} e_l e_q e_s e_p)].$$

The factor T may be removed from either side.

$$\mathcal{A} = \mathcal{A}' T = T \mathcal{A}'',$$

$$\tilde{\mathcal{A}} = T \tilde{\mathcal{A}}' = \tilde{\mathcal{A}}'' T,$$

where $$\mathcal{A}' = [(\Sigma a_{pq} \gamma_{pst} e_q)], \quad \mathcal{A}'' = [(\Sigma a_{pq} \gamma_{qtps})],$$

$$\tilde{\mathcal{A}}' = [(\Sigma a_{pq} \gamma_{tqs} e_p)], \quad \tilde{\mathcal{A}}'' = [(\Sigma a_{pq} \gamma_{qspt})].$$

Since $\tilde{\mathcal{A}}''$ is the transpose of the scalar matrix $\mathcal{A}''$, $\mathcal{A}$ and $\tilde{\mathcal{A}}$ may be taken as equivalent.

Equations of invariance.

As before, if the suffix 1 denotes the invariant of the transformed linear form, and $\mathcal{L}$, $\tilde{\mathcal{L}}$, etc. denote the second invariants of the linear substitution, it may be proved that

$$\mathcal{A}_1 = \mathcal{L}' \mathcal{A}, \qquad \tilde{\mathcal{A}}_1 = \tilde{\mathcal{A}} \tilde{\mathcal{L}}',$$

$$\mathcal{A}_1' = \mathcal{L}' \mathcal{A}', \qquad \tilde{\mathcal{A}}_1' = \tilde{\mathcal{A}}' \tilde{\mathcal{L}}',$$

$$\mathcal{A}_1'' = \mathcal{L}'' \mathcal{A}'', \quad \tilde{\mathcal{A}}_1'' = \tilde{\mathcal{A}}'' \tilde{\mathcal{L}}''.$$

Obviously the determinants of these invariants must be zero in the same circumstances as the determinants of A, $\tilde{A}$, etc., namely for a singular linear form. For a non-singular form, the matrices A' and $\mathcal{A}'$ will have inverses, hence the determinants cannot be zero, and conversely.

The properties of these invariants may be expressed otherwise.

THEOREM VIII. *The linear operators* $\Sigma a_{pq} e_p (\ ) e_q$ *are simply isomorphic as regards addition and multiplication with the corresponding matrices* A', A'', $\tilde{A}'$, *or* $\tilde{A}''$.

With the matrices $\tilde{A}'$, $\tilde{A}''$, $\mathcal{A}'$, $\mathcal{A}''$, the order of multiplication is reversed.

In quaternions, for the linear form,

$$x = a_1 X + a_2 Xi + a_3 Xj + a_4 Xk,$$

where

$$a_r = a_{1r} + a_{2r} i + a_{3r} j + a_{4r} k,$$

we have

$$A' = \begin{bmatrix} a_1, & a_2, & a_3, & a_4 \\ -a_2, & a_1, & a_4, & -a_3 \\ -a_3, & -a_4, & a_1, & a_2 \\ -a_4, & a_3, & -a_2, & a_1 \end{bmatrix},$$

$$\tilde{\mathcal{A}}' = \begin{bmatrix} a_1, & -a_2, & -a_3, & -a_4 \\ a_2, & a_1, & a_4, & -a_3 \\ a_3, & -a_4, & a_1, & a_2 \\ a_4, & a_3, & -a_2, & a_1 \end{bmatrix},$$

$$A'' = \begin{bmatrix} a_{11}-a_{22}-a_{33}-a_{44}, & a_{12}+a_{21}+a_{34}-a_{43}, & & \\ & & a_{13}-a_{24}+a_{31}+a_{42}, & a_{14}+a_{23}-a_{32}+a_{41} \\ -a_{12}-a_{21}+a_{34}-a_{43}, & a_{11}-a_{22}+a_{33}+a_{44}, & & \\ & & -a_{14}-a_{23}-a_{32}+a_{41}, & a_{13}-a_{24}-a_{31}-a_{42} \\ -a_{13}-a_{24}-a_{31}+a_{42}, & a_{14}-a_{23}-a_{32}-a_{41}, & & \\ & & a_{11}+a_{22}-a_{33}+a_{44}, & -a_{12}+a_{21}-a_{34}-a_{43} \\ -a_{14}+a_{23}-a_{32}-a_{41}, & -a_{13}-a_{24}+a_{31}-a_{42}, & & \\ & & a_{12}-a_{21}-a_{34}-a_{43}, & a_{11}+a_{22}+a_{33}-a_{44} \end{bmatrix}.$$

It will be seen that in the special case of real quaternions $\tilde{\mathcal{A}}'$ may be obtained from A' by changing the sign of the first row and of the first column.

It is interesting to see whether, in the general algebra, a simple transformation enables us to obtain the second invariant from the first. It is required to find a matrix P such that

$$\tilde{\mathcal{A}}' = PA' P^{-1}$$

for every linear substitution.

Since $$A' = [(\Sigma a_{pq}\gamma_{qst}e_p)],$$

$$\tilde{A}' = [(\Sigma a_{pq}\gamma_{tqs}e_p)],$$

it is sufficient to find a scalar matrix P such that

$$[(\gamma_{tqs})]\,P = P\,[(\gamma_{qst})] \quad \text{for all } q.$$

Representing the transpose of P by $\tilde{P}$, we have

$$\tilde{P}\,[(\gamma_{sqt})] = [(\gamma_{qts})]\,\tilde{P} \quad \text{for all } q.$$

But $$[(\gamma_{qts})] = T^{-1}e_q T, \quad [(\gamma_{sqt})] = Te_q T^{-1}.$$

Hence $$T^{-1}e_q T\tilde{P} = \tilde{P}Te_q T^{-1},$$

i.e. $$e_q T\tilde{P}T = T\tilde{P}Te_q.$$

The condition is satisfied if $T\tilde{P}T$ commutes with e_q for every q. Clearly $T\tilde{P}T$ as well as P must be a scalar matrix.

If $$P = [(\lambda_{ts})], \quad \tilde{P} = [(\lambda_{st})],$$

then $$[(e_t e_s)][(\lambda_{st})][(e_t e_s)]$$

must be scalar,

i.e. $$[(\Sigma \lambda_{uv} e_u e_s e_t e_v)]$$

must be scalar.

This is true if the linear form

$$\Sigma \lambda_{uv} e_u x e_v$$

is always scalar for every value of the variable x. We shall show later that it is always possible to obtain such a form and we shall obtain the matrix $[(\lambda_{ts})]$ more explicitly.

For quaternions, the scalar part of x is

$$\tfrac{1}{4}(x - ixi - jxj - kxk),$$

hence $$P = \begin{bmatrix} \frac{1}{4}, & & & \\ & -\frac{1}{4}, & & \\ & & -\frac{1}{4}, & \\ & & & -\frac{1}{4} \end{bmatrix}.$$

This gives the rule, already obtained by inspection, of changing the sign of the first row and column of A' to obtain $\tilde{A}'$.

THEOREM IX. *If T is non-singular there are no linear identities.*

If there is a linear identity, let it be $\Sigma l_{pq} e_p x e_q = 0$. If $x = \Sigma x_r e_r$, this may be written

$$\Sigma l_{pq} \gamma_{prqu} x_r = 0$$

for all values of x and u. Hence

$$A'' = [(\Sigma l_{pq} \gamma_{ptqs})] = 0,$$

$$A' = TA'' T^{-1} = [(\Sigma l_{pq} \gamma_{qst} e_p)] = 0.$$

Equating coefficients of e_p, we have

$$[(\Sigma l_{pq} \gamma_{qst})] = 0 \quad \text{for all } q,$$

$$[(\Sigma l_{pq} \gamma_{qts})] = 0 \quad \text{for all } p,$$

$$\Sigma l_{pq} e_q = T\,[(\Sigma l_{pq} \gamma_{qts})]\, T^{-1} = 0 \quad \text{for all } p.$$

Hence

$$l_{pq} = 0 \quad \text{for all } p \text{ and } q.$$

THEOREM X. *If T is non-singular, every set of n simultaneous linear equations in n variables corresponds to one linear equation in the algebra.*

In the absence of linear identities, this may be deduced immediately by comparing the number of scalar coefficients in the linear form

$$\Sigma l_{pq} e_p x e_q$$

with the n^2 coefficients of the n linear equations in n real variables.

The actual coefficients of the form may be found as follows.

Consider n simultaneous equations

$$X_r = \Sigma a_{ru} x_u.$$

It is required to express this in the form

$$X = \Sigma l_{pq} e_p x e_q,$$

where

$$X = \Sigma X_r e_r, \quad x = \Sigma x_r e_r.$$

Let $A'' = [(a_{st})]$. It is required to find values for l_{pq} such that

$$A'' = [(\Sigma l_{pq}\gamma_{ptqs})],$$

$$A' = TA''T^{-1} = [(\Sigma l_{pq}\gamma_{qst}e_p)]$$

$$= \Sigma B_p e_p,$$

where B_p is a scalar matrix. Its transpose will be written $\tilde{B}_p$, so that

$$\tilde{B}_p = [(\Sigma l_{pq}\gamma_{qts})].$$

Hence $$T\tilde{B}_p T^{-1} = \Sigma l_{pq} e_q.$$

By comparing coefficients of e_q we may obtain the coefficients l_{pq}. With T non-singular, we have the following results.

THEOREM XI. *There are no bilinear identities of the form*

$$\Sigma l_{pqr} e_p x e_q y e_r = \Sigma m_{pqr} e_p x e_q y e_r.$$

A bilinear identity of this form, considered first as a linear form in y, would enable us to compare coefficients and so to obtain a linear identity in the variable x.

The bilinear identities of the form

$$\Sigma l_{pqr} e_p x e_q y e_r = \Sigma m_{pqr} e_p y e_q x e_r$$

will be obtained later.

The bilinear form $\Sigma l_{pqr} e_p x e_q y e_r$, equated to a constant, is equivalent to n bilinear equations in the two sets of variables x_r, y_r. As in Theorem X, by comparison of the number of coefficients, we have

THEOREM XII. *Every set of n simultaneous bilinear equations in two sets, each of n variables, is equivalent to one bilinear equation in the algebra.*

Further, the quadratic form $\Sigma l_{pqr} e_p x e_q x e_r$ is equivalent to n simultaneous quadratics in n variables. Since any quadratic form may be obtained from a corresponding bilinear form by equating the variables, we have

THEOREM XIII. *Any set of n simultaneous quadratic equations in n variables corresponds to one (not necessarily unique) quadratic equation in the algebra.*

The extension to multilinear forms and forms of degree r presents no difficulty. Hence Theorems XII and XIII may be generalized and stated thus:

THEOREM XIV. *Any set of n simultaneous multilinear equations in sets of n real variables is equivalent to one multilinear equation in the algebra. Any set of n simultaneous r-th degree equations in n variables corresponds to one equation of the r-th degree in the algebra.*

Invariants of the extended linear form.

Consider the linear form

$$\Sigma a_{pq} e_p x e_q + \Sigma a_r e_r.$$

The invariant A of the linear terms may be extended to include the independent term $\Sigma a_r e_r$ by bordering the matrix with another row and column. We thus get

$$A^* = \left[\begin{array}{ccc:cc} (\Sigma a_{pq} e_p e_t e_q e_s) & & & \Sigma a_r e_r e_s & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \hdashline & 0 & & & 1 \end{array}\right].$$

The factor

$$T_1 = \left[\begin{array}{cc:c} (e_t e_s) & & 0 \\ \cdots & & \cdots \\ \hdashline 0 & & 1 \end{array}\right]$$

may be taken out on the left or on the right. We then have

$$A^* = A'^* T_1 = T_1 A''^*,$$

$$A'^* = \left[\begin{array}{ccc:cc} (\Sigma a_{pq} \gamma_{qst} e_p) & & & \Sigma a_r e_r e_s & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \hdashline & 0 & & & 1 \end{array}\right],$$

$$A''^* = \left[\begin{array}{ccc:c} (\Sigma a_{pq} \gamma_{ptqs}) & & & a_s \\ \cdots & \cdots & \cdots & \cdots \\ \hdashline & 0 & & 1 \end{array}\right].$$

These satisfy the same invariance equations as the invariants of the homogeneous linear form, namely

$$A_1^* = A^* L''^*, \quad A_1'^* = A'^* L'^*, \quad A_1''^* = A''^* L''^*.$$

The invariants $\tilde{A}$, $\tilde{A}'$, $\tilde{A}''$ may be extended similarly, the independent terms appearing in the last row instead of the last column. Extensions of the invariants of the second type $\mathcal{A}$, $\tilde{\mathcal{A}}$, etc., are not quite so obvious.

We can always find an extension, however, by transforming A'^* by the matrix

$$P_1 = \left[\begin{array}{c|c} P & 0 \\ \cdots & \cdots \\ 0 & 1 \end{array}\right].$$

Since this extension is of considerable importance in the theory of the bilinear and quadratic forms, we shall show how to find it more exactly and find some properties of the matrix P.

Clearly the condition already obtained for the matrix $[(\lambda_{st})]$, namely that $\Sigma \lambda_{pq} e_p x e_q$ is always scalar, does not define the matrix uniquely, since there will be many such scalar forms. For example, in quaternions, if $x = x_1 + x_2 i + x_3 j + x_4 k$, then

$$\begin{aligned} 4x_1 &= \quad x - ixi - jxj - kxk, \\ 4x_2 &= -ix - xi - kxj + jxk, \\ 4x_3 &= -jx - xj - ixk + kxi, \\ 4x_4 &= -kx - xk - jki + ixj \end{aligned}$$

are all scalar forms, as is any linear combination of these.

Any such matrix will transform A' into $\tilde{A}'$, and hence give an extension of $\tilde{A}'$. But these extensions will be different. They may be obtained one from another by a transformation of the form

$$\tilde{A}_2'^* = Q \tilde{A}_1'^* Q^{-1},$$

where

$$Q = \left[\begin{array}{c|c} 1 & 0 \\ \cdots & \cdots \\ 0 & q \end{array}\right],$$

q being any number of the algebra.

We shall define P more exactly in order to find that extension of $\tilde{A}'$ most suitable for the theory of the bilinear and quadratic forms.

LEMMA I. *A basis of the algebra may be chosen so that e_1 is the modulus and $\gamma_{pq1} = \gamma_{qp1}$ for all p and q.*

Obviously e_1 may be chosen as the modulus. To satisfy the second condition it must first be noted that the equation

$$xy - yx = 1$$

has no solution in a finite algebra over a non-modular field.

The *spur* of a number of the algebra of the form $xy-yx$ in any matrix representation with elements in a field, *i.e.* the sum of the terms in the leading diagonal in the representation, is zero. In any algebra over a non-modular field the spur of the modulus is not zero. Hence we may change the basis of the algebra by adding a suitable multiple of the modulus to each of $e_2, e_3, \ldots, e_n$, so as to make the spur of each of these zero in the representation. In this representation, and with the changed basis of the algebra, the spur of $\Sigma a_r e_r$ is the spur of a_1, and if the spur is zero, so also is a_1.

Since, then, the spur of $e_p e_q - e_q e_p$ is zero, and

$$e_p e_q - e_q e_p = \Sigma(\gamma_{pqr} - \gamma_{qpr}) e_r,$$

we clearly have $$\gamma_{pq1} - \gamma_{qp1} = 0.$$

This change of basis of the algebra is not essential to the theory, but it greatly simplifies the proofs, which would be very cumbersome without it. Most results are independent of a change of basis of the algebra. All the matrix invariants of linear forms are invariant for a change of basis of the algebra, as may be readily verified. For a change of basis $e_p = \Sigma a_{pq} \epsilon_q$, each invariant is multiplied on the right by $[(a_{ts})]$ or $[(a_{st})]^{-1}$ and on the left by $[(a_{st})]$ or $[(a_{ts})]^{-1}$, according to the invariant chosen.

THEOREM XV. *If* $x = \Sigma x_r e_r$, *then* x_1 *can be expressed as a linear function of* x,

$$x_1 = \Sigma \lambda_{mn} e_m x e_n.$$

For this linear form $\Sigma \lambda_{mn} e_m x e_n$, we must choose the matrix invariant A'' to have unity in the first row, first column, and zero elsewhere. From this the coefficients λ_{mn} in the linear form may be deduced as in Theorem X.

Definition. We define the matrix P as $[(\lambda_{ts})]$, the scalars λ_{mn} being chosen in accordance with Theorem XV. We define $\tilde{A}'^*$ by means of the equation $\tilde{A}'^* = P_1 A'^* P_1^{-1}$. The extended invariants A^*, $\tilde{A}^*$, etc., are obtained from this by the usual equations.

The following results are easily verified.

THEOREM XVI.

$$\Sigma \lambda_{mn} e_m x y e_n = \Sigma \lambda_{mn} e_m y x e_n.$$

This follows from the fact that $\gamma_{pq1} = \gamma_{qp1}$.

THEOREM XVII. $[(e_t x)]\,[(\lambda_{st})]\,[(y e_s)]$ *is equal to the scalar part of* xy.

These two results are of considerable importance in later theory. We shall prove later that P is a symmetric matrix. Thus P and $\tilde{P}$ are the same matrix, and in the last result P or $[(\lambda_{ts})]$ may be substituted for $[(\lambda_{st})]$.

The solution of the linear equation.

The solution of linear equations by means of the determinants of the invariants A' is known†. The solution is also obtainable from the invariant A''*.

It is required to make a substitution

$$x = \Sigma x_r e_r$$

so that the invariant A_1''* shall have unity in the last row, last column, and zero elsewhere.

We have

$$\begin{bmatrix} (\Sigma a_{pq}\gamma_{ptqs}) & \vdots & a_s \\ \cdots\ \cdots\ \cdots & \vdots & \cdots \\ 0 & \vdots & 1 \end{bmatrix} \begin{bmatrix} x_s \\ \cdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdots \\ 1 \end{bmatrix}.$$

Hence $$x_1 : x_2 : \ldots : x_n : 1 :: |(\Sigma a_{pq}\gamma_{ptqs}) : a_s|.$$

Conjugate transformations in quaternions‡.

If x is subjected to the transformation

$$x = \Sigma a_{pq} e_p X e_q,$$

$\bar{x}$, the conjugate of x, is subjected to another transformation termed the conjugate transformation.

$$\bar{x} = \Sigma b_{pq} e_p \bar{X} e_q = \Sigma a_{pq} \bar{e}_q \bar{X} \bar{e}_p.$$

Take $$e_1 = 1, \quad e_2 = i, \quad e_3 = j, \quad e_4 = k,$$

then $$\bar{e}_1 = 1, \quad \bar{e}_2 = -i, \quad \bar{e}_3 = -j, \quad \bar{e}_4 = -k.$$

It is clear, then, that b_{pq} may be obtained from a_{pq}, by first changing the order of the suffixes p and q, and secondly changing the sign of the terms a_{1r}, a_{r1} $(r = 2, 3, 4)$.

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† A. R. Richardson, *Proc. London Math. Soc.* (2), 28 (1928), 395–420.

‡ In this, as in other parts of the paper, brevity is obtained by considering real quaternions as illustrating the results for the general division algebra.

To obtain the invariants of the conjugate transformation, it must be noted that, first, to take the matrix $\mathcal{A}''$ in the place of A'' is equivalent to changing the order of the suffixes p and q, and secondly, transposing the matrix is equivalent to changing the sign of the terms a_{1r}, a_{r1} $(r = 2, 3, 4)$.

Hence we obtain

$$B'' = \tilde{\mathcal{A}}'', \quad \mathfrak{B}'' = \tilde{A}'',$$

$$\tilde{B}'' = \mathcal{A}'', \quad \tilde{\mathfrak{B}}'' = A''.$$

It should be noticed that there is no correspondence between A' and $\tilde{\mathfrak{B}}'$, since

$$A' = TA''T^{-1},$$

but

$$\tilde{\mathfrak{B}}' = T^{-1}\tilde{\mathfrak{B}}''T.$$

For the special case of quaternions, then, there are extended invariants of the extended linear form, corresponding to $\mathcal{A}''$ and $\tilde{\mathcal{A}}''$, for we can put $\mathcal{A}''^* = \tilde{B}''^*$, *i.e.*

$$\mathcal{A}''^* = \left[\begin{array}{cccc|c} & & (\Sigma a_{pq}\gamma_{qtps}) & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1, & -a_2, & -a_3, & -a_4 & 1 \end{array}\right].$$

Similarly we can obtain $\tilde{\mathcal{A}}''^*$. There will be corresponding extensions of $\mathcal{A}$ and $\mathcal{A}'$ which can be obtained from $\mathcal{A}''^*$ by using the equations

$$\mathcal{A}^* = \mathcal{A}'^* T_1 = T_1 \mathcal{A}''^*.$$

These extensions could also have been obtained from A^* by transforming by the matrix P_1.

Contragredient substitutions.

Consider a non-singular homogeneous linear substitution

$$x = \Sigma a_{pq} e_p X e_q. \tag{1}$$

There will be an inverse substitution

$$X = \Sigma b_{pq} e_p x e_q. \tag{2}$$

Now consider a third substitution

$$y = \Sigma c_{pq} e_p Y e_q, \tag{3}$$

in which $c_{pq} = b_{qp}$.

The substitutions (1) and (3) are said to be contragredient substitutions.

The relation between the invariants is readily obtained. Clearly, since (1) and (2) are inverse substitutions,

$$A''B'' = A'B' = \mathfrak{A}''\mathfrak{B}'' = \mathfrak{A}'\mathfrak{B}' = 1. \tag{4}$$

Again, since the only distinction between (2) and (3) is the order of the factors e_p, e_q in each term,

$$B' = \mathfrak{C}', \quad B'' = \mathfrak{C}'',$$
$$\mathfrak{B}' = C', \quad \mathfrak{B}'' = C''.$$

Hence

THEOREM XVIII. *A necessary and sufficient condition that two substitutions*

$$x = \Sigma a_{pq} e_p X e_q,$$
$$y = \Sigma c_{pq} e_p Y e_q$$

should be contragredient is given by any of the equations

$$A''\mathfrak{C}'' = 1, \quad A''\mathfrak{C}'' = 1,$$
$$A'\mathfrak{C}' = 1, \quad A'\mathfrak{C}' = 1,$$
$$\mathfrak{A}''C'' = 1, \quad \mathfrak{A}''C'' = 1,$$
$$\mathfrak{A}'C' = 1, \quad \mathfrak{A}'C' = 1.$$

The two forms $\Sigma a_{pq} e_p x e_q$, $\Sigma c_{pq} e_p y e_q$ are said to be contragredient forms under the same conditions.

THEOREM XIX. *If two contragredient forms are subjected to contragredient substitutions, the resulting forms will be contragredient.*

In terms of the invariants, we have

$$A_1'' = A''L'', \quad \mathfrak{C}_1'' = \mathfrak{M}''\mathfrak{C}''.$$

Hence the two conditions

$$A''\mathfrak{C}'' = 1, \quad L''\mathfrak{M}'' = 1$$

give immediately $$A_1''\mathfrak{C}_1'' = 1,$$

which proves the theorem.

Orthogonal substitutions.

A homogeneous linear substitution

$$x = \Sigma a_{pq} e_p X e_q$$

on the variable $X = \Sigma X_r e_r$ is said to be orthogonal if it leaves

$$\Sigma X_r^2$$

invariant.

Putting $x = \Sigma x_u e_u$, we have

$$x_u = \Sigma a_{pq} \gamma_{prqu} X_r,$$

$$\Sigma x_u^2 = \Sigma a_{pq} a_{p'q'} \gamma_{prqu} \gamma_{p'r'q'u} X_r X_{r'}.$$

Hence, if $$\Sigma x_u^2 = \Sigma X_r^2,$$

it follows that $$\Sigma a_{pq} a_{p'q'} \gamma_{prqu} \gamma_{p'r'q'u} = 1 \quad \text{if} \quad r = r',$$

$$= 0 \quad \text{if} \quad r \neq r'.$$

Thus $[(\Sigma a_{pq} \gamma_{ptqs})][(\Sigma a_{pq} \gamma_{psqt})] = 1$, and hence we have

THEOREM XX. *A necessary and sufficient condition for a substitution* $x = \Sigma a_{pq} e_p X e_q$ *to be orthogonal is that* $A'' \tilde{A}'' = 1$.

This condition is, of course, dependent upon the chosen basis of the algebra. If the basis is changed the condition is changed. It may be shown, however, that, for a change of basis of the algebra $e_p = \Sigma \alpha_{pp'} \epsilon_{p'}$,

$$A_1'' = [(\alpha_{st})] A'' [(\alpha_{st})]^{-1},$$

$$\tilde{A}_1'' = [(\alpha_{ts})]^{-1} \tilde{A}'' [(\alpha_{ts})].$$

Hence the condition for orthogonality is unchanged by an orthogonal transformation on the basis of the algebra, since

$$[(\alpha_{st})][(\alpha_{ts})] = 1$$

for such a transformatiom, and

$$A'' \tilde{A}'' = 1$$

implies that $$A_1'' \tilde{A}_1'' = 1.$$

For quaternions the usual basis 1, i, j, k gives the norm of x equal to Σx_r^2.

THEOREM XXI. *Orthogonal substitutions form a group.*

For if $C'' = A'' B''$, and hence $\tilde{C}'' = \tilde{B}'' \tilde{A}''$, clearly $A'' \tilde{A}'' = 1$ and $B'' \tilde{B}'' = 1$ give $C'' \tilde{C}'' = 1$.

Invariants of orthogonal substitutions.

A linear form with invariant B'' is transformed to a form with invariant C'' by an orthogonal transformation with invariant A''.

Hence $$C'' = B''\,A'', \quad \tilde{C}'' = \tilde{A}''\,\tilde{B}''.$$

Then, clearly,

$$C''\,\tilde{C}'' = B''\,A''\,\tilde{A}''\,\tilde{B}'' = B''\,\tilde{B}''.$$

Hence $B''\,\tilde{B}''$ is an absolute invariant.

Again $$\tilde{C}''\,C''-\lambda = \tilde{A}''(\tilde{B}''\,B''-\lambda)\,A''.$$

Hence the equation $$|(\tilde{B}''\,B''-\lambda)| = 0$$

is invariant, and is easily shown to be identical with the equation

$$|(B''\,\tilde{B}''-\lambda)| = 0.$$

Since $B''\,\tilde{B}''$ is a symmetric matrix, the roots of this equation are real and positive. For real quaternions the roots λ_i will be in the field of the algebra, as also will be the square roots $\sqrt{\lambda_i}$.

THEOREM XXII. *The group of linear substitutions is generated by orthogonal and diagonal substitutions.*

Definition. A substitution is said to be diagonal if its invariant A'' is a diagonal matrix.

The proof of Theorem XXII now follows.

Let A'' be the invariant of any linear transformation, and let the roots of $|(A''\,\tilde{A}''-\lambda)| = 0$ be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Suppose for simplicity that these roots are all distinct.

The matrix $[(A''\,\tilde{A}''-\lambda_r)]$ is singular ($r = 1, 2, \ldots, n$). Hence we can find scalars $\xi_{r1}, \xi_{r2}, \ldots, \xi_{rn}$, not all zero, such that

$$[(A''\,\tilde{A}''-\lambda_r)][(\xi_{rs})] = 0,$$

$[(\xi_{rs})]$ being a column matrix. Further, these may be chosen so that

$$\sum_p \xi_{rp}^2 = 1.$$

Since $A''\,\tilde{A}''$ is a symmetric matrix,

$$[(\xi_{rt})][(A''\,\tilde{A}''-\lambda_r)] = 0,$$

$[(\xi_{rt})]$ being a one-rowed matrix.

Hence $$[(\xi_{rt})][(A''\tilde{A}''-\lambda_r)][(\xi_{r's})]=0,$$

$$[(\xi_{rt})][(A''\tilde{A}''-\lambda_{r'})][(\xi_{r's})]=0,$$

and by subtraction, since $\lambda_r \neq \lambda_{r'}$,

$$[(\xi_{rt})][(\xi_{r's})]=0 \quad (r \neq r'),$$

$$=1 \quad (r=r').$$

Hence $$[(\xi_{st})][(\xi_{ts})]=1.$$

Again, $$[(A''\tilde{A}'')][(\xi_{rs})]=[(\lambda_r \xi_{rs})];$$

hence $$[(A''\tilde{A}'')][(\xi_{ts})]=[(\xi_{ts})][(\lambda_s \gamma_{st})],$$

or $$[(\xi_{st})][(A''\tilde{A}'')][(\xi_{ts})]=[(\lambda_s \gamma_{st})].$$

Thus $[(\xi_{st})]$ is the invariant B'' of an orthogonal substitution such that

$$B''A''\tilde{A}''\tilde{B}''=[(\lambda_s \gamma_{st})].$$

Consider the diagonal substitution with invariant

$$C''=[(\lambda_s^{\frac{1}{2}}\gamma_{st})].$$

Then $$C''\tilde{C}''=[(\lambda_s \gamma_{st})],$$

$$C''^{-1}B''A''\tilde{A}''\tilde{B}''\tilde{C}''^{-1}=[1],$$

and $C''^{-1}B''A''=D''$, the invariant of an orthogonal substitution.

Hence $$A''=\tilde{B}''C''D'',$$

where $\tilde{B}''$, D'' are the invariants of orthogonal substitutions and C'' is the invariant of a diagonal substitution.

It follows that any substitution may be generated by the consecutive applications of an orthogonal, a diagonal, and a second orthogonal substitution.

In quaternions, these substitutions will belong to the algebra, since the roots λ_i of the characteristic equation of $A''\tilde{A}''$ and also their square roots, belong to the field of the algebra. In other division algebras, where the field is not that of the real numbers, the same thing cannot be predicted, and the roots of the characteristic equation, and their square roots, may exist only in an extension of the field. In this case the required substitutions will exist only in an extension of the algebra corresponding to this extension of the field.

Quaternions.

The general orthogonal substitution in quaternions with determinant $+1$ is given by

$$x = \alpha X \beta,$$

α and β being quaternions of unit norm. The restriction that the determinant shall be $+1$ and not -1 is easily justified, since the substitution in which A'' has $-1, +1, +1, +1$ in the leading diagonal and zero elsewhere is both a diagonal and an orthogonal substitution. Hence, if necessary, it may be combined with both the diagonal and the orthogonal substitution to ensure that the latter has determinant $+1$.

The group of orthogonal substitutions has two obvious sub-groups, commutative with one another, namely

$$x = \alpha X,$$

$$x = X\beta.$$

The general diagonal transformation is given by

$$x = \lambda X + \mu i X i + \nu j X j + \epsilon k X k,$$

where the coefficients λ, μ, ν, and ϵ are scalar. It is easily shown that the general linear group can be generated by substitutions of the three types

$$(a) \quad x = \alpha X,$$

$$(b) \quad x = X\beta,$$

$$(c) \quad x = X - \gamma(X - iXi - jXj - kXk),$$

where α and β are quaternions of unit norm, and γ is scalar.

Other sub-groups of the linear group.

There are three other sub-groups of the linear group with interesting properties. In some respects each resembles the group of orthogonal substitutions.

The first sub-group is the group of substitutions which leave the form $\Sigma e_r x^2 e_r$ invariant.

Under the substitution

$$x = \Sigma a_{pq} e_p X e_q,$$

$\Sigma e_r x^2 e_r$ becomes

$$\Sigma a_{pq} a_{p'q'} e_r e_p X e_q e_{p'} X e_{q'} e_r = \Sigma e_u X a_{pq} a_{p'q'} \gamma_{rpu} \gamma_{q'rv} e_q e_{p'} X e_v$$

$$= \Sigma e_u X^2 e_u,$$

if $$\Sigma a_{pq} a_{p'q'} \gamma_{rpu} \gamma_{q'rv} e_q e_{p'} = \gamma_{uv},$$

i.e. $$A' \tilde{A}' = 1.$$

Hence a substitution will belong to this sub-group provided that its invariants satisfy the equation

$$A' \tilde{A}' = 1.$$

If a linear form with invariant B' is transformed to a form with invariant C' by a substitution of this sub-group with invariant A', then, since

$$C' = B' A', \quad \tilde{C}' = \tilde{A}' \tilde{B}', \quad A' \tilde{A}' = 1,$$

we clearly have $$C' \tilde{C}' = B' \tilde{B}',$$

and $B' \tilde{B}'$ is an absolute invariant under these substitutions.

The second sub-group consists of those substitutions which are identical with their contragredient substitutions.

Clearly a necessary and sufficient condition for a substitution with invariant A'' to belong to this sub-group is that

$$A'' \mathcal{A}'' = 1.$$

In exactly the same manner as above it can be shown that, if a linear form with invariant B'' is transformed by a substitution of this sub-group, $B'' \mathfrak{B}''$ is an absolute invariant.

The third sub-group is that which leaves the norm unaltered.

Invariants in the algebra.

The above invariants are matrices and not, of course, invariants in the algebra, but invariants in the algebra may be found from them.

In the case of absolute invariant matrices, each element of the matrix is an absolute invariant in the algebra.

For scalar matrices, since the determinant of the product of matrices is the product of the determinants, the determinant of the matrix gives us an invariant in the algebra, in fact a scalar invariant.

In the case of matrices with elements in the algebra, some difficulty arises owing to the introduction of permutators. In a division algebra the right- or left-hand determinant of the matrix may be obtained, but the determinant of the product of two matrices is obtained from the determinants of the matrices by multiplying these together with a

permutator. That the introduction of these permutators is unavoidable is illustrated by the following example.

Suppose P to be an invariant belonging to the algebra, and suppose the transform of P to be Pl. Then the transform of P^2 is $PlPl$, which may be written $P^2(P^{-1}lPl^{-1})l^2$. Thus, unless a permutator is disregarded, P is an invariant but P^2 is not. Obviously the permutator is of unit norm, and although in nearly all cases it will contain both the variables and the coefficients of the form as well as those of the linear substitution, it will be disregarded. The following definition of a concomitant will be adopted.

If a relation $\theta(a, x) = 0$ between the coefficients a and the variables x in a number of polynomials involves also the condition that the relation $\theta(A, X) = 0$ holds between the corresponding coefficients and variables after an arbitrary linear substitution, then $\theta(a, x)$ will be termed a *concomitant* of the forms under the linear substitution.

It follows that this definition resembles that for invariant equations in ordinary algebra. In fact, if a concomitant is equated to zero, there will arise n ordinary simultaneous equations. Thus $||A'|$ is an invariant of a linear form. The vanishing of $||A'|$ is the necessary and sufficient condition that the linear form shall be singular.

An alternative is available to the use of these left- and right-hand determinants, and fortunately this alternative is not restricted to division algebras. Since every algebra is simply isomorphic with a matrix algebra we can find invariants in the field over which the algebra is defined by replacing the elements of our matrices by the corresponding matrices in scalars. We can, therefore, determine complete scalar invariants and covariants from the determinants of the compound matrices. Sets of matrices simply isomorphic with the elements of the algebra are given by the matrices R_x and S_x on pages 93–95 of Dickson's "Algebras and their arithmetics".

The resultant of two linear forms in a single real quaternion.

The theory of the resultant of m linear forms in n variables $(m > n)$ can be illustrated sufficiently by reference to the case of two linear forms in a single real quaternion variable.

The general linear form $a_1x+a_2xi+a_3xj+a_4xk+d$, if non-singular, will have a single real quaternion as a zero; if it is singular, there may be no zero, as with the equation $xi-ix = 1$, or there may be an infinite number†.

† A. R. Richardson, *Proc. London Math. Soc.* (2), 28 (1928), 416.

Also linear forms in a single variable may have common zeros. For example, of the following equations, the first contains all the roots of the others.

(i) $$(i+j)x+(1-k)xi+(j-i)xj+(1+k)xk+2(1-i)=0$$

has the general solution

$$x=k+(3-i+j-k)X+(1-i+j+k)Xi+(-1-i-j+k)Xj$$
$$+(-1-i-j+k)Xk,$$

X being an arbitrary quaternion.

(ii) $$2x+(1+i-j+k)xi+(1-i+j-k)xj=2(i+j)$$

has the general solution

$$x=j+(2-j-k)X+(-2i+j-k)Xi+(-1+i)Xj+(1+i)Xk.$$

(iii) $$x(1+j)+ixi-ixk=2j,$$

has the general solution

$$x=j+X-iXi-Xj+iXk.$$

We shall determine a necessary and sufficient condition in order that two linear forms in a single variable may have all their roots in common.

Let the linear forms be

$$A: \quad a_1x+a_2xi+a_3xj+a_4xk+d=0, \tag{1}$$

$$B: \quad b_1x+b_2xi+b_3xj+b_4xk+c=0. \tag{2}$$

The common roots, if any, satisfy the equations

$$\begin{aligned}
a_1x+a_2xi+a_3xj+a_4xk+d &= 0,\\
-a_2x+a_1xi+a_4xj-a_3xk+di &= 0,\\
-a_3x-a_4xi+a_1xj+a_2xk+dj &= 0,\\
-a_4x+a_3xi-a_2xj+a_1xk+dk &= 0,\\
b_1x+b_2xi+b_3xj+b_4xk+c &= 0,\\
-b_2x+b_1xi+b_4xj-b_3xk+ci &= 0,\\
-b_3x-b_4xi+b_1xj+b_2xk+cj &= 0,\\
-b_4x+b_3xi-b_2xj+b_1xk+ck &= 0.
\end{aligned}$$

It is therefore necessary for the matrices

$$\left.\begin{array}{cccc|cccc}
a_1, & a_2, & a_3, & a_4, & a_1, & a_2, & a_3, & a_4, \; d \\
-a_2, & a_1, & a_4, & -a_3, & -a_2, & a_1, & a_4, & -a_3, \; di \\
-a_3, & -a_4, & a_1, & a_2, & -a_3, & -a_4, & a_1, & a_2, \; dj \\
-a_4, & a_3, & -a_2, & a_1, & -a_4, & a_3, & -a_2, & a_1, \; dk \\
b_1, & b_2, & b_3, & b_4, & b_1, & b_2, & b_3, & b_4, \; c \\
-b_2, & b_1, & b_4, & -b_3, & -b_2, & b_1, & b_4, & -b_3, \; ci \\
-b_3, & -b_4, & b_1, & b_2, & -b_3, & -b_4, & b_1, & b_2, \; cj \\
-b_4, & b_3, & -b_2, & b_1, & -b_4, & b_3, & -b_2, & b_1, \; ck
\end{array}\right\} \qquad (3)$$

to have the same left rank less than or equal to 4.

The condition is also sufficient. If they have the same left rank, then real quaternions X, Y, Z, W, not all zero, exist such that

$$\begin{aligned}
a_1X+a_2Y+a_3Z+a_4W+d &= 0, \\
-a_2X+a_1Y+a_4Z-a_3W+di &= 0, \\
-a_3X-a_4Y+a_1Z+a_2W+dj &= 0, \\
-a_4X+a_3Y-a_2Z+a_1W+dk &= 0,
\end{aligned}$$

and also the four similar equations involving b_1, b_2, b_3, b_4, and c. Multiply these equations respectively by 1, $-i$, $-j$, $-k$ on the right and add; if we write $x=\frac{1}{4}(X-Yi-Zj-Wk)$, we find that x satisfies the equations

$$a_1x+a_2xi+a_3xj+a_4xk+d=0$$

and

$$b_1x+b_2xi+b_3xj+b_4xk+c=0.$$

Hence the condition is both necessary and sufficient.

The matrices (3) are invariant, in our previous sense, under the linear substitution

$$x=l_1X+l_2Xi+l_3Xj+l_4Xk,$$

being multiplied on the right by the matrix in which

$$\begin{array}{cccc}
l_1, & l_2, & l_3, & l_4, \\
-l_2, & l_1, & l_4, & -l_3, \\
-l_3, & -l_4, & l_1, & l_2, \\
-l_4, & l_3, & -l_2, & l_1,
\end{array}$$

occurs in the upper left-hand corner, and zeros elsewhere except in the remaining diagonal elements, which are 1.

The question can be dealt with from a different aspect. Since the left rank of the matrices (3) is less than or equal to 4,

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4, & d \\ -a_2, & a_1, & a_4, & -a_3, & di \\ -a_3, & -a_4, & a_1, & a_2, & dj \\ -a_4, & a_3, & -a_2, & a_1, & dk \\ b_1, & b_2, & b_3, & b_4, & c \end{vmatrix} = 0.$$

Hence

$$\begin{vmatrix} a_1, & -a_2, & -a_3, & -a_4, & b_1 \\ a_2, & a_1, & -a_4, & a_3, & b_2 \\ a_3, & a_4, & a_1, & -a_2, & b_3 \\ a_4, & -a_3, & a_2, & a_1, & b_4 \\ d, & di, & dj, & dk, & c \end{vmatrix} = 0.$$

Therefore real quaternions $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ exist such that

$$\begin{aligned} \alpha_1\alpha_1-\alpha_2\alpha_2-\alpha_3\alpha_3-\alpha_4\alpha_4 &= b_1, \\ \alpha_1\alpha_2+\alpha_2\alpha_1-\alpha_3\alpha_4+\alpha_4\alpha_3 &= b_2, \\ \alpha_1\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_1-\alpha_4\alpha_2 &= b_3, \\ \alpha_1\alpha_4-\alpha_2\alpha_3-\alpha_3\alpha_2+\alpha_4\alpha_1 &= b_4, \\ \alpha_1 d+\alpha_2 di+\alpha_3 dj+\alpha_4 dk &= c. \end{aligned}$$

These are the conditions that

$$\begin{aligned} &\alpha_1(a_1x+a_2xi+a_3xj+a_4xk+d)+\alpha_2(a_1x+a_2xi+a_3xj+a_4xk+d)i \\ &\quad+\alpha_3(a_1x+a_2xi+a_3xj+a_4xk+d)j+\alpha_4(a_1x+a_2xi+a_3xj+a_4xk+d)k \\ &\qquad= b_1x+b_2xi+b_3xj+b_4xk+c. \end{aligned} \tag{4}$$

In other words a linear operator $\phi : \alpha_1(\ )+\alpha_2(\ )i+\alpha_3(\ )j+\alpha_4(\ )k$ exists such that $\phi A = B$.

Equation (4) shows that every zero of A is also a zero of B. If ϕ is non-singular, the only possible zero of $\phi A = 0$ is $A = 0$. If, however, ϕ is singular, B will have zeros which are not zeros of A.

It is evident that

$$\begin{bmatrix} b_1, & b_2, & b_3, & b_4, & c \\ -b_2, & b_1, & b_4, & -b_3, & ci \\ -b_3, & -b_4, & b_1, & b_2, & cj \\ -b_4, & b_3, & -b_2, & b_1, & ck \\ 0, & 0, & 0, & 0, & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1, & a_2, & a_3, & a_4, & 0 \\ -a_2, & a_1, & a_4, & -a_3, & 0 \\ -a_3, & -a_4, & a_1, & a_2, & 0 \\ -a_4, & a_3, & -a_2, & a_1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} a_1, & a_2, & a_3, & a_4, & d \\ -a_2, & a_1, & a_4, & -a_3, & di \\ -a_3, & -a_4, & a_1, & a_2, & dj \\ -a_4, & a_3, & -a_2, & a_1, & dk \\ 0, & 0, & 0, & 0, & 1 \end{bmatrix}.$$

Hence a necessary and sufficient condition for A and B to have all their zeros in common, including cases in which neither has a zero, is that a non-singular operator ϕ can be found such that $\phi A = B$.

In commutative algebra the condition may be expressed by equating the resultant to zero, but with non-commutative algebra the resultant belongs to two linear forms in two variables, for the equation

$$a_1 x + a_2 xi + a_3 xj + a_4 xk + dy = 0$$

may have a solution for which $y \neq 1$.

The corresponding resultant is

$$R = \begin{Vmatrix} a_1, & a_2, & a_3, & a_4, & d, & 0, & 0, & 0 \\ -a_2, & a_1, & a_4, & -a_3, & 0, & d, & 0, & 0 \\ -a_3, & -a_4, & a_1, & a_2, & 0, & 0, & d, & 0 \\ -a_4, & a_3, & -a_2, & a_1, & 0, & 0, & 0, & d \\ b_1, & b_2, & b_3, & b_4, & c, & 0, & 0, & 0 \\ -b_2, & b_1, & b_4, & -b_3, & 0, & c, & 0, & 0 \\ -b_3, & -b_4, & b_1, & b_2, & 0, & 0, & c, & 0 \\ -b_4, & b_3, & -b_2, & b_1, & 0, & 0, & 0, & c \end{Vmatrix},$$

and the condition $R = 0$ is necessary and sufficient in order that

$$a_1 x + a_2 xi + a_3 xj + a_4 xk + dy = 0,$$
$$b_1 x + b_2 xi + b_3 xj + b_4 xk + cy = 0$$

may have solutions x, y in common, but it does not follow that among these there are solutions for which $y = 1$.

The bilinear form.

The general bilinear form in two variables x and y may be written

$$\Sigma(a_{pqr}\, e_p\, x e_q\, y e_r + b_{pqr}\, e_p\, y e_q\, x e_r). \tag{1}$$

It has been shown† that, in quaternions, sufficient bilinear identities exist to express any bilinear form in the form

$$f = \Sigma\, l_{pqr}\, e_p\, x e_q\, y e_r. \tag{2}$$

We shall now show how this can be accomplished in the general algebra.

Bilinear identities.

LEMMA I. *If* $[(\gamma_{st1})]$ *is singular, the algebra contains a nilpotent invariant sub-algebra.*

If $[(\gamma_{st1})]$ is singular, there exists a non-zero number of the algebra $x = \Sigma\, x_r e_r$ such that

$$[x_t][(\gamma_{st1})] = 0.$$

Let $y = \Sigma\, y_r e_r$ be any other number of the algebra. The basis being chosen as in the Lemma to Theorem XV, the spurs of $e_2, \ldots, e_n$ in any scalar matrix representation are zero. Then in this matrix representation of the algebra, since

$$xy = \Sigma\, [x_t][(\gamma_{str})][y_s]\, e_r,$$

the spur of xy is equal to

$$[x_t][(\gamma_{st1})][y_s]$$

times the spur of the modulus, and is consequently zero. Hence the spur of x and of every multiple of x is zero.

Let the complex roots of the characteristic equation of x be $\lambda_1, \ldots, \lambda_m$. Since the spur of x is zero,

$$\Sigma\, \lambda_r = 0.$$

Again the spur of x^p is zero, and hence

$$\Sigma\, \lambda_r{}^p = 0.$$

† D. E. Littlewood, *Proc. London Math. Soc.* (2), 32 (1931), 312–320.

It follows that all the roots of the characteristic equation are zero, and x is nilpotent. Similarly every multiple of x is nilpotent, x is properly nilpotent, and there exists an invariant nilpotent sub-algebra†.

LEMMA II. *For any algebra containing an invariant nilpotent sub-algebra, T is singular.*

Let e_p belong to the invariant nilpotent sub-algebra. Let $x = x_p e_p$.

If T is non-singular, by Theorem X we may express x_p as a linear function of x. But every linear function of x belongs to the invariant nilpotent sub-algebra, and x_p, being scalar, is not nilpotent. Hence T is singular.

Since we are concerned with algebras in which T is non-singular, we may assume that $[(\gamma_{st1})]$ also is non-singular.

Put $$[(\gamma_{st1})]^{-1} = [(\alpha_{st})].$$

Then, if $$x = \Sigma x_r e_r,$$

we have $$x_1 = \Sigma \lambda_{mn} e_m x e_n.$$

Again, $$\Sigma \lambda_{mn} e_m e_u x e_n = \Sigma x_r \gamma_{ur1}.$$

Hence we have

$$\Sigma \lambda_{mn} \alpha_{uv} e_m e_u x e_n e_v = \Sigma \alpha_{uv} x_r \gamma_{ur1} e_v = x, \tag{3}$$

since $$\Sigma \alpha_{pq} \gamma_{rp1} = \gamma_{rq}.$$

The bilinear identities may now be obtained. They are

$$xy = \Sigma \lambda_{mn} \alpha_{uv} e_m e_u x y e_n e_v$$
$$= \Sigma \lambda_{mn} \alpha_{uv} e_m y e_u x e_n e_v,$$

since the scalar part of $e_u xy$ is equal to the scalar part of $y e_u x$ (Theorem XVI).

Also $$e_p x e_q y e_r = \Sigma \lambda_{mn} \alpha_{uv} e_m e_q y e_r e_u e_p x e_n e_v.$$

Hence we have

THEOREM XXIII.

$$\Sigma l_{pqr} e_p x e_q y e_r = \Sigma l_{pqr} \lambda_{mn} \alpha_{uv} \gamma_{mqp'} \gamma_{rupq'} \gamma_{nvr'} e_{p'} y e_{q'} x e_{r'}.$$

† Dickson, *Algebras and their arithmetics* (1923), 47.

This is the general expression for the bilinear identities and enables us to express any bilinear form in the form (2). That these are the only bilinear identities follows from Theorem XI.

THEOREM XXIV. *The matrix* $P = [(\lambda_{ts})]$ *is a symmetric matrix.*

Since $\gamma_{st1} = \gamma_{ts1}$, $[(\gamma_{st1})]$ is a symmetric matrix and hence so also is $[(\alpha_{st})]$.

From (3) in Theorem XXIII we have

$$x = \Sigma \lambda_{mn} \alpha_{uv} e_m e_u x e_n e_v.$$

Since there are no linear identities we may equate coefficients, and hence

$$\left.\begin{aligned} \Sigma \lambda_{mn} \alpha_{uv} \gamma_{mup} \gamma_{nvq} &= 1 \quad \text{if } p = q = 1, \\ &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{4}$$

From this, since $\alpha_{uv} = \alpha_{vu}$,

$$\left.\begin{aligned} \Sigma \lambda_{mn} \alpha_{uv} \gamma_{nuq} \gamma_{mvp} &= 1 \quad \text{if } p = q = 1, \\ &= 0 \quad \text{otherwise.} \end{aligned}\right\} \tag{5}$$

Equation (4) is equivalent to n^2 linear equations which may be taken as defining the n^2 scalars λ_{mn}. By comparison of (4) and (5) we may deduce that

$$\lambda_{mn} = \lambda_{nm},$$

i.e. that P is a symmetric matrix, provided that the determinant of the coefficients of λ_{mn} in the n^2 equations is not zero.

This determinant is

$$|(\Sigma \alpha_{uv} \gamma_{sut} \gamma_{s'vt'})|.$$

But

$$\begin{aligned} [(\Sigma \alpha_{uv} \gamma_{sut} \gamma_{s'vt'})]\, [(\Sigma \lambda_{mn} \gamma_{smt} \gamma_{s'nt'})]\, TT' \\ &= [(\Sigma \alpha_{uv} \lambda_{mn} \gamma_{sumt} \gamma_{s'vnt'})]\, [(e_t e_s)]\, [(e_{t'} e_{s'})] \\ &= [(\Sigma \alpha_{uv} \lambda_{mn} e_t e_s e_u e_m e_{t'} e_{s'} e_v e_n)] \\ &= [(e_t e_s e_{t'} e_{s'})] = TT', \end{aligned}$$

where $$T = [(e_t e_s)], \quad T' = [(e_{t'} e_{s'})].$$

Hence the matrix $[(\Sigma \alpha_{uv} \gamma_{sut} \gamma_{s'vt'})]$ is non-singular and our theorem is proved.

This suggests that there is room for investigation of a new type of product of linear forms.

If $\Sigma a_{mn} e_m x e_n$, $\Sigma b_{pq} e_p x e_q$ are two linear forms, we may define another type of product of these forms by

$$\Sigma a_{mn} b_{pq} e_m e_p x e_n e_q.$$

There is a matrix invariant for this type of product, namely

$$[(\Sigma a_{mn} \gamma_{smt} \gamma_{s'nt'})].$$

Division is possible and unique provided that this matrix is non-singular.

THEOREM XXV.

$$[\alpha_{st}] = n\,[\lambda_{st}],$$

where $[\alpha_{st}] = [\gamma_{st1}]^{-1}$, $[\lambda_{st}]$ *is the matrix* P *of Theorem XV, and* n *is the order of the algebra.*

We show first that $[\alpha_{st}]$ is a scalar multiple of $[\lambda_{st}]$.

We have
$$\begin{aligned} TPT &= [e_t e_s]\,[\lambda_{st}]\,[e_t e_s] \\ &= [(\Sigma \lambda_{uv} e_u e_s e_t e_v)] \\ &= [\gamma_{st1}]. \end{aligned}$$

Hence $TPT\,[\alpha_{st}] = 1$. It is sufficient to show that $TPTP$ is a scalar.

Now
$$\begin{aligned} TPTPT &= [(\Sigma \lambda_{uv} \lambda_{pq} e_q e_s e_v e_p e_t e_u)] \\ &= [(\Sigma \lambda_{uv} \lambda_{pq} e_q e_v e_s e_p e_t e_u)] \quad \text{(Theorem XVI)} \\ &= [(\Sigma \lambda_{uv} \lambda_{pq} e_q e_v e_p e_t e_s e_u)] \\ &= [(\Sigma \lambda_{u1} e_t e_s e_u)] \quad (\text{since } \Sigma \lambda_{pq} e_q e_v e_p = \gamma_{v1}) \\ &= Tz, \end{aligned}$$

where $z = \Sigma \lambda_{u1} e_u$.

Again,
$$\begin{aligned} TPTPT &= [(\Sigma \lambda_{uv} \lambda_{pq} e_q e_v e_p e_t e_s e_u)] \\ &= [(\Sigma \lambda_{uv} \lambda_{pq} e_q e_v e_t e_s e_p e_u)] \\ &= [(\Sigma \lambda_{uv} \lambda_{pq} e_q e_t e_s e_v e_p e_u)] \\ &= [(\Sigma \lambda_{1q} e_q e_t e_s)] \\ &= zT. \end{aligned}$$

Since z is commutative with T, it is commutative with every number of the algebra. Hence either z is scalar or there is a linear identity

$$xz - zx = 0.$$

The latter is incompatible with the assumption that T is non-singular. Hence z is scalar.

We now show that $z = 1/n$. In the identity

$$x_1 = \Sigma \lambda_{pq} e_p x e_q,$$

put $x = 1$. We get

$$1 = \Sigma \lambda_{pq} e_p e_q.$$

Hence $$\Sigma \lambda_{pq} \gamma_{pq1} = 1.$$

But $$\Sigma \alpha_{pq} \gamma_{qv1} = \gamma_{pv},$$

$$\Sigma \alpha_{pq} \gamma_{qp1} = \Sigma \gamma_{pp} = n.$$

It follows that $z = 1/n$, and

$$[a_{st}] = n\,[\lambda_{st}].$$

COROLLARY. *We may rewrite equation* (3) *of Theorem XXIII in the form*

$$\Sigma \lambda_{mn} \lambda_{uv} e_m e_u x e_n e_v = \frac{x}{n} = \Sigma \lambda_{mn} \lambda_{uv} e_m x e_u e_n e_v$$

$$= \Sigma \lambda_{mn} \lambda_{uv} e_m e_u e_n x e_v.$$

Again, $$TPTPT = \frac{T}{n}.$$

Hence $$TPTP = \frac{1}{n},$$

$$T^{-1} = nPTP.$$

This gives an easy way of finding the inverse of T. In quaternions,

$$T^{-1} = 4 \begin{bmatrix} \frac{1}{4} & & & \\ & -\frac{1}{4} & & \\ & & -\frac{1}{4} & \\ & & & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1, & i, & j, & k \\ i, & -1, & -k, & j \\ j, & k, & -1, & -i \\ k, & -j, & i, & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & & & \\ & -\frac{1}{4} & & \\ & & -\frac{1}{4} & \\ & & & -\frac{1}{4} \end{bmatrix}.$$

Hence $$4T^{-1} = \begin{bmatrix} 1, & -i, & -j, & -k \\ -i, & -1, & -k, & j \\ -j, & k, & -1, & -i \\ -k, & -j, & i, & -1 \end{bmatrix}.$$

Invariants of the bilinear form.

We consider the bilinear form in the form (2). If x and y are subjected to the substitutions

$$x = \Sigma a_{mn} e_m X e_n,$$

$$y = \Sigma b_{mn} e_m Y e_n,$$

then f becomes

$$f_1 = \Sigma l_{pqr} a_{mn} b_{m'n'} e_p e_m X e_n e_q e_{m'} Y e_{n'} e_r.$$

Hence $$l'_{p'q'r'} = \Sigma l_{pqr} a_{mn} b_{m'n'} \gamma_{pmp'} \gamma_{nqm'q'} \gamma_{n'rr'}.$$

THEOREM XXVI. *The matrices*

$$\left[\left(\frac{\partial^2 f}{\partial x_s \partial y_t}\right)\right], \quad \left[\left(\frac{\partial^2 f}{\partial x_t \partial y_s}\right)\right]$$

are invariants of the bilinear form.

If $$x = \Sigma x_r e_r,$$

then† $$\frac{\partial x}{\partial x_r} = e_r.$$

Similarly $$\frac{\partial y}{\partial y_r} = e_r.$$

Hence $$\theta = \left[\left(\frac{\partial^2 f}{\partial x_s \partial y_t}\right)\right] = [(\Sigma l_{pqr} e_p e_s e_q e_t e_r)],$$

$$\bar{\theta} = \left[\left(\frac{\partial^2 f}{\partial x_t \partial y_s}\right)\right] = [(\Sigma l_{pqr} e_p e_t e_q e_s e_r)].$$

For the transformed bilinear form we have

$$\begin{aligned}\theta_1 &= [(\Sigma l_{pqr} a_{mn} b_{mn} e_p e_m e_s e_n e_q e_m e_t e_n e_r)] \\ &= [(\Sigma a_{mn} \gamma_{msnu} l_{pqr} e_p e_u e_q e_v e_r b_{m'n'} \gamma_{m'tn'v})] \\ &= [(\Sigma a_{mn} \gamma_{msnt})] [(\Sigma l_{pqr} e_p e_s e_q e_t e_r)] [(\Sigma b_{m'n'} \gamma_{m'tn's})] \\ &= \tilde{A}'' \theta B''.\end{aligned}$$

Similarly we have $$\bar{\theta}_1 = \tilde{B}'' \bar{\theta} A''.$$

Hence θ and $\bar{\theta}$ are invariants of the bilinear form.

† This defines the operator $\partial/\partial x_r$.

The actual bilinear form f may be obtained from either of these invariants by means of the equations

$$f = [(x_t)]\,\theta\,[(y_s)],$$
$$f = [(y_t)]\,\theta\,[(x_s)].$$

The factor T may be taken out of either θ or $\bar{\theta}$ on the right or on the left, or on both sides of θ. Thus

$$T^{-1}\theta T^{-1} = [(\Sigma\, l_{pqr}\gamma_{ps}e_q\gamma_{tr})]$$
$$= [(\Sigma\, l_{sqt}e_q)],$$
$$T^{-1}\bar{\theta} = [(\Sigma\, l_{pqr}\gamma_{ptqs}e_r)],$$
$$\bar{\theta}T^{-1} = [(\Sigma\, l_{pqr}\gamma_{qsrt}e_p)].$$

The equations of invariance of these are easily obtained, namely

$$T^{-1}\theta_1 T^{-1} = A'(T^{-1}\theta T^{-1})B',$$
$$T^{-1}\bar{\theta}_1 = B'(T^{-1}\bar{\theta})A'',$$
$$\bar{\theta}_1 T^{-1} = B''(\theta T^{-1})A'.$$

Other invariants of the bilinear form.

It is clear that the elements of the matrices θ and $\bar{\theta}$ are the same save for the order of the terms in the product e_p, e_s, e_q, e_t, and e_r. This suggests that other permutations of these terms in the product may lead to other invariants, just as the different linear invariants were obtained by different permutations of e_p, e_t, e_q, and e_s.

By trial, it is found that the following matrices are invariants, while other permutations of the terms do not lead to invariants.

THEOREM XXVII. *The matrices*

$$\chi = [(\Sigma\, l_{pqr}e_re_te_qe_se_p)],$$
$$\psi = [(\Sigma\, l_{pqr}e_re_pe_te_qe_s)],\quad \tilde{\psi} = [(\Sigma\, l_{pqr}e_te_re_pe_se_q)],$$
$$\phi = [(\Sigma\, l_{pqr}e_qe_te_re_pe_s)],\quad \tilde{\phi} = [(\Sigma\, l_{pqr}e_te_qe_se_re_p)]$$

are invariant.

The following equations of invariance may be verified:

$$\chi_1 = \mathfrak{B}'\chi\tilde{\mathfrak{A}}',$$
$$\psi_1 = \mathfrak{B}'\psi A'',\quad \tilde{\psi}_1 = \tilde{A}''\tilde{\psi}\tilde{\mathfrak{B}}',$$
$$\phi_1 = \mathfrak{A}'\phi B'',\quad \tilde{\phi}_1 = \tilde{B}''\tilde{\phi}\tilde{\mathfrak{A}}'.$$

The factor T may be taken out of any of these from either side with a corresponding change in the equation of invariance. It should be noted especially that

$$\psi'' = T^{-1}\psi = [(\Sigma l_{pqr}\gamma_{rptqs})],$$

$$\phi'' = T^{-1}\phi = [(\Sigma l_{pqr}\gamma_{qtrps})],$$

$$\tilde{\psi}'' = \tilde{\psi}T^{-1} = [(\Sigma l_{pqr}\gamma_{rpsqt})],$$

$$\tilde{\phi}'' = \tilde{\phi}T^{-1} = [(\Sigma l_{pqr}\gamma_{qsrpt})].$$

These are scalar matrices. The first is the transpose of the third, and the second the transpose of the fourth. Hence ψ and $\tilde{\psi}$ may be taken as equivalent, as also ϕ and $\tilde{\phi}$. The determinants of ψ, $\tilde{\psi}$, ϕ, and $\tilde{\phi}$ are zero simultaneously, since the vanishing of any one of these is the condition for

$$\Sigma l_{pqr} e_r e_p x e_q$$

to be a singular linear form.

If the matrix χ is known, the bilinear form is completely defined.

$$[(x_t)]\chi[(y_s)] = \Sigma l_{pqr} e_r y e_q x e_p,$$

and if the form $\Sigma l_{pqr} e_r y e_q x e_p$ is known, the original form $\Sigma l_{pqr} e_p x e_q y e_r$ may be deduced from the coefficients.

Knowledge of the other invariants ψ, $\tilde{\psi}$, ϕ, $\tilde{\phi}$ does not define the bilinear form completely. These are all absolutely invariant under the transformation on the bilinear form

$$f_1 = \alpha^{-1} f \alpha,$$

where α is any number of the algebra. Hence knowledge of the invariant matrices cannot convey all the information about the bilinear form. This can be remedied by modifying the invariants as follows:

$$\psi(\xi) = [(\Sigma l_{pqr} e_r \xi e_p e_t e_q e_s)],$$

$$\tilde{\psi}(\xi) = [(\Sigma l_{pqr} e_t e_r \xi e_p e_s e_q)],$$

$$\phi(\xi) = [(\Sigma l_{pqr} e_q e_t e_r \xi e_p e_s)],$$

$$\tilde{\phi}(\xi) = [(\Sigma l_{pqr} e_t e_q e_s e_r \xi e_p)],$$

where ξ is an arbitrary number of the algebra. The equations of invariance are unaltered.

As before, the determinants of $\psi(\xi)$ and $\phi(\xi)$ are zero simultaneously, and, if $\psi(\xi)$ is given as a matrix function of ξ, the bilinear form is completely defined.

The effect of a linear transformation on the bilinear form.

Let the bilinear form be subjected to the transformation

$$f_1 = \Sigma a_{mn} e_m f e_n.$$

Then it may be proved that

$$\theta_1 = \Sigma a_{mn} e_m \theta e_n,$$

$$\bar{\theta}_1 = \Sigma a_{mn} e_m \bar{\theta} e_n,$$

$$\chi_1 = \Sigma [(\gamma_{smt})] \chi [(\gamma_{nts})] a_{mn},$$

$$\psi_1(\eta) = \psi(\xi), \quad \tilde{\psi}_1(\eta) = \tilde{\psi}(\xi),$$

$$\phi_1(\eta) = \phi(\xi), \quad \tilde{\phi}_1(\eta) = \tilde{\phi}(\xi),$$

where

$$\eta = \Sigma a_{mn} e_n \xi e_m.$$

Hence none of these invariants is properly invariant under a linear transformation on the form.

Rectangular matrix invariants.

It is apparently impossible to obtain a non-singular *square* matrix which is invariant for transformations both on the form and on the variable, but the following *rectangular* matrices (n by n^2 and n^2 by n) are invariant for both of these:

$$[(\Sigma l_{pqr} e_p e_s e_q e_t e_r e_{s'})], \quad [(\Sigma l_{pqr} e_{t'} e_p e_t e_q e_s e_r)],$$

$$[(\Sigma l_{pqr} e_p e_t e_q e_s e_r e_{s'})], \quad [(\Sigma l_{pqr} e_{t'} e_p e_s e_q e_t e_r)],$$

$$[(\Sigma l_{pqr} e_t e_p e_s e_q e_{s'} e_r)], \quad [(\Sigma l_{pqr} e_p e_t e_q e_{t'} e_r e_s)],$$

as are also the twelve others which can be obtained from these by a cyclic permutation of the elements e_p, e_q, and e_r.

Of these the following are especially important:

$$\Theta = [(\Sigma l_{pqr} e_p e_t e_q e_{t'} e_r e_s)], \quad \tilde{\Theta} = [(\Sigma l_{pqr} e_t e_p e_s e_q e_{s'} e_r)],$$

$$\Phi = [(\Sigma l_{pqr} e_{t'} e_q e_t e_r e_s e_p)], \quad \tilde{\Phi} = [(\Sigma l_{pqr} e_q e_s e_r e_t e_p e_{s'})],$$

$$\Psi = [(\Sigma l_{pqr} e_{t'} e_r e_t e_p e_s e_q)], \quad \tilde{\Psi} = [(\Sigma l_{pqr} e_r e_t e_p e_s e_q e_{s'})].$$

The factor T may be removed from these either on the right or on the left to give scalar matrices

$$\Theta'' = [(\Sigma l_{pqr} \gamma_{ptqt'rs})], \quad \ldots.$$

The importance of these invariants lies in the fact that, for a substitution on the variables, both factors appear on the right, or both on the left,

e.g., $$\Theta_1 = \Theta A''(B'')'.$$

$(B'')'$ is the usual invariant of the substitution on y, but the rows and columns used correspond to s' and t', the matrix being repeated identically for the different rows and columns corresponding to s and t, for $s = t$. Similarly the matrix A'' is repeated identically for the different rows and columns corresponding to s' and t' for $s' = t'$.

If the columns of Θ corresponding to t and t' are interchanged, the resulting matrix will be denoted by $(\Theta)'$, *e.g.*,

$$(\Theta'')' = [(\Sigma l_{pqr}\gamma_{pt'qtrs})].$$

Similarly for $\tilde{\Theta}$, Φ, $\tilde{\Phi}$, Ψ, $\tilde{\Psi}$.

Invariants of the bilinear form under special substitutions.

Cogredient substitutions.

Let x and y be subjected to cogredient substitutions, *i.e.*

$$A'' = B''.$$

Then $$\theta_1 = \tilde{A}''\theta A'',$$

$$\bar{\theta}_1 = \tilde{A}''\bar{\theta}A''.$$

Hence $\theta+\lambda\bar{\theta}$ is invariant, where λ is an arbitrary scalar. The values of λ, if any, for which this matrix is singular are absolute invariants. In other words,

$$|\theta+\lambda\bar{\theta}| = 0$$

is an invariant equation.

Since the elements of this matrix are in the algebra, this equation is difficult to deal with, for the theory of such equations is not known. A preferable form is given by

$$|\psi''-\lambda\phi''| = 0.$$

This also is clearly an invariant equation in λ. Or, better still, we have

$$|\psi''(\xi)-\lambda\phi''(\xi)| = 0,$$

$$|\psi''(\xi)-\phi''(\eta)| = 0,$$

which are invariant equations in ξ and λ, and in ξ and η respectively.

Contragredient substitutions.

Let x and y be subjected to contragredient substitutions, *i.e.*

$$A'' \mathfrak{B}'' = 1, \quad \mathfrak{A}'' B'' = 1.$$

Then $$\psi_1'' = \mathfrak{A}'' \psi'' B''.$$

Hence $$\psi_1'' - \lambda = \mathfrak{A}''(\psi'' - \lambda) B'',$$

and $$|\psi'' - \lambda| = 0$$

is an invariant equation.

Further, $$|\psi''(\xi) - \lambda| = 0$$

is an invariant equation in λ and ξ.

The invariants of the bilinear form under contragredient substitutions may be dealt with in another way, by using the rectangular matrix invariants.

Consider $$\Phi'' = [(\Sigma l_{pqr} \gamma_{qtrspt'})];$$

we have $$\Phi_1'' = \Phi'' A''(\tilde{\mathfrak{B}}'')'.$$

Similarly $$(\tilde{\Phi}_1'')' = (\tilde{A}'')' \mathfrak{B}''(\tilde{\Phi}'')'.$$

Hence $\Phi''(\tilde{\Phi}'')'$ is an absolute invariant. Similarly $(\Phi'')'(\tilde{\Phi}'')$ is an absolute invariant, but this is equivalent to the above being obtained from it by an interchange of rows and a corresponding interchange of columns.

By the same means, it may be proved that

$$\Psi''(\tilde{\Psi}'')' \quad \text{and} \quad (\Psi'')' \tilde{\Psi}''$$

are absolutely invariant.

Orthogonal substitutions.

We now find the invariants of the bilinear form when the variables are subjected to arbitrary orthogonal substitutions. We have

$$A'' \tilde{A}'' = B'' \tilde{B}'' = 1.$$

Hence $\Theta'' \tilde{\Theta}''$ is an absolute invariant, or equivalently $\Theta\tilde{\Theta}$.

Other sub-groups.

The sub-group of substitutions which are identical with their contragredient substitutions also leads to an absolute matrix invariant of the bilinear form.

We have $$A''\mathfrak{A}''=1, \quad \tilde{\mathfrak{B}}''\tilde{B}''=1.$$

Hence $\Psi''\tilde{\Psi}''$ is absolutely invariant.

The method does not seem to be applicable to the sub-group of substitutions which leaves $\Sigma e_r x^2 e_r$ invariant.

Extensions of the bilinear invariants.

Consider the extended bilinear form

$$f=\Sigma l_{pqr}e_p x e_q y e_r+\Sigma l_{pq}e_p x e_q+\Sigma l'_{qr}e_q y e_r+\Sigma l_q e_q.$$

The invariant θ may be extended to include the linear and independent terms as follows:

$$\theta^*=\left[\begin{array}{cccc|ccc} (\Sigma l_{pqr}e_p e_s e_q e_t e_r) & & & & \Sigma l_{pq}e_p e_s e_q & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \multicolumn{4}{c|}{\Sigma l'_{qr}e_q e_t e_r} & \multicolumn{3}{c}{\Sigma l_q e_q} \end{array}\right].$$

It is readily seen that

$$\theta_1^*=\tilde{A}''^*\theta^* B''^*,$$

and

$$f=[(x_t:1)]\theta\left[\begin{array}{c}(y_s)\\ \cdots \\ 1\end{array}\right].$$

Similarly we can extend $\bar{\theta}$, so as to get

$$\bar{\theta}^*=\left[\begin{array}{cccc|ccc} (\Sigma l_{pqr}e_p e_t e_q e_s e_r) & & & & \Sigma l'_{qr}e_q e_s e_r & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \multicolumn{4}{c|}{\Sigma l_{pq}e_p e_t e_q} & \multicolumn{3}{c}{\Sigma l_q e_q} \end{array}\right].$$

The following equations may be verified:

$$\bar{\theta}_1^*=\tilde{B}''^*\bar{\theta}^* A''^*,$$

$$f=[(y_t:1)]\bar{\theta}^*\left[\begin{array}{c}(x_s)\\ \cdots \\ 1\end{array}\right].$$

The extensions of χ, ψ, and ϕ present more difficulty, since these involve the extensions of $\mathfrak{A}'$ or $\mathfrak{B}'$. To obtain the terms in the bordering row and column, we take a homogeneous bilinear form and subject x and y to non-homogeneous linear substitutions. This will show us what the extension must be if such extension exists at all. For completeness it would be necessary to verify that the bordered matrix obtained is invariant for a

substitution in any bilinear form, but, since this verification is perfectly straightforward and rather lengthy, it will not be included here.

Let the substitutions be

$$x = \Sigma a_{mn} e_m X e_n + \Sigma a_u e_u,$$

$$y = \Sigma b_{mn} e_m Y e_n + \Sigma b_v e_v.$$

Then

$$\begin{aligned}\Sigma l_{pqr} e_p x e_q y e_r &= \Sigma l_{pqr} a_{mn} b_{m'n'} e_p e_m X e_n e_q e_{m'} Y e_{n'} e_r + \Sigma l_{pqr} a_{mn} b_v e_p e_m X e_n e_q e_v e_r \\ &\quad + \Sigma l_{pqr} a_u b_{m'n'} e_p e_u e_q e_{m'} Y e_{n'} e_r \\ &\quad + \Sigma l_{pqr} a_u b_v e_p e_u e_q e_v e_r \\ &= \Sigma m_{pqr} e_p X e_q Y e_r + \Sigma m_{pq} e_p X e_q + \Sigma m'_{qr} e_q Y e_r + \Sigma m_q e_q;\end{aligned}$$

$$\phi_1^* = A_1'^* \phi^* B''^*$$

$$= \begin{bmatrix} (\Sigma a_{mn} \gamma_{mst} e_n) & \vdots & 0 \\ \cdots \quad \cdots \quad \cdots & \vdots & \cdots \\ \Sigma a_u \lambda_{zt} e_z e_u & \vdots & 1 \end{bmatrix} \begin{bmatrix} (\Sigma l_{pqr} e_q e_t e_r e_p e_s) & \vdots & 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \vdots & \cdots \\ 0 & \vdots & 0 \end{bmatrix} \begin{bmatrix} (\Sigma b_{m'n'} \gamma_{m'tn's}) & \vdots & b_s \\ \cdots \quad \cdots \quad \cdots & \vdots & \cdots \\ 0 & \vdots & 1 \end{bmatrix}.$$

It is clear that the element in the s-th row in the bordered column must be

$$\Sigma l_{pqr} a_{mn} b_v e_n e_q e_v e_r e_p e_m e_s = \Sigma m_{pq} e_q e_p e_s.$$

Again, the element in the t-th column in the bordered row is

$$\begin{aligned}\Sigma l_{pqr} \lambda_{zv} a_u b_{m'n'} e_z e_u e_q e_{m'} e_t e_{n'} e_r e_p e_v &= \Sigma l_{pqr} \lambda_{zv} a_u b_{m'n'} e_z e_p e_u e_q e_{m'} e_t e_{n'} e_r e_v \\ &= \Sigma m'_{qr} \lambda_{zv} e_z e_q e_t e_r e_v.\end{aligned}$$

Lastly, the element in the last row and last column is

$$\begin{aligned}\Sigma l_{pqr} \lambda_{zw} a_u b_v e_z e_u e_q e_v e_r e_p e_w &= \Sigma l_{pqr} \lambda_{zw} a_u b_v e_z e_p e_u e_q e_v e_r e_w \\ &= \Sigma m_q \lambda_{zw} e_z e_q e_w,\end{aligned}$$

i.e. the scalar part of the independent term.

Hence, for the bilinear form

$$\Sigma m_{pqr} e_p x e_q y e_r + m_{pq} e_p x e_q + m'_{qr} e_q y e_r + m_q e_q,$$

we have the invariant

$$\phi^* = \begin{bmatrix} (\Sigma m_{pqr} e_q e_t e_r e_p e_s) & \vdots & \Sigma m_{pq} e_q e_p e_s \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \vdots & \cdots \quad \cdots \quad \cdots \\ \Sigma m'_{qr} \lambda_{zv} e_z e_q e_t e_r e_v & \vdots & \Sigma \lambda_{zw} m_q e_z e_q e_w \end{bmatrix}.$$

There is a corresponding extension of $\phi(\xi)$. We give the extended invariant $\psi^*(\xi)$ which can be obtained in the same way:

$$\psi^*(\xi) = \begin{bmatrix} (\Sigma m_{pqr} e_r \xi e_p e_t e_q e_s) & \Sigma m'_{qr} e_r \xi e_q e_s \\ \cdots \quad \cdots \quad \cdots \quad \cdots & \cdots \quad \cdots \quad \cdots \\ \Sigma m_{pq} \lambda_{zv} e_z \xi e_p e_t e_q e_v & \Sigma m_q \lambda_{zw} e_z \xi e_q e_w \end{bmatrix}.$$

We can make the substitution on the homogeneous bilinear form

$$x = \Sigma x_r e_r, \quad y = \Sigma y_r e_r.$$

Hence it is readily shown that the scalar part of any multiple of the bilinear form may be obtained from either of the invariants $\phi(\xi)$ or $\psi(\xi)$.

We get $$[(e_t x)]\,[(\lambda_{st})]\,\phi(\xi)\,[(y_s)] = [(x_t)]\,TP\phi(\xi)\,[(y_s)]$$
$$= \Sigma \lambda_{zw} e_z \xi f e_w,$$

i.e. the scalar part of ξf.

But $$[(x_t)]\,\theta\,[(y_s)] = f,$$

$$[(x_t)]\,\Sigma \lambda_{zw} e_z \xi\theta e_w\,[(y_s)] = \text{the scalar part of } \xi f.$$

Hence $$TP\phi(\xi) = \Sigma \lambda_{uv} e_u (\xi\theta) e_v.$$

This formula, which shows that ϕ is dependent on θ, may be proved directly. We have

$$\begin{aligned} TP\phi(\xi) &= [(e_t e_s)]\,[(\lambda_{st})]\,[(\Sigma l_{pqr} e_q e_t e_r \xi e_p e_s)] \\ &= [(\Sigma l_{pqr} \lambda_{uv} e_u e_s e_q e_t e_r \xi e_p e_v)] \\ &= [(\Sigma l_{pqr} \lambda_{uv} e_u \xi e_p e_s e_q e_t e_r e_v)] \\ &= \Sigma \lambda_{uv} e_u\,[(\Sigma l_{pqr} \xi e_p e_s e_q e_t e_r)]\,e_v \\ &= \Sigma \lambda_{uv} e_u \xi\,[(\Sigma l_{pqr} e_p e_s e_q e_t e_r)]\,e_v \\ &= \Sigma \lambda_{uv} e_u \xi\theta e_v. \end{aligned}$$

We can treat the invariant $\psi(\xi)$ similarly, and prove that

$$TP\psi(\xi) = \Sigma \lambda_{uv} e_u \xi \bar{\theta} e_v.$$

Both of these equations hold, with small modification, for the extended invariants. T and P are replaced by T_1 and P_1, which are obtained by bordering T and P by zeros save in the last row and last column, in which we have unity.

We may obtain the extension of χ similarly, by making non-homogeneous substitutions in a homogeneous bilinear form. We must have in the bordered row

$$\Sigma \lambda_{uv} m'_{qr} e_u e_q e_t e_r e_v,$$

and in the bordered column

$$\Sigma \lambda_{uv} m_{pq} e_u e_p e_s e_q e_v.$$

The element in the last row and last column is

$$\Sigma \lambda_{uv} \lambda_{wz} m_q e_u e_w e_q e_v e_z.$$

Hence we can obtain

$$\Sigma \lambda_{uv} \lambda_{wz} e_u e_w f e_v e_z = [(y_l)]\, TP\chi PT\, [(x_s)].$$

Hence
$$TP\chi PT = \Sigma \lambda_{uv} \lambda_{wz} e_u e_w \bar{\theta} e_v e_z,$$

with similar equations for the extended forms.

We have thus expressed all the other invariants in terms of θ and $\bar{\theta}$.

Since, as we have shown, knowledge of χ completely defines the bilinear form, it cannot vanish for a non-zero form. It follows then, from the last equation, that

$$\Sigma \lambda_{uv} \lambda_{wz} e_u e_w x e_v e_z$$

is a non-singular linear form. This may be contrasted with

$$\Sigma \lambda_{uv} e_u x e_v,$$

which is, of course, a singular form.

In the special case of quaternions,

$$\Sigma \lambda_{uv} \lambda_{wz} e_u e_w x e_v e_z = \tfrac{1}{4} x.$$

By the corollary to Theorem XXV we have, in the general algebra,

$$\Sigma \lambda_{uv} \lambda_{wz} e_u e_w x e_v e_z = \frac{x}{n}.$$

Hence
$$nTP\chi PT = \bar{\theta}.$$

This may be proved directly. We have

$$\begin{aligned} nTP\chi PT &= n\,[e_t e_s][\lambda_{st}][(\Sigma l_{pqr} e_r e_t e_q e_s e_p)][\lambda_{st}][e_t e_s] \\ &= [(n \Sigma \lambda_{uv} \lambda_{wz} l_{pqr} e_u e_s e_r e_w e_q e_v e_p e_t e_z)] \\ &= [(n \Sigma \lambda_{uv} \lambda_{wz} l_{pqr} e_u e_s e_r e_w e_p e_t e_q e_v e_z)] \\ &= [(n \Sigma \lambda_{uv} \lambda_{wz} l_{pqr} e_u e_p e_t e_q e_s e_r e_w e_v e_z)] \\ &= [(\Sigma l_{pqr} e_p e_t e_q e_s e_r)] = \bar{\theta}. \end{aligned}$$

The homogeneous quadratic form.

Since the quadratic form is a particular case of the bilinear form when the variables are identical, the invariants of the bilinear form are also invariants of the quadratic form.

Suppose that $$f = \Sigma l_{pqr} e_p x e_q x e_r.$$

Since the variables are identical, they will, of course, be cogredient, and the invariants of the bilinear form under cogredient substitutions will be invariants of the quadratic form, *e.g.*,

$$|\psi''(\xi) - \lambda\phi''(\xi)| = 0$$

is an invariant equation.

These, however, have not the same importance for the quadratic form, since the existence of quadratic identities necessitates a different treatment.

The quadratic identities.

We have $$f = [(x_t)]\,\theta\,[(x_s)],$$

also $$f = [(x_t)]\,\bar{\theta}\,[(x_s)].$$

Now $\bar{\theta}$ is the transposed matrix of θ. Hence, if θ is a skew-symmetric matrix, it follows that

$$\theta + \bar{\theta} = 0,$$

and $$2f = [(x_t)][\theta + \bar{\theta}][(x_s)] = 0,$$

for all values of x. Hence we have

THEOREM XXVIII. *If θ is a skew-symmetric matrix it is the invariant of a quadratic identity, and conversely.*

The proof of the converse may be put in *reductio ad absurdum* form as follows.

Let θ be the invariant of a quadratic identity, and not a skew-symmetric matrix. θ may be expressed as the sum of a skew-symmetric matrix θ_1 and a non-zero symmetric matrix θ_2. Since θ_1 is the invariant of a quadratic identity, $\theta_2 = \theta - \theta_1$ is also the invariant of a quadratic identity. Hence

$$[x_t]\,\theta_2\,[x_s] = 0$$

for all values of $x = \Sigma x_r e_r$.

Putting x equal to $e_1, e_2, \ldots, e_n$ in succession, it follows that the terms in the leading diagonal are all zero. Since θ is non-zero, there is a term, say in the p-th row and q-th column, that is not zero. Then, for $x = e_p + e_q$, we have

$$[x_l]\,\theta_2\,[x_s] \neq 0,$$

which is a contradiction.

THEOREM XXIX. *If f is a quadratic identity, $\psi(\xi)+\phi(\xi) = 0$. Conversely, if $\psi(\xi)+\phi(\xi) = 0$, for all values of ξ, then f is a quadratic identity.*

Since
$$TP\psi(\xi) = \Sigma\lambda_{uv}\, e_u(\xi\theta)\, e_v,$$
$$TP\phi(\xi) = \Sigma\lambda_{uv}\, e_u(\xi\bar{\theta})\, e_v,$$

if
$$\theta+\bar{\theta} = 0,$$

it follows that
$$TP\,[\psi(\xi)+\phi(\xi)] = 0,$$

i.e.
$$\psi(\xi)+\phi(\xi) = 0.$$

Conversely, if
$$\Sigma\lambda_{uv}\, e_u[\xi(\theta+\bar{\theta})]\, e_v = 0$$

for all values of ξ, then

$$\theta+\bar{\theta} = 0,$$

and f must be a quadratic identity.

The reduced form of the quadratic.

Let the quadratic form

$$f = \Sigma l_{pqr}\, e_p\, xe_q\, xe_r$$

have invariant θ. Then

$$\theta = \tfrac{1}{2}(\theta+\bar{\theta})+\tfrac{1}{2}(\theta-\bar{\theta}).$$

$\frac{1}{2}(\theta-\bar{\theta})$ is a skew-symmetric matrix, and hence it is the invariant $[(\Sigma m_{pqr}\, e_p\, e_s\, e_q\, e_l\, e_r)]$ of a quadratic identity $\Sigma m_{pqr}\, e_p\, xe_q\, xe_r$. Now put

$$f_1 = \Sigma(l_{pqr}-m_{pqr})\, e_p\, xe_q\, xe_r.$$

Then
$$f_1 = f$$

for all values of x, and the invariant θ_1 of f_1 is $\frac{1}{2}(\theta+\bar{\theta})$, a symmetric matrix. Hence

$$\theta_1 = \bar{\theta}_1.$$

f_1 is said to be the reduced form of f. A homogeneous quadratic is said to be in reduced form if $\theta-\bar{\theta}$ is zero.

THEOREM XXX. *The transform of a quadratic in reduced form is also in reduced form.*

This follows from the fact that $\theta-\bar{\theta}$ is invariant.

THEOREM XXXI. *For a quadratic in reduced form* $\psi(\xi)-\phi(\xi)=0$, *and conversely.*

The proof is exactly the same as the proof that

$$\psi(\xi)+\phi(\xi)=0$$

for a quadratic identity, save that the positive sign is replaced by a negative sign.

Henceforward it will be assumed that all homogeneous quadratics, and, in fact, the homogeneous quadratic terms in any quadratic, are put into reduced form. The importance of this form is here illustrated.

Owing to the existence of these quadratic identities, *i.e.* expressions which are zero for all values of the variable, the expression for a quadratic is ambiguous. Any quadratic identity may be added to the form so as to alter the form itself and yet leave the values of the form unaltered. Thus, if two quadratics are equal, it is not permissible to equate coefficients. It is possible by means of the quadratic identities to eliminate certain of the terms. Thus in quaternions we may make the left-hand coefficients of

$$x^2i, \quad xixi, \quad x^2j, \quad xjxj, \quad x^2k, \quad xkxk$$

zero. By this means the ambiguity may be removed and then coefficients may be equated. However, if a substitution is made on the variable, these terms will reappear, and will have to be eliminated all over again.

However, the expression in the reduced form is unique, coefficients may be equated, and if a substitution is made either on the variable or on the form itself, the form remains reduced.

The invariants of the quadratic in reduced form are

$$\theta=[(\Sigma l_{pqr}e_p e_s e_q e_t e_r)]=[(\Sigma l_{pqr}e_p e_t e_q e_s e_r)],$$

$$\chi=[(\Sigma l_{pqr}e_r e_t e_q e_s e_p)],$$

$$\psi(\xi)=[(\Sigma l_{pqr}e_r \xi e_p e_t e_q e_s)]=[(\Sigma l_{pqr}e_q e_t e_r \xi e_p e_s)],$$

to which may be added

$$\tilde{\psi}(\xi) = [(\Sigma l_{pqr} e_t e_r \xi e_p e_s e_q)] = [(\Sigma l_{pqr} e_t e_q e_s e_r \xi e_p)],$$

which is equivalent to ψ, by transposition.

To express a quadratic in reduced form.

Let $f = \Sigma l_{pqr} e_p x e_q x e_r$ be any quadratic form.

Express $\Sigma l_{pqr} e_p x e_q y e_r$ in the form $\Sigma m_{pqr} e_p y e_q x e_r$ by means of the bilinear identities.

Then $$\Sigma l_{pqr} e_p x e_q x e_r = \Sigma \tfrac{1}{2}(l_{pqr}+m_{pqr}) e_p x e_q x e_r.$$

Also $$\Sigma \tfrac{1}{2}(l_{pqr}+m_{pqr}) e_p x e_q y e_r = \Sigma \tfrac{1}{2}(l_{pqr}+m_{pqr}) e_p y e_q x e_r,$$

so that

$$[\{\Sigma \tfrac{1}{2}(l_{pqr}+m_{pqr}) e_p e_s e_q e_t e_r\}] = [\{\Sigma \tfrac{1}{2}(l_{pqr}+m_{pqr}) e_p e_t e_q e_s e_r\}].$$

Hence $\Sigma \tfrac{1}{2}(l_{pqr}+m_{pqr}) e_p x e_q x e_r$ is the reduced form of $\Sigma l_{pqr} e_p x e_q x e_r$.

Rectangular matrix invariants.

There are the same rectangular matrix invariants for the quadratic as for the bilinear form. In particular we have

$$\Theta = [(\Sigma l_{pqr} e_p e_t e_q e_{t'} e_r e_s)].$$

For a quadratic in reduced form,

$$(\Theta)' = \Theta.$$

Again $$\Psi = [(\Sigma l_{pqr} e_{t'} e_r e_t e_p e_s e_q)],$$

$$\Phi = [(\Sigma l_{pqr} e_{t'} e_q e_t e_r e_s e_p)].$$

For a quadratic in reduced form,

$$\Psi = \Phi.$$

Corresponding to these are $\tilde{\Theta}$, $\tilde{\Psi}$, and $\tilde{\Phi}$.

Invariants under orthogonal and other substitutions.

As before, $$\Theta\,\tilde{\Theta}$$

is an absolute invariant under orthogonal substitutions.

For the group of substitutions identical with their contragredient substitutions,

$$\Psi''\tilde{\Psi}''$$

is an absolute invariant.

We next consider diagonal substitutions. In the special case of real quaternions these present little difficulty. With the usual basis 1, i, j, k for a diagonal substitution,

$$x = \Sigma a_{pq} e_p X e_q,$$

we have $$A'' = \tilde{A}'' = \mathcal{A}'' = \tilde{\mathcal{A}}''.$$

Hence the invariants θ, $T^{-1}\chi T^{-1}$, $T^{-1}\psi$, and $\tilde{\psi}T^{-1}$ are all transformed by the same equations. From these we may obtain invariant equations by taking the determinant of a linear combination. In the case of the first two, the matrices contain elements in the algebra, but it is probably sufficient to take the invariant equation

$$|\psi''(\xi) - \lambda\tilde{\psi}''(\eta)| = 0.$$

This is an invariant equation in λ for all values of ξ and η.

The extended quadratic form.

Let $$f = \Sigma l_{pqr} e_p x e_q x e_r + 2\Sigma l_{pr} e_p x e_r + \Sigma l_q e_q$$

be a quadratic form.

Provided that the quadratic terms are in reduced form, we may take the extended invariants of the bilinear form unaltered, *i.e.*

$$\theta^* = \bar{\theta}^* = \begin{bmatrix} (\Sigma l_{pqr} e_p e_s e_q e_t e_r) & \cdots & \Sigma l_{pr} e_p e_t e_r \\ \cdots & \cdots & \cdots \\ \Sigma l_{pr} e_p e_s e_r & \cdots & \Sigma l_q e_q \end{bmatrix},$$

$$\psi^*(\xi) = \phi^*(\xi) = \begin{bmatrix} (\Sigma l_{pqr} e_q e_t e_r \xi e_p e_s) & \cdots & \Sigma l_{pr} e_r \xi e_p e_s \\ \cdots & \cdots & \cdots \\ \Sigma \lambda_{uv} l_{pr} e_u \xi e_p e_t e_r e_v & \cdots & \Sigma \lambda_{uv} l_q e_u \xi e_q e_v \end{bmatrix},$$

$$\chi^* = \begin{bmatrix} (\Sigma l_{pqr} e_r e_t e_q e_s e_p) & \cdots & \Sigma \lambda_{uv} l_{pr} e_u e_p e_s e_r e_v \\ \cdots & \cdots & \cdots \\ \Sigma \lambda_{uv} l_{pr} e_u e_p e_t e_r e_v & \cdots & \Sigma \lambda_{uv} \lambda_{wz} l_q e_u e_w e_q e_v e_z \end{bmatrix}.$$

Covariants of the quadratic form.

By treating the bilinear form as a linear form in one of the variables, we may obtain the invariants of this linear form as a covariant of the bilinear form containing the other variable.

These will also be covariants of the quadratic form. We thus have eight covariants of the quadratic form

$$f = \Sigma l_{pqr} e_p x e_q x e_r,$$

namely

$$L = [(\Sigma l_{pqr} e_p x e_q e_t e_r e_s)], \quad \tilde{L} = [(\Sigma l_{pqr} e_t e_p x e_q e_s e_r)],$$

$$M = [(\Sigma l_{pqr} e_p e_t e_q x e_r e_s)], \quad \tilde{M} = [(\Sigma l_{pqr} e_t e_p e_s e_q x e_r)],$$

$$N = [(\Sigma l_{pqr} e_q x e_r e_t e_p e_s)], \quad \tilde{N} = [(\Sigma l_{pqr} e_t e_q x e_r e_s e_p)],$$

$$Q = [(\Sigma l_{pqr} e_r e_t e_p x e_q e_s)], \quad \tilde{Q} = [(\Sigma l_{pqr} e_t e_r e_s e_p x e_q)].$$

These are transformed according to the equations

$$L_1 = LA'', \quad \tilde{L}_1 = \tilde{A}'' \tilde{L},$$

$$M_1 = MA'', \quad \tilde{M}_1 = \tilde{A}'' \tilde{M},$$

$$N_1 = \mathscr{A}' N, \quad \tilde{N}_1 = \tilde{N} \tilde{\mathscr{A}}',$$

$$Q_1 = \mathscr{A}' Q, \quad \tilde{Q}_1 = \tilde{Q} \tilde{\mathscr{A}}'.$$

L and $\tilde{L}$ are equivalent to transposed scalar matrices, and so on.

For the reduced form, since

$$\Sigma l_{pqr} e_p x e_q y e_r = \Sigma l_{pqr} e_p y e_q x e_r,$$

the eight will be equivalent to four, since

$$L = M, \quad N = Q,$$

$$\tilde{L} = \tilde{M}, \quad \tilde{N} = \tilde{Q}.$$

The determinants of these four will vanish together, namely when

$$\Sigma l_{pqr} e_p x e_q y e_r$$

is a singular linear form in y. In fact, it may be proved that

$$\tilde{N} = TPLT^{-1} P^{-1}.$$

Covariants of the extended quadratic form.

If the quadratic terms are in reduced form, we may extend the matrices L and N to include the linear and independent terms. If it is not in reduced form, we extend $\frac{1}{2}(L+M)$ and $\frac{1}{2}(N+Q)$ instead.

For the form

$$f = \Sigma l_{pqr} e_p x e_q x e_r + \Sigma m_{pr} e_p x e_r + \Sigma n_q e_q,$$

the quadratic terms being in reduced form, the following is a matrix covariant:

$$L^* = \left[\begin{array}{ccccccc|ccccc} (\Sigma l_{pqr} e_p x e_q e_t e_r e_s + \Sigma m_{pr} e_p e_t e_r e_s) & & & & & & & \Sigma m_{pr} e_p x e_r e_s + \Sigma n_r e_r e_s & & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \multicolumn{7}{c|}{0} & \multicolumn{5}{c}{1} \end{array}\right].$$

There is a corresponding covariant $\tilde{L}^*$. N^* and $\tilde{N}^*$ can be found from these by means of the equations

$$\tilde{N}^* = T_1 P_1 L^* T_1^{-1} P_1^{-1}, \quad N^* = P_1^{-1} T_1^{-1} \tilde{L}^* P_1 T_1.$$

These covariants are transformed according to the equations

$$L_1^* = L^* A''^*, \quad \tilde{L}_1^* = \tilde{A}''^* \tilde{L}^*,$$

$$N_1^* = \mathscr{A}'^* N^*, \quad \tilde{N}_1^* = \tilde{N}^* \tilde{\mathscr{A}}'^*.$$

The semi-hermitian forms in quaternions.

Consider the form $\Sigma l_{pqr} e_p x e_q \bar{x} e_r,$

in which $\bar{x}$ is the conjugate of x. This will be called the left-handed semi-hermitian form. There is a corresponding right-handed form

$$\Sigma l_{pqr} e_p \bar{x} e_q x e_r,$$

but it is evidently sufficient to consider the left-handed form.

It is obviously a particular case of the bilinear form, and it will have the same invariants. The interesting point to examine is the part played by the identities in this form and the corresponding modification of the invariants, and to find, if possible, a reduced form corresponding to the reduced form of the quadratic.

The variables x and $\bar{x}$ are subjected to conjugate substitutions

$$x = \Sigma a_{pq} e_p X e_q,$$

$$\bar{x} = \Sigma b_{pq} e_p \bar{X} e_q,$$

and $$\mathfrak{A}'' = \tilde{B}'', \quad A'' = \tilde{\mathfrak{B}}''.$$

Invariants of the semi-hermitian form are transformed as follows:

$$\phi_1'' = \mathfrak{A}'' \phi'' B'',$$

$$\tilde{\phi}_1'' = \tilde{B}'' \tilde{\phi}'' \tilde{\mathfrak{A}}''.$$

Hence $\phi''(\xi) - \tilde{\phi}''(\xi)$ is invariant. Its determinant gives an invariant equation.

Now $$\phi''(\xi) = (TPT)^{-1} \Sigma \lambda_{uv} e_u \xi \theta e_v,$$

$$\tilde{\phi}''(\xi) = \Sigma \lambda_{uv} e_u \xi \bar{\theta} e_v (TPT)^{-1}.$$

Hence

$$\phi''(\xi) - \tilde{\phi}''(\xi) = \Sigma \lambda_{uv} e_u \xi [(TPT)^{-1} \theta - \bar{\theta}(TPT)^{-1}] e_v.$$

Thus we may take, instead of $\phi''(\xi) - \tilde{\phi}''(\xi)$,

$$(TPT)^{-1} \theta - \bar{\theta}(TPT)^{-1}.$$

Now $$(TPT)^{-1} = [(\gamma_{st1})]^{-1}$$

$$= \begin{bmatrix} 1, & & & \\ & -1, & & \\ & & -1, & \\ & & & -1 \end{bmatrix},$$

for quaternions.

Now consider the identity

$$0 = x\bar{x} - \bar{x}x = x\bar{x} - \Sigma \lambda_{pn} a_{qv} e_p x e_q \bar{x} e_n e_v.$$

We have

$$\theta = [(e_s e_t)] - [(e_t e_s)] = \begin{bmatrix} 0, & 0, & 0, & 0 \\ 0, & 0, & 2k, & -2j \\ 0, & -2k, & 0, & 2i \\ 0, & 2j, & -2i, & 0 \end{bmatrix}.$$

Hence $$(TPT)^{-1} \theta + \bar{\theta}(TPT)^{-1} = 0,$$

and, since every identity may be obtained from this by substitution on the variable or on the form,

$$(TPT)^{-1}\theta+\bar{\theta}(TPT)^{-1}=0,$$

or

$$\phi''(\xi)+\tilde{\phi}''(\xi)=0,$$

for every identity in this form.

Evidently the invariant θ of the general identity is given by

$$\theta=\begin{bmatrix} 0, & a, & b, & c \\ a, & 0, & d, & e \\ b, & -d, & 0, & f \\ c, & -e, & -f, & 0 \end{bmatrix},$$

where a, b, c, d, e, and f are any numbers of the algebra.

There is evidently a reduced form for which

$$\phi''(\xi)=\tilde{\phi}''(\xi).$$

RULED FORMS IN FOUR DIMENSIONS

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The object of this paper is to give an introduction to the study of line congruences in space of four dimensions*, and the forms which they generate. Up to the present no systematic investigation of the subject has been made; a few generalities have been given by James†, and, as regards particular systems, Segre‡ has made a complete classification of cubic forms in [4] and the congruences contained in them. Some brief notes on ruled quartic forms have been given by Togliatti§ and Marletta‖, and Sisam¶ has published a note on certain types of ruled quintics. These latter, however, do not come within the scope of the present discussion.

It must be borne in mind that there exist two types of ruled form, differing greatly in character: the "simple" type, such that through an arbitrary point of the form there passes a *single* generating line—this gives rise on projection to the ordinary type of congruence in [3]; and the "compound", in which case there is a (constant) number of such lines. Here we are concerned with the simple ruled form. Beginning with a description of its general features, we consider the various ways in which congruences in ordinary space may be derived from it by projection. We give a classification of the elementary types of ruled form having plane sections of genus 1, 2, or 3; the discussion is necessarily brief, partly for lack of space, but also because the subject presents no particular difficulties. We then proceed to obtain enumerative properties of ruled forms by means

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* For line systems in general, see James, *Proc. London Math. Soc.* (2), 24 (1925), 359. For the representation of lines in [4], see Todd, *ibid.*, 30 (1930), 513. For other references see §10 below.

† "Some results for the geometry of line systems in space of four dimensions", *ibid.*, 28 (1928), 161.

‡ Segre, *Memorie di Torino* (2), 39 (1889), 1.

§ Togliatti, *Rend. Lincei* (5), 30 $(1921)_1$, 252, and $(1921)_2$, 22.

‖ Marletta, *Giorn. di Mat.*, 40 (1902), 265.

¶ Sisam, *Messenger of Math.*, 58 (1928), 89.

of the functional method*, from which follow some interesting consequences. In conclusion, we consider some properties of multiple tangents which have a single point of contact, and the ruled forms generated by the quadrisecants of surfaces in four dimensions.

Introduction.

1. A *ruled form* F is a threefold in [4] which is generated by a congruence of ∞^2 lines, as distinct from a *plane-form*, which contains ∞^3 lines lying in a single infinity of planes†. Through a point of F there pass κ rays in general, where the maximum value of κ is six, the number of four-point tangents at the given point. The *order* of the congruence is the number of rays of the system which meet an arbitrary line, and this number, divided by κ, gives the order n of F. The *class* l is the number of rays in an arbitrary space. In general, it is necessary to distinguish between the two cases $\kappa = 1$, $\kappa > 1$; we shall call the corresponding systems "simple" and "compound" respectively, and unless otherwise stated, it is to be assumed that κ is unity.

It may happen that F contains a number of distinct congruences, simple or otherwise. A familiar example is that of the nodal cubic form, considered in detail by Segre‡. In the case of six nodes, the congruence of order 18 and class 27 ($\kappa = 6$) breaks up into two simple systems, each of order 3 and class 6, and a residual system of order 12 and class 15, for which $\kappa = 4$. With the acquisition of a seventh node and, therefore, of two planes, there is a reduction in class leading to two systems (3, 5), and the compound system breaks up into a simple system (3, 5) together with a residual system (9, 10) having $\kappa = 3$. Further particulars of this form are given later (§ 11).

Again, consider the intersection of two general quadric forms in [5]; since these contain ∞^5 lines, the intersection is a threefold F containing ∞^2 lines, and four of these pass through any point of F. Projecting on to [4] from an external point we obtain a ruled quartic form containing a congruence of class 16, for which $\kappa = 4$. If, however, one of the quadrics is a line-cone, *i.e.* contains a double line, its equation may be reduced to the form $x_1 x_2 = x_3 x_4$, so that evidently it contains two systems of spaces [3]; moreover, F acquires two nodes in consequence. It follows

* Certain results in the present paper are taken from the writer's previous work, *Proc. London Math. Soc.* (2), 32 (1931), 72. This is referred to as **C**.

† For a discussion of plane-forms, see Roth, *Proc. London Math. Soc.* (2), 33 (1931), 115. Besides these types we exclude also those congruences which lie on a quadric form.

‡ Segre, *Memorie di Torino* (2), 39 (1889), 1.

that, if both quadrics are line-cones, the lines on F fall into four simple systems, each of order and class 4, and F has four nodes, through each of which passes a pencil of lines of each system.

2. *Singularities of a congruence.* Apart from essential singularities the form F contains a double surface through an arbitrary point of which pass 2κ rays; similarly there are, in general, a triple curve of points through which pass 3κ rays, and a finite number of quadruple points, through which pass 4κ rays. This is the type of form which we consider in the present paper; it has the characters of the general projection of a ruled threefold in [5] which is without singularities. But it must be noted that not all ruled forms are necessarily of this type; thus there are ruled forms having multiple loci of different character; for example, the ruled quintics considered by Sisam*. The case which we propose to discuss is, however, that presented by a general congruence in [4].

In this connection we may consider briefly the characteristics of a special ruled form of some interest—the reciprocal of a general surface in [4]. Suppose, in the usual notation, that the surface is of order n, rank a, and class m, with j apparent pinch-points; then the reciprocal is of order m and rank a, and the congruence which it contains is simple, of order m and class j; an arbitrary space section is of rank a and class n. The form possesses only ∞^2 tangent spaces, n of them passing through an arbitrary line; the tangent space is the same at all points of a generating ray, and each such ray is met by $m-2$ others. There are d bitangent spaces, corresponding to the improper nodes of the surface, in number $d=\binom{n}{2}-\frac{1}{2}a-\frac{1}{2}j$, but no others. Evidently the projection of the congruence on ordinary space has a ruled focal surface (§ 5).

The essential singularities of the congruence may be of three kinds. Through a multiple point of order h there passes a cone of rays of order h; such a point is necessarily a singularity of F, and in the sequel it will be supposed to be an ordinary double point†. There may exist one or more planes containing ∞^1 rays of the congruence; such planes must lie on F. In general a plane on F gives rise to a number of nodes; for if π is such a plane, not meeting the double surface in a curve, the spaces of the pencil through π meet F elsewhere in surfaces of order $n-1$, and these define in π a pencil of curves of that order, having $(n-1)^2$ base-points. Finally, there may be spaces containing scrolls of

* Sisam, *Messenger of Math.*, 58 (1928), 89.

† This is the only case which occurs in the present work.

rays of order h; these may be ∞^1 in number and h may take various values.

3. *Characters of the congruence.* A simple congruence depends upon five additive characters which are conveniently defined in terms of an arbitrary space section. Besides the order n and class l these are a, the rank; j, the number of pinch-points; and m, the class of a space section. If there are nodes, we assume a sixth character C, representing the number of these points. Another important character is the class w of the form itself; but, as will appear, this is not independent of the others.

4. *James's theorem.* In discussing the properties of ruled forms we require the following general result, due to James:*

An arbitrary ray of a simple congruence of order n is met by $n-3$ other rays, i.e. is $(n-3)$-secant to the double surface.

The proof given by James follows immediately from Schubert's well-known lemma† concerning the coincidences of a system of ∞^2 point-pairs in a plane. If an arbitrary point of the plane, considered as belonging to one set, determines α or β points respectively of the other, while γ pairs lie on a given line, then the number of coincident pairs whose join passes through a given point is $\alpha+\beta+\gamma$.

Taking a ray ρ of the congruence and two fixed spaces σ, σ' meeting in a plane ϖ, we obtain in ϖ a system of ∞^2 point-pairs for which $\alpha=\beta=n-1$ and $\gamma=l-1$. Deducting a number $n+l$ of irrelevant coincidences we have the above result.

An immediate corollary is the fact that any plane passing through ρ is bitangent to F, and hence that an arbitrary ray of a congruence in [3] is met by two consecutive rays. It should be noted that this bitangent property holds also for compound congruences; thus, in all cases, the rays of the congruence project into bitangents of the apparent contour.

5. *The apparent contour.* Suppose that F is generated by a simple congruence; then this congruence projects from an external point O into a general congruence γ of order n and class l in ordinary space; and the surface of contact Φ of tangents drawn to F from O projects into the focal surface ϕ of the projected congruence. For the plane joining O to

* James, *loc. cit.*, 165.

† Schubert, *Math. Annalen*, 26 (1886), 52.

an arbitrary ray ρ of F meets F in a residual curve of order $n-1$; and, of the $n-1$ intersections with ρ, $n-3$ lie on the double surface (§ 4). The remaining intersections are points of contact of the plane with F, so that on projection we obtain the result that the projected ray is bitangent to the projection of Φ; and the points of contact project into intersections with two consecutive rays.

Now Φ is of order* a, rank m, and class w; hence, if a' denotes the rank of the projected congruence†, we have

$$a = a' - 2l, \quad w = a' - 2n,$$

from the known formulae. Eliminating a', we obtain the result

$$w = a + 2(l - n).$$

This gives *the class of a ruled form in the simple case.*

The rank of Φ is also the rank of ϕ. The character j', representing the number of focal planes of γ passing through a given point and having a focus in a given plane, is at once determined by comparison of previous results, and is given by

$$j' = j + 4l.$$

The ∞^2 pairs of rays, intersecting on the double surface of F, give rise to a finite number d of planes passing through O, each of which contains two rays of F; and these project into double rays of γ. The value of d is thus given by

$$d = \binom{n}{2} + \binom{l}{2} - \tfrac{1}{2}j - 2l,$$

in general. The original congruence may possess apparent double rays arising from the presence of multiple points; these will project into double rays of γ. Thus an h-ple cone with H apparent double generators gives rise to H double rays on projection.

The double curve on ϕ is the projection of the curve of contact of the bitangents drawn to F from O, and the cuspidal curve arises similarly from the contacts of the inflexional tangents. Hence‡ the orders of these curves are given by

$$\mu_{2-2} = \tfrac{1}{2}a^2 + a - 3j - 2m,$$

$$\mu_3 = -a + 2j + m.$$

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* Note that, for the apparent contour of any form in higher space, the i-th rank is equal to the $(i+1)$-th rank of the given form.

† See **C**, § 6.

‡ See **C**, 80.

The double curve contains a number ${}_3\mu_2$ of triple points, arising from the tritangents drawn to F from O; the two curves contain a number μ_4 of common stationary points, arising from the four-point tangents drawn to F. There is also a number μ_{3-2} of points common to the curves, which are projected by the 3—2 tangents drawn from O. These numbers are known* to be

$$ {}_3\mu_2 = \binom{a}{3} + \tfrac{3}{2}a^2 - a(3j+2m) - 198a + 85j + 59m + \tfrac{1}{3}(a+j-m) + 152n + 8l + 8C, $$

$$ \mu_4 = -54a + 22j + 12m + 44n + 4l + 4C, $$

$$ \mu_{3-2} = a\mu_3 + 228a - 96j - 60m - 180n - 12l - 12C. $$

Similar formulae are obtained in the dual case.

Suppose instead that the point O is an ordinary point of F; then the characters of γ are

$$ n-1, \quad l, \quad a-2, \quad j-4, \quad m. $$

There is also a number of additional singularities; for the rays consecutive to the ray ρ passing through O project into a pencil whose centre is the projection of ρ and whose plane is the intersection of the tangent space to F at O with the space of projection. Moreover, the cone of rays at an h-ple point of F will meet ρ in a certain number η of points, and the congruence thus acquires, on projection, a multiple ray of order η.

Similarly, if we project from a ν-ple point O of F, through which pass in general ν rays, the projected congruence γ has characters

$$ n-\nu, \quad l, \quad a-2\nu, \quad j-4\nu, \quad m. $$

Each of the ν tangent spaces at O gives rise to a singular pencil of γ, as described above†; and multiple rays of γ are obtained in the same manner.

6. *The compound case.* Consider the projection from an external point O. If F contains a single compound system, at each ordinary point of F there are κ rays defining $\binom{\kappa}{2}$ planes, each of which contains a pair of rays. Thus there are ∞^3 such planes; ∞^1 of them pass through O and project into double rays of γ. It follows that *the projected congruence*

* *Ibid.*, 81, 82. The necessary changes in the notation are effected by writing $a+2l$ for a and $j+4l$ for j in these results, which apply to congruences in [3].

† If the double surface is ruled, the generator passing through the point of projection, which is not a ray of the congruence, gives rise to an additional singularity. In the case of a double quadric there will be two such singular points in the projection.

possesses a double scroll. Apart from these planes, there are, as before, the planes defined by pairs of rays meeting on the double surface of F, and these furnish a finite number of double rays of γ which do not lie on the double scroll.

If O is an ordinary point of F, the κ rays passing through O project into multiple points of γ lying on a conic, the projection of the cone of inflexional tangents at O; the plane of the conic is projected from the tangent space at O. For the rays meeting one of these rays form a scroll of order r, say, and this projects into a cone of order $r-(\kappa-1)$. Each of the κ rays projects into a node of the focal surface ϕ.

Similar considerations apply to the projection from a multiple point, as described in the case of a simple system.

7. *Simple system with residual systems.* A case of some interest is provided by the combination of the two previous types. Suppose that F contains one simple congruence C (n, l) and s compound congruences C_i (n_i, l_i, κ_i), so that

$$\kappa = 1+\sum_1^s \kappa_i.$$

Consider the projection, from an ordinary point O of F, of the congruence C. Each of the κ rays passing through O gives a singular point in the projection, since the rays of C indefinitely near to the ray ρ at O project into a singular pencil, as before (§ 5 *supra*). Suppose that the rays of C meeting each of the κ_i rays of C_i at O generate a scroll of order r_i; then these scrolls project into singular cones of order $r_i-(\kappa_i-1)$; and all the κ singular points thus obtained lie in the projection of the tangent space at O and on the projection of the quadric cone of inflexional tangents at O. If F contains multiple points, these project into singular points of C, through which pass double rays (§ 5).

8. *Case of singularities.* A singular point or plane of order h projects from an external point into a singular point or plane of the same order. Thus, if a congruence has a_h h-ple points, the projected congruence also has a_h h-ple points. As regards the singular spaces, suppose that there exist ∞^1 such spaces, each containing a scroll of rays of order h. If we project from an external point O, a finite number β_h of these spaces passing through O give rise to singular planes of order h in the projection. In particular there may be a directrix plane on F, met by all rays of the congruence, in which case the system of spaces forms a pencil, for which $h = n-1$, $\beta_{n-1} = 1$. If, instead, O is supposed to lie on F, there will be a

reduced number β_h' of such spaces through O, projecting into singular planes of order h, together with $\beta_h - \beta_h'$ spaces projecting into singular planes of order $h-1$; and so for other special positions of O. A singular pencil counts both as a singular point and plane of order unity.

9. *Dual congruences.* Two congruences in [3] are said to be dual when the focal surface of one is dual to the focal surface of the other; this amounts to an interchange of the characters n and l, the other three, a', j', and m, remaining as before. From the relations given in § 5 it follows that to obtain the dual congruence in [4]—if it exists—we interchange the characters as follows:

$$\begin{array}{cccccccc} n & a & j & m & w & l & a_h & \beta_h \\ l & w & j+4(l-n) & m & a & n & \beta_h & a_h. \end{array}$$

This process leaves unchanged the genus of a plane section of the corresponding ruled form, and the class of a space section. Evidently, when we have determined the singularities of a congruence, we are able to write down those of its dual; in this manner are found the singular planes mentioned in the following examples, and in a similar manner they may be determined in the sequel.

Consider, for example, the elliptic congruence (6, 3), which is the dual of (3, 6), and for which therefore $\beta_3 = 6$ (§ 11). This projects from a point of the form into (5, 3)', the dual of (3, 5)', having $\beta_1 = 1$, $\beta_2 = 3$, $\beta_3' = 3$. Of these singularities the first arises from the tangent space at the point of projection (§ 5), while the remainder represent the reduced form of the previous result; we have, in fact, $\beta_3' + \beta_2 = \beta_3$. Again, consider (5, 3)'', which has $\beta_4 = 1$, $\beta_2 = 6$, the first singularity arising from a directrix plane; this projects into the elliptic congruence (4, 3) from a point of itself, having $\beta_3 = 1$, $\beta_1 = 3$, $\beta_2' = 4$; one of the singular pencils is obtained as before, while the others are due to the projection of two of the six quadrics of the quintic congruence (§ 13).

As a second example illustrative of the foregoing principles, we project one of the two simple congruences (6, 3) on the Del Pezzo sextic (§ 14) from a point of the double surface. This is ruled, so that through the point of projection pass five lines, two of which belong to the congruence in question; the latter give rise to singular pencils, and the rays of the residual congruence meeting them generate cubic scrolls (*loc. cit.*). Finally, the rays of either congruence which meet the generator of the double scroll generate a quadric, so that, on projection, we obtain, for the point singularities of (4, 3), $a_1 = 3$, $a_2 = 2$. The plane singularities are dual to those of (3, 4), whence (§ 11) $\beta_1 = 3$, $\beta_2 = 4$, $\beta_3 = 1$.

Special types of ruled forms.

10. *Del Pezzo forms**. First in order of simplicity are those forms which have elliptic plane sections, *i.e.* whose space sections are projections of Del Pezzo surfaces. Not all of these are ruled and, in general, only one is simple; in the remaining cases simple congruences arise from the acquisition of nodes.

All these forms are rational, by Fano's theorem†. The simplest types are those represented by means of quadrics in a space S_3; the dual loci have been discussed elsewhere‡, but from a different point of view. The forms thus obtained are of class 4; the first type, represented by ∞^4 quadrics in general position, is of order 8 and is non-ruled. If the quadrics have a common point P, the form is of order 7 and contains a simple system of lines, corresponding to the lines passing through that point. P corresponds to a plane π which is obviously a directrix of the congruence; the ∞^1 cones of the system of quadrics having their vertices at P represent the residual sections by spaces through π, so that these are rational sextic scrolls. The congruence is of class 2 (as are all the congruences obtained by this method of representation), since any quadric of the system, not a cone, contains two lines through P.

If the quadrics have two common points P_1, P_2, corresponding to planes π_1, π_2, the form is a sextic and contains two simple sextic congruences, each having a plane for directrix. The planes meet in a node of the form, which is the vertex of a singular pencil for each congruence; the ∞^3 quadrics containing $P_1 P_2$ represent the space sections passing through the node.

If there are three base-points P_1, P_2, P_3, corresponding to planes π_1, π_2, π_3, the form is a trinodal quintic§, the nodes being the intersections, in pairs, of the three planes. There is a fourth plane containing the nodes, represented by $P_1 P_2 P_3$, which is not a directrix plane; each of the three quintic congruences (5, 2) has three singular pencils with vertices at the nodes.

In the case of four base-points‖ P_i $(i = 1, 2, 3, 4)$ there are four simple congruences (4, 2) and six nodes which give rise to singular pencils in each

* Since this paper was written there has appeared a complete discussion of Del Pezzo congruences in higher space, by Todd, *Proc. London Math. Soc.* (2), 33 (1932), 328. For the Del Pezzo forms, see Scorza, *Annali di Mat.* (3), 15 (1908), 224.

† Fano, *Annali di Mat.* (3), 24 (1915), 49.

‡ "A class of quartic forms in four dimensions", Roth, *Proc. London Math. Soc.* (2), 30 (1930), 305.

§ See Todd, *Proc. London Math. Soc.* (2), 30 (1930), 540.

‖ This is the projection of the V_3^4 in [5] whose points represent the lines of a tetrahedral complex.

congruence. Besides the directrix planes there are four planes, each containing three nodes. The remaining type, for which there are five base-points, is treated in the next section*.

11. *Cubic congruences.* These have been analysed in complete detail by Segre; for convenience we reproduce the principal results. The general cubic form contains a compound congruence (18, 27) for which $\kappa = 6$; in the case of six arbitrary nodes this breaks up into two simple systems (3, 6) and a compound system (12, 15, 4). Each node is a triple point for the simple systems. With the acquisition of a seventh node these degenerate into systems (3, 5)′ and there arises a third system (3, 5)″, of different type. The form contains two planes, in each of which lie four nodes (§ 2). The form with eight nodes contains five planes and four systems (3, 4) of the same type. If there are nine nodes, the form contains nine planes and six systems (3, 3) having the same singularities. The maximum number of nodes is ten; in that case we have a rational form with the above representation by quadrics passing through five base-points P_i. There are six systems (3, 2), each having a singular pencil at each node; of these systems, five are obtained as before, the sixth being represented by twisted cubics passing through P_i. There are fifteen planes on the form, each containing four nodes. Each system of lines meets five planes.

The list of singularities for these congruences is as follows:

(3, 6): $a_3 = 6$.

(3, 5)′: $a_3 = 3,\ a_2 = 3,\ a_1 = 1$.

(3, 5)″: $a_2 = 6,\ a_4 = 1$.

(3, 4): $a_3 = 1,\ a_2 = 4,\ a_1 = 3$.

(3, 3): $a_2 = 3,\ a_1 = 6$.

(3, 2): $a_1 = 10$.

12. *Elliptic quartic congruences.* As remarked in § 1, the intersection of two general quadric forms in [5] is ruled, four lines passing through any one of its points. Thus its projection F on [4] contains a compound congruence (16, 16) for which $\kappa = 4$. Its representation on a space S_3 is obtained by projecting from a line of itself; in this manner, space sections

* It is interesting to note that, from the known values of the classes of these forms, we can deduce those of the general Del Pezzo forms of the same order; at present no other method is available. For the only singularities in the present case are ordinary nodes; whence we obtain the following results:

n	8	7	6	5	4	3
w	4	4	6	10	16	24

of F are made to correspond to ∞^4 cubic surfaces passing through a quintic c_5 of genus 2, and plane sections correspond to elliptic quartics c_4 meeting the base-curve c_5 in eight points. The lines of F correspond to chords of c_5, four of which pass through an arbitrary point of S_3. The double surface on F is represented by a quadric, as follows by projection.

With the acquisition of nodes the base-curve degenerates into various parts, whose intersections correspond to the nodes; and the compound system of lines breaks up accordingly. We propose to examine the simple congruences to be obtained in this manner. In the case of three nodes, c_5 consists of a rational quartic γ_4 with a trisecant τ; curve c_4 meets γ_4 in seven points and τ in one point, so that τ represents a plane on F containing the three nodes. There is one simple congruence* (4, 5) corresponding to the secants of γ_4 which meet τ, and there is a residual compound congruence (12, 10) having $\kappa = 3$.

A space section of F passing through a node contains four lines through the node, as is known from the theory of quartic surfaces. Hence each node of F is a *triple* point for (4, 5); the plane of the nodes is a directrix. It follows from the plane representation of the cubic surface that the rays of (4, 5) meeting a given ray of (12, 10) generate a cubic scroll. Thus, projecting the simple congruence from an arbitrary point of F, we obtain a cubic congruence having three triple points, three double points arising from the cubic scrolls just mentioned, and one singular pencil due to the rays consecutive to the ray of (4, 5) through the point of projection (§ 5). This is the congruence (3, 5)′ given above.

Suppose that γ_4 breaks up into a twisted cubic γ_3 together with a secant τ'; it is easily seen that τ meets γ_3 twice and τ' once; and c_4 meets γ_3 five times, τ once, and τ' twice. F now contains four nodes, and two planes π_1, π_2 corresponding to τ and the plane of τ, τ'. There are two simple systems (4, 4)′, which are found to be of the same type, corresponding to:

(1) secants of γ_3 and τ; these all meet π_1;

(2) chords of γ_3; these meet π_2; and

(3) a compound system (8, 6) having $\kappa = 2$, corresponding to secants of γ_3 and τ'.

We find that the rays of (1) which meet a given ray of (2) generate a quadric, and that the rays of (1) meeting a given ray of (3) generate a

* All the results of this nature which are given in the sequel are easily deduced from the plane representation of the cubic surface, general or nodal.

cubic scroll; and similarly for the system (2). Each congruence has the singularities $a_3 = 1$, $a_2 = 2$, $a_1 = 1$ at the nodes.

Projecting from a point of F we obtain the congruence (3, 4) from either system.

If, instead, the base-curve c_5 consists of γ_3 and two chords τ_1, τ_2, we obtain a type of F with four nodes and four identical systems (4, 4)'', as in § 1. The lines τ_1, τ_2 are each met by c_4 in two points and represent quadrics on F. The four systems are represented as follows:

(1) secants of τ_1 and τ_2; (2) chords of γ_3; (3) secants of γ_3 and τ_1; (4) secants of γ_3 and τ_2.

Each node of F is a double point for each congruence, two congruences having a quadric cone of rays in common at such a point. We find that the rays of (1) or (2) meeting a ray of (3) or (4) generate a quadric; that the rays of (1) meeting a ray of (2) generate a quartic scroll; and similarly for (3) and (4). Projecting any one of these congruences from a point of F we obtain the congruence (3, 4).

The congruence (4, 3) is the dual of the latter. Suppose that in the last case γ_3 breaks up into a conic γ_2 and a secant τ_3 each meeting τ_1 and τ_2. It is found that c_4 meets τ_3 in one point; thus τ_3 represents a plane containing three of the five nodes of F; the planes (τ_2, τ_3) and (τ_1, τ_3) also represent planes of F, and so does the plane of γ_2. Each of the previous systems (4, 4) consists of a congruence (4, 3) and a plane. At three of the nodes the double points are replaced by pencils; and the rays of (1) meeting a ray of (2) generate a cubic scroll. Each congruence has one directrix plane, as is obvious from the space representation.

It is readily seen that the above investigation exhausts all possible types of simple congruences to be obtained from F, with the exception of (4, 2), which has already been considered. The complete list of singularities is as follows:

(4, 5): $a_3 = 3$. (4, 3): $a_2 = 2$, $a_1 = 1$.

(4, 4)': $a_3 = 1$, $a_2 = 2$, $a_1 = 1$. (4, 2): $a_1 = 6$.

(4, 4)'': $a_2 = 4$.

13. *Elliptic quintic congruences.* The Del Pezzo quintic F in its projection on [4] is represented* by means of ∞^4 cubic surfaces passing through a rational quartic γ_4. Plane sections of F correspond to elliptic curves c_5

* The representations of the Del Pezzo forms considered here are given by Semple, *Proc. Camb. Phil. Soc.*, 25 (1929), 145.

meeting γ_4 in ten points. The system of lines on F, for which $\kappa = 3$, corresponds to chords of γ_4. The double quintic on F is represented by a sextic surface containing γ_4 as a double curve.

As before, we investigate the simple congruences which arise from the presence of nodes. If γ_4 breaks up into a twisted cubic γ_3 and a secant τ, there is one node on F; c_5 meets τ in three points and γ_3 in seven. There is now a simple system (1) (5, 4) represented by chords of γ_3, and a compound system (2) represented by secants of γ_3 and τ. The node of F is a triple point for (1). The rays of this system meeting a ray of (2) generate a cubic scroll; thus, projecting from a point of F, we obtain the previous type (4, 4)′.

If γ_3 degenerates, so that the base-curve consists of a conic γ_2 and two secants τ_1, τ_2, there are two nodes and three simple systems (5, 3); c_5 meets τ_1 and τ_2 in three points each and γ_2 in four points, whence the plane of γ_2 represents a plane on F. The congruences are as follows:

(1) represented by secants of τ_1 and γ_2,

(2) represented by secants of τ_2 and γ_2,

(3) represented by secants of τ_1 and τ_2.

Of these, (1) and (2) are of the same type, (5, 3)′ say; each has one simple and one double singular point at the nodes. The third system (5, 3)″ has double points at the nodes. The rays of (1) meeting a ray of (2) generate a cubic scroll, and those of (1) or (2) meeting a ray of (3) generate a quadric. Thus each type projects from a point of F into the same congruence (4, 3).

The remaining type (5, 2) is already known. The list of singularities is as follows:

(5, 4): $\alpha_3 = 1$. (5, 3)″: $\alpha_2 = 2$.

(5, 3)′: $\alpha_2 = 1$, $\alpha_1 = 1$. (5, 2): $\alpha_1 = 3$.

14. *Elliptic sextic congruences.* The general Del Pezzo sextic F in [4] is represented by means of ∞^4 cubic surfaces which pass through a twisted cubic γ_3 and have a fixed node on the curve. Hence F contains two simple congruences (6, 3) corresponding to (1) chords of γ_3 and (2) lines passing through the node. The double surface on F is represented by a surface of order 9 on which γ_3 is a triple curve. There are no isolated singularities. The rays of either congruence meeting a given ray of the other generate a cubic scroll, so that projection from a point of F gives the type (5, 3)′.

There are two types of class 2; the first, $(6, 2)'$, which has already been discussed, is represented by means of quadrics through two base-points; there are two such congruences on the form, each with a singular pencil. The second type $(6, 2)''$ is represented by means of cubic surfaces passing through three skew lines; the form contains three simple congruences of this type, corresponding to secants of the base-lines taken in pairs. The rays of one congruence meeting a given ray of another generate a quadric, so that $(6, 2)'$ and $(6, 2)''$ both project into $(5, 2)$ from a point of the form.

15. *Castelnuovo forms.* We consider now the ruled forms whose plane sections are of genus 2; their space sections are well-known surfaces and have been discussed by Castelnuovo*. For these we have $n \leqslant 8$, $\kappa = 2$, and the possible singularities are simple or double points or planes; their interest is therefore more restricted. It follows from Fano's theorem that they are rational; we shall show that they may be represented† by means of quartic surfaces f_4 having a common double line g and a simple base-curve γ; the lines of the congruence correspond to secants of g and γ.

Consider the plane representation of one such surface f_4, by means of quartics with a fixed node and passing through eight base-points or, symbolically, $(4, 2, 1^8)$. In this representation the double line g corresponds to the cubic $(3, 1, 1^8)$, so that the intersection, residual to g, of f_4 with another surface of the system is represented by

$$4(4, 2, 1^8) - 2(3, 1, 1^8) \equiv (10, 6, 2^8).$$

In the case of the quartic form F_4, this must consist of a curve of genus 2, corresponding to a plane section, and a residual base-curve γ meeting the lines on f_4. We thus see that the plane section corresponds to a curve $(4, 2, 1^8)$ and γ to $c_6 \equiv (6, 4, 1^8)$. Hence γ is of order 8 and genus 4, meeting g in six points.

For the form F_n of order n and class $l = 24 - 2n$, we modify the previous representation by making c_6 acquire nodes at one, two, three, or four of the simple base-points; then plane sections of F_n correspond to nodal quartics not passing through these points. Thus, in general, F_n $(4 \leqslant n \leqslant 8)$ is represented by means of surfaces f_4 passing through a double line g and a simple curve γ_{12-n}^{8-n} meeting g in $10-n$ points; plane sections of F_n correspond to curves ${c_n}^2$, and the lines on F_n to secants of g and γ, two passing through an arbitrary point of the form. The remaining lines on f_4 are chords of γ and thus do not represent lines on F_n.

* Castelnuovo, *Rend. di Palermo*, 4 (1890), 73.

† For the representation of any hyperelliptic threefold see Enriques, *Math. Annalen*, 46 (1895), 181.

Simple congruences are obtained, as before, by breaking up γ; as an example, we give a brief discussion of F_4, for which the base-curve is γ_8^4. The form contains a double (directrix) plane corresponding to the unique cubic surface passing through g and γ_8. Now, suppose that γ_3 consists of two rational quartics γ_4, γ_4' each trisecant to g, meeting in five points; then F contains five nodes and two simple systems (4, 8) of the same type. Each node is the vertex of a quadric cone of rays common to the two systems. The rays of one system meeting a given ray of the other generate a quadric, as is obvious from the representation of f_4. Hence, projecting F_4 from a point of itself, we obtain a congruence (3, 8) for which $\alpha_1 = 2$, $\alpha_2 = 5$. We may proceed, as before, to break up γ_4 and γ_4' into simpler curves, thus deriving additional nodes and reducing the class of the congruences; and so in general. The complete list of singularities is found to be as follows:

$$(3, l): \quad \alpha_1 = \beta_1 = 2(9-l), \quad \alpha_2 = l-3, \quad \beta_2 = 0,$$
$$(4, l): \quad \alpha_1 = \beta_1 = 2(8-l), \quad \alpha_2 = l-3, \quad \beta_2 = 1,$$
$$(5, l): \quad \alpha_1 = \beta_1 = 2(7-l), \quad \alpha_2 = l-3, \quad \beta_2 = 2, \text{ etc.}$$

The dual congruences give the plane singularities. The values of l are given by $3 \leqslant l \leqslant 12-n$; although, for $n = 3$, the maximum value cannot be attained by projection.

As in § 10, we may determine the class w of F_n by considering the number of nodes acquired in the case of a simple congruence contained in it; this number is easily seen to be $C \equiv 21-2n-l$, whence

$$w = 2(n+1)+2(l-n)+2(21-2n-l) = 4(11-n).$$

We return to the representation of F_n to consider the case $n = 9$. The class of a simple congruence $(9, l)_2$ must be 3, since it is represented on a Castelnuovo surface, which is of maximum order 12. Then F_9, if it exists, must contain two simple congruences $(9, 3)_2$, since, by projecting from a point of itself, we obtain two congruences $(8, 3)_2$, each with two nodes, as shown by the previous formula. Such a form cannot be represented by the method already given, and we may show that it cannot be represented in any other way. For Enriques* has proved that all forms of hyperelliptic curve section may be represented by means of surfaces f_m of order m with a common $(m-2)$-ple line g and a simple base-curve γ_x meeting g in $x-2$ points. The method of obtaining the representation shows that, if the forms are ruled, their generating lines correspond to secants of g. Now a surface f_m is rational and is represented by means of

* Enriques, *loc. cit.*, 199.

curves $(m;\ m-2,\ 1^{3m-4})$, *i.e.* of order m, with an $(m-2)$-ple base-point and $3m-4$ simple base-points: g corresponds to $(m-1;\ m-3,\ 1^{3m-4})$. Thus the residual intersection of two surfaces f_m is represented by

$$m(m;\ m-2,\ 1^{3m-4})-(m-2)(m-1;\ m-3,\ 1^{3m-4})=(3m-2;\ 3m-6,\ 2^{3m-4}).$$

In the present case this must break up into a variable portion of genus 2 and a curve representing γ_x; and, in order to yield the configuration of lines on a Castelnuovo surface, the only possible degeneration is

$$(4;\ 2,\ 1^i)+(3m-6;\ 3m-8,\ 1^i,\ 2^{3m-4-i}),$$

for which $m=4$, $i=3$, if $n=9$.

16. *Bordiga forms.* There is a class of surfaces of order n $(5\leqslant n\leqslant 16)$ and sectional genus 3 which are represented by means of quartic curves having $16-n$ base-points, or $(4,\ 1^{16-n})$. The sextic type in [4] was first studied by Bordiga*. The forms F_n having these surfaces for space sections are rational and may be represented by means of quintic surfaces f_5 having a double cubic γ_3 and a simple base-curve γ_{13-n}; the rays of the congruence correspond to chords of γ_3, so that $\kappa=1$.

Consider the representation of f_5 by means of quartics having eleven base-points or, symbolically, $(4,\ 1^{11})$. The double cubic γ_3 corresponds to $(7,\ 2^{11})$, so that the residual intersection of two surfaces f_5 is a curve

$$5(4,\ 1^{11})-2(7,\ 2^{11})=(6,\ 1^{11}).$$

For F_5 this consists of a curve $c_4\equiv(4,\ 1^{11})$, representing a plane section, and a conic c_2 representing a rational γ_8 which meets γ_3 in fourteen points. The class l of F_5 is 11.

To obtain the representation of F_n $(5\leqslant n\leqslant 11)$ we suppose c_2 to pass through $n-5$ of the base-points and c_4 through the remainder; only for $n=11$ does this impose a restriction upon the form of f_5. Thus the representation is by means of surfaces f_5 having a common double cubic γ_3 and a simple base-curve γ_{13-n} which is rational and meets γ_3 in $2(12-n)$ points; plane sections of F_n correspond to curves $c_n{}^3$ meeting γ_3 in $2n-4$ points and γ_{13-n} in 8 points. The class l of the congruence is $16-n$.

For $n=11$, c_2 must pass through six of the base-points; the conic c_2' through the remaining five forms with c_2 a degenerate $(4,\ 1^{11})$, c_2 representing a conic on f_5 which evidently meets γ_3 in two points, and these

* Bordiga, *Memorie dei Lincei* (4), 4 (1887), 182; the quintic type had, however, been considered by Clebsch, *Math. Annalen*, 1 (1869), 253. A general discussion of Bordiga surfaces is given by Castelnuovo, *Atti di Torino*, 25 (1890), 695. See also Scorza, *Annali di Mat.* (3), 16 (1908), 255.

also lie on the cubic corresponding to c_2'. It is clear that no greater value of n is possible, since for $n = 12$ we should have on f_5 a line γ_1 not meeting the double curve*.

The only way in which the class l of the congruence may be reduced is for the form to acquire planes. We thus find for the singularities

$$(5, l)_3 : \alpha_1 = 11-l; \quad (6, l)_3 : \alpha_1 = 10-l, \quad \text{and so on.}$$

The quartic type $(4, l)_3$ is obtained by projecting F_5 from an ordinary point of itself, so that we obtain $\alpha_1 = 12-l$ in this case. The cubic type $(3, l)_3$, which has been investigated by Fano†, is obtained by projecting from a double point. Now the double surface on F_5 is a cubic scroll, and the generator of the scroll passing through the point of projection gives rise to a singularity α_{l-1} (§ 5). This follows from the fact that the surface f_5, whose double curve consists of a line and a secant conic, contains ten lines meeting both the line and the conic. Thus $(3, l)_3$ has the singularities

$$\alpha_{l-1} = 1, \quad \alpha_1 = 13-l.$$

The values of l are given by $3 \leqslant l \leqslant 16-n$.

Enumerative formulae for congruences.

17. *The functional process.* The results which follow are obtained, as in previous work‡, either by correspondence methods or by the functional process. The latter depends upon the fact that the rays of the congruence may be represented by the points of a surface in [9], so that the process as applied to the surface may be equally well applied to the congruence. The characters n, a, j, m, l, C which define a simple congruence are all additive, and will be employed in the functional equations about to be given. In the general solutions of these appear six terms linear in the characters, the coefficients of which must be determined from known special cases; and it will readily be seen that, where the result is independent of C, the coefficients of l and C are identical.

We denote by $(n, l)_p$ a simple congruence for which the corresponding ruled form has plane sections of genus p. In a number of cases it is

* We obtain f_5 by projecting from a point of itself the Bordiga sextic f_6 in [4]. This is represented by curves $(4, 1^{10})$, whence it follows that f_6 contains twisted cubics, corresponding to conics $(2, 1^5)$, which meet five of its lines. By projecting f_6 from a general point of one of these cubics we obtain the special case considered, in which f_5 contains a conic meeting six of its lines, represented by $(2, 1^6)$. The type $(2, 1^7)$, which contains seven lines meeting an eighth, can only arise from projecting from a point of itself a conic of f_6 which meets six lines of the surface; and such conics do not exist.

† Fano, *Memorie di Torino* (2), 51 (1901), 1.

‡ *E.g.*, **C**, 78–82.

possible, as well as convenient, to solve by using the same congruences, namely

$$(3, l)_1, \quad (4, l)_1, \quad (5, l)_1, \quad (4, l)_2, \quad (5, l)_3.$$

We thus have to deal with linear equations in which coefficients of corresponding unknowns are the same in each case; the resulting simplicity in computation is very considerable. Unless otherwise stated, it is understood that these are the special congruences employed in the solution. The notation follows that of previous work.

We give below a number of congruences which will be employed in the work; the character t denotes, as usual, the order of the triple curve, and κ is the number of lines passing through an arbitrary point of the form, in the case where there are also residual congruences, simple or otherwise.

n	l	a	j	m	t	C	κ
$(3, l)_1$	$2 \leqslant l \leqslant 6$	6	0	12	0	$12-l$	6
$(4, l)_1$	$2 \leqslant l \leqslant 5$	8	4	12	0	$8-l$	4
$(5, l)_1$	$2 \leqslant l \leqslant 4$	10	8	12	1	$5-l$	3
$(6, l)_1$	$2 \leqslant l \leqslant 3$	12	12	12	4	$3-l$	2
$(7, l)_1$	2	14	16	12	10	0	1
$(4, l)_2$	$3 \leqslant l \leqslant 8$	10	4	20	0	$13-l$	2
$(5, l)_2$	$3 \leqslant l \leqslant 7$	12	8	20	0	$11-l$	2
$(6, l)_2$	$3 \leqslant l \leqslant 6$	14	12	20	2	$9-l$	2
$(5, 11)_3$	11	14	10	27	0	0	1
$(6, 10)_3$	10	16	14	27	1	0	1

The characters given above are special cases of the following general results. For the general Del Pezzo form of order n,

$$a = 2n, \quad j = 4(n-3), \quad m = 12, \quad C = \tfrac{1}{2}(n^2-15n)+30-l.$$

For the Castelnuovo form we have

$$a = 2(n+1), \quad j = 4(n-3), \quad m = 20, \quad C = 21-2n-l, \quad \kappa = 2.$$

For the Bordiga form,

$$a = 2(n+2), \quad j = 2(2n-5), \quad m = 27, \quad l = 16-n, \quad \kappa = 1.$$

18. *The double surface.* The characters of the double surface Γ are determined by means of the same functional equations as were employed for the purpose in the case of plane-forms. Thus the order of Γ is given by

$$\rho_0 = \binom{n}{2} - \tfrac{1}{2}a, \tag{1}$$

and the rank by

$$\rho_1 = \int an' + a'n = an - a - \tfrac{5}{2}j - m. \tag{2}$$

The class is given by

$$\rho_2 = \int mn' + aa' + m'n = mn + \binom{a}{2} + 45a - \tfrac{51}{2}j - 18m - 41n - 3(l+C). \quad (3)$$

The number of apparent pinch-points is

$$\omega_2 = \int jn' + aa' + j'n = jn + \tfrac{1}{2}a^2 + 62a - 32j - 14m - 50n - 6(l+C). \quad (4)$$

The triple curve T, in points of which concur three distinct rays, is of order

$$t = \binom{n}{3} - \tfrac{1}{2}an + \tfrac{1}{3}(2a+2j+m), \quad (5)$$

and rank

$$t_1 = \int a\rho_0' + a'\rho_0 + n\rho_1' + n'\rho_1$$

$$= a\binom{n}{2} - \tfrac{1}{2}a^2 - n(a + \tfrac{5}{2}j + m) - 72a + 40j + 20m + 58n + 6(l+C). \quad (6)$$

It contains a number χ of quadruple points, quadruple also for F, in which concur four distinct rays; thus

$$\chi = \int tn' + t'n + \rho_0\rho_0'$$

$$= \binom{n}{4} - \tfrac{1}{2}a\binom{n}{2} + \tfrac{1}{8}a^2 + \tfrac{53}{4}a - \tfrac{33}{4}j - \tfrac{9}{2}m - 11n + \tfrac{1}{3}n(2a+2j+m) - (l+C). \quad (7)$$

The class of immersion of T in F is given by

$$\mathbf{t} = \int a\rho_0' + a'\rho_0 + nz' + n'z,$$

where

$$z = an - a - 2j - m$$

is the class of immersion of the double curve on a space section of F. Hence

$$\mathbf{t} = a\rho_0 - n(a+2j+m) - 55a + 31j + 17m + 44n + 4(l+C). \quad (8)$$

Of these results, in all except the last the special cases are immediately available. Now $\mathbf{t}$ must vanish when $t = 0$; and, for $(5, l)_1$, it is equal to the number of intersections of the triple curve with the third polar of an arbitrary point, *i.e.* a quadric; hence $\mathbf{t} = 2$. In the general case, in which the form has a number of quadruple points, there exists the relation

$$t(n-3) = \mathbf{t} + 4\chi,$$

analogous to the Cayley-Zeuthen equation for surfaces,

$$\rho_0(n-2) = z + 3t.$$

19. *The pinch-curve.* There is a curve J on Γ of pinch-points in which two consecutive rays of the congruence meet. Its order is j; and its rank ρ, which must be linear in the characters, is known for $(3, l)_1$, $(4, l)_1$, $(5, l)_1$, and $(4, l)_2$. For the remaining case, we use $(5, l)_3$ and hence obtain

$$\rho = -27a+12j+6m+22n+2(l+C). \tag{9}$$

The classes of immersion of J in Γ and F are then found to be

$$j_0 = -41a+19j+6m+34n+6(l+C), \tag{9a}$$

$$j_1 = -17a+8j+3m+14n+2(l+C). \tag{9b}$$

The curves J and T meet in a number of points which are stationary on the latter; this number σ is given by

$$\sigma = \int jn'+j'n = jn+34a-18j-6m-28n-4(l+C), \tag{10}$$

vanishing in all cases in which $t = 0$. These points are evidently included in the ω_2 points given by (4).

20. *Some special cases.* It is of interest to note the various particulars deduced from the previous equations. Thus we find that the form $(5, l)_1$ contains a double surface whose characters are $\rho_0 = 5$, $p_1 = 8$, $\rho_2 = 5$, $\omega_2 = 6$, $t = 1$; it is therefore a rational quintic scroll having a triple line. The double surface on $(6, l)_1$ has characters $\rho_0 = 9$, $\rho_1 = 18$, $\rho_2 = 9$, $\omega_2 = 18$, $t = 4$, and is thus an elliptic scroll containing a triple curve of order 4 which, by (6), is a rational quartic. The form $(7, 2)_1$ has a double surface which is found to be a scroll of order 14 and genus 3, containing a triple decimic on which there are three quadruple points.

Of the Castelnuovo forms, $(4, l)_2$ contains a double plane and $(5, l)_2$ a double Segre surface, with a pinch-curve of order 8 and genus 4. The double surface on $(6, l)_2$ has characters $\rho_0 = 8$, $\rho_1 = 20$, $\rho_2 = 28$, $\omega_2 = 20$, $t = 2$, $t_1 = 2$, and therefore contains a triple conic. The surface itself is evidently a special case of a type of rational surface with space sections of genus 3; it is represented on a plane by means of quintics with a triple base-point and eight simple base-points; the general type has been considered by Castelnuovo*.

The Bordiga quintic $(5, 11)_3$ contains a double cubic scroll, and the sextic form $(6, 10)_3$ has a double surface whose characters are $\rho_0 = 7$, $\rho_1 = 18$, $\rho_2 = 27$, $\omega_2 = 18$, $t = 1$. This is a special case of the

* Castelnuovo, *loc. cit.* (§ 15).

rational surfaces represented by means of plane quartics having simple base-points, investigated by Castelnuovo (*loc. cit.*).

These results may be generalized by calculating the characters of the double surface for the general form of each type given in § 17. We thus find, for the Del Pezzo form of order n,

$$\left.\begin{gathered}\rho_0=\tfrac{1}{2}n(n-3)=\rho_2,\quad \rho_1=2(n-3)^2,\quad \omega_2=3(n-3)(n-4),\\ \sigma=2(n-3)(n-4),\quad t=\tfrac{1}{6}(n-2)(n-3)(n-4),\\ t_1=(n-3)(n-4)(n-5),\quad \chi=\binom{n-3}{4}.\end{gathered}\right\}\tag{11}$$

For the Castelnuovo form,

$$\left.\begin{gathered}\rho_0=\tfrac{1}{2}(n^2-3n-2),\quad \rho_1=2(n-1)(n-4),\quad \rho_2=2(n-4)(n+1),\\ \omega_2=2(n-4)(3n-13),\quad \sigma=4(n-4)(n-5),\\ t=\tfrac{1}{6}n(n-4)(n-5),\quad t_1=(n-4)(n-5)^2,\quad \chi=\binom{n-3}{4}.\end{gathered}\right\}\tag{12}$$

For the Bordiga form,

$$\left.\begin{gathered}\rho_0=\tfrac{1}{2}(n^2-3n-4),\quad \rho_1=2(n^2-4n-3),\quad \rho_2=2n(n+1)-57,\\ \omega_2=2(3n^2-25n+51),\quad \sigma=(n-5)(4n-18),\\ t=\tfrac{1}{6}(n-5)(n^2-4n-6),\quad t_1=(n-2)(n-5)(n-6),\quad \chi=\frac{n}{4}\binom{n-5}{3}.\end{gathered}\right\}\tag{13}$$

In the two latter cases the invariant*

$$p_a=\rho_0-1-\frac{2\rho_1}{3}+\frac{\rho_2}{6}+\frac{\omega_2}{12}$$

vanishes identically. We thus deduce that

The double surface of a Del Pezzo form is a scroll;

and it is suggested that

The double surface of a Castelnuovo or a Bordiga form is rational.

We can establish the rationality directly by obtaining the plane representations of these surfaces, as has already been done in certain special cases†. For all Castelnuovo forms and for Bordiga forms of order 5 or 6 we have $3\rho_0>\rho_1$, so that‡ the double surfaces are certainly rational.

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* Enriques, *Lezioni sulla teoria delle superficie algebriche*, Padova (1932), 214.

† See Roth, *Proc. Camb. Phil. Soc.*, 28 (1932), 300.

‡ Enriques, *loc. cit.*, 445.

21. *The scroll* $S(\pi)$. The rays of the congruence, supposed simple, which meet a given plane π generate a scroll of order $n+l$, meeting π in a curve of order n. The scroll projects from a point of π into the scroll (g) of rays meeting a given line g in the space of the projection; hence its properties are well known*. The scroll $S(\pi)$ has ρ_0 improper nodes in π; thus, if y is the number of improper nodes not lying in π, we have

$$y+\rho_0=\binom{n+l}{2}-\tfrac{3}{2}(a+2l)+n+l=\binom{n+l}{2}-\tfrac{3}{2}a+n-2l. \qquad (14)$$

The number of tangents which may be drawn from a given point P is found by supposing P to be in π, whence we obtain the number

$$j_0=a+w=2(a-n+l). \qquad (15)$$

Consider now two such scrolls $S(\pi)$, $S(\pi')$ corresponding to two arbitrary planes; these have in common $n+l$ generators (the number of rays meeting both planes), and each generator absorbs $1+2z$ intersections of the two scrolls, where z is the class of immersion of the generator in either scroll†, *i.e.* the order of the form of tangent planes to the scroll in points of the generator. Since this form is simply the space defined by the generator and one consecutive to it, z is equal to unity. Hence the two scrolls intersect in $(n+l)(n+l-3)$ points external to the common generators.

From this result we may obtain the order N of the pinch-scroll generated by pairs of consecutive rays meeting on the pinch-curve J. For the number of such intersections is equal to the order of the scroll S' generated by the rays which meet the generators of $S(\pi)$; and S' meets π in $2\rho_0+2y+N$ points, whence

$$N=(n+l)(n+l-3)-2\rho_0-2y=3a-4n+2l, \qquad (16)$$

on substitution. If there are a_h h-ple points, the modified value is

$$N=3a-4n+2l-\Sigma a_h h. \qquad (16')$$

To obtain the order M of the form of planes defined by the consecutive pairs, we introduce a second character ζ, representing the order of the form of planes defined by rays which meet on a given space. Consider the correspondence between points of a space σ' lying on rays which intersect on a given space σ. These rays generate a scroll whose order is clearly $l(n-3)+2\rho_0$, so that the pairs of points in σ' lie on a curve of this

* See **C**, 78.

† The method and results which follow are due to James, *loc. cit.*, 171.

order; the lines joining corresponding points generate a scroll of order ζ. Now the united points of the correspondence are $j+2\zeta$ in number, whence

$$2l(n-3)+2\rho_0 = j+2\zeta.$$

Again, consider the pairs of rays which intersect and are cospatial with an arbitrary line g. By taking a plane through g it follows that these generate a scroll of order $n(n-3)+2y$; also the planes defined by such pairs generate a form of order $y+d$, where d has the same significance as in § 5.

The section of the figure by an arbitrary space gives a (1, 1) correspondence on the section of the scroll having coincidences in number

$$M+2\zeta = 2n(n-3)+4y-2(y+d).$$

Substituting and reducing, we thus have

$$M = -a+2j-2n+6l, \tag{17}$$

and

$$\zeta = \binom{n}{2}+nl-\tfrac{1}{2}(a+j)-3l. \tag{18}$$

22. *Some properties of multiple tangents.* By means of the theorem given in § 4 the following results may be obtained*.

I. The inflexional tangents at points of a ray ρ of F (supposed simple) generate a quadric cone containing ρ. By considering the Hessian of the section by an arbitrary space through ρ, we see that ρ is elsewhere bitangent to the Hessian; thus there are two points at which ρ coincides with other inflexional tangents to the section. Hence the inflexional tangents which meet ρ are such that the planes containing ρ and the consecutive tangents generate a quadric line-cone.

II. The spaces defined by ρ and consecutive rays of F envelop a quadric cone, which is identical with the above. This property holds for all values of κ. Since these spaces are tangent to F along ρ, it follows that the class of immersion of a ray in F is equal to two, as may be deduced from the functional method, since the number in question must be a constant and it is determined by the cubic form.

III. Consider an arbitrary plane section of F (simple or compound) passing through ρ, and meeting F elsewhere in a curve of order $n-1$; of the points in which this meets ρ, $n-3$ are intersections with the double surface, and the remaining two are contacts with F. Thus any plane

* The Theorems I–IV which follow are due to James, *loc. cit.*

through ρ is bitangent to F; hence an arbitrary plane through ρ contains two inflexional tangents with their points of contact on ρ.

IV. The ∞^2 inflexional tangents meeting ρ generate a form which is of the same order for all congruences (simple or compound). This follows from the functional method; and the order in question may be found by considering the cubic form. Taking a space section through ρ, the order is that of the scroll of inflexional tangents at points of ρ to the surface thus obtained. Now a plane through ρ contains two such tangents and, moreover, ρ is a simple directrix and a double generator on the scroll, since there exist two planes passing through ρ for which the residual section touches ρ. Hence the form of inflexional tangents is of order 5, and ρ is triple on the form, being a simple directrix and double generator.

V. The four-point tangents to F form a complex ∞^3 of order (§ 5)

$$\mu_4 = -54a+22j+12m+44n+4(l+C) \qquad (19')$$

and class* $$\nu_4 = -10a+11j+6m-4n. \qquad (19)$$

At an ordinary point of F there are five such tangents (as follows from the case of the cubic congruences, for which $\kappa = 1$). The numbers of 4-point tangents which may be drawn from a simple, double, ... point of F are at once found to be

$$t_4 = \mu_4-16, \quad \delta_4 = \mu_4-32, \quad \tau_4 = \mu_4-48, \quad \chi_4 = \mu_4-64, \qquad (20)$$

by using appropriate special cases.

23. *The quadrisecants of a surface.* A general surface f in S_4, of characters ρ_0, ρ_1, ρ_2, ω_2 suppose, has ∞^2 quadrisecants which generate a ruled form F of order†

$$n = 3\binom{\rho_0}{4} - \rho_1\binom{\rho_0}{2} + \tfrac{1}{3}\rho_0(5\rho_1+2\omega_2+\rho_2) + \tfrac{1}{8}\rho_1{}^2 - \tfrac{1}{4}(7\rho_1+11\omega_2+6\rho_2) \qquad (21)$$

and class

$$l = 2\binom{\rho_0}{4} - \tfrac{1}{2}\rho_1\binom{\rho_0}{2} + \rho_1\rho_0 + \tfrac{1}{8}\rho_1{}^2 - \tfrac{11}{4}\rho_1 + \rho_0. \qquad (22)$$

The number of quadrisecants passing through an arbitrary point of f is

$$q = \binom{\rho_0}{3} - \binom{\rho_0}{2} + \tfrac{7}{6}\rho_1 + \tfrac{1}{3}(2\omega_2+\rho_2) + 2\rho_0 - 6 + \epsilon, \qquad (23)$$

where ϵ is unity in the case of a scroll, but is otherwise zero.

* Roth, *Proc. Camb. Phil. Soc.*, 26 (1930), 52.

† Roth, *Proc. Camb. Phil. Soc.*, 25 (1929), 390.

In general F contains a quintuple surface, the locus of quintisecants of f, and this has a finite number of sextuple generators, which are sextisecants of f. It is of interest to determine those surfaces for which F has only singular loci of the usual type and, in particular, for which F is simply ruled. A first requisite is, therefore, the non-existence of quintisecants and sextisecants; a number of surfaces having this property may be obtained as the complete or partial intersection of two forms of order 3 or 4, though not all such intersections are of the desired type.

We give below a list of surfaces and scrolls up to the eighth order, for which F is a simple or compound form of general character; surfaces all of whose quadrisecants lie in planes are, of course, excluded from the present investigation. Since all quartic and quintic surfaces lie on a pair of cubic forms, they are necessarily without quadrisecants unless these lie in planes.

We note* first that F must have nodes at the d improper nodes of f, in number

$$d = \binom{\rho_0}{2} - \tfrac{1}{2}\rho_1 - \tfrac{1}{2}\omega_2, \tag{24}$$

and δ nodes at ordinary points of f, where

$$\delta = \rho_0\{n(n-2)-\rho_0\} - \rho_1 n + 2(\rho_0+\rho_1+\omega_2). \tag{25}$$

The list is then as follows:

	ρ_0	ρ_1	ρ_2	ω_2	n	l	q
(1)	6	12	12	12	3	3	1
(2)	6	10	6	8	3	6	1
(3)	7	20	33	22	3	2	1
(4)	7	18	27	18	3	5	1
(5)	7	18	26	20	3	5	2
(6)	7	16	20	16	7	9	2
(7)	7	14	7	14	7	14	2
(8)	8	26	54	28	3	5	1
(9)	8	24	40	28	7	10	2
(10)	8	24	48	24	6	10	2
(11)	8	24	39	30	6	10	3
(12)	8	24	38	32	15	10	4
(13)	8	22	34	24	11	16	3
(14)	8	20	28	20	16	23	4
(15)	8	18	8	20	16	31	4
(16)	8	22	28	24	4	16	1

* Severi, *Memorie di Torino* (2), 52 (1903), 61.

We append a brief description of these surfaces.

(1) The surface is the projection of a Del Pezzo sextic and is the intersection of a cubic and a quartic form residual to an elliptic sextic scroll. Its quadrisecants generate the cubic form, which is nine-nodal, having three nodes at the improper nodes of the surface and six elsewhere.

(2) This is a rational sextic scroll, obtained as the intersection of a cubic and a quartic form residual to a scroll of precisely the same type. Its quadrisecants generate the cubic form in question, which has nodes at the six improper nodes of the scroll; the form contains a second simple system of generators* which are quadrisecant to the residual scroll.

(3) This surface is the intersection of a cubic and a quartic form residual to an elliptic quintic scroll; it is rational and is represented by means of plane sextics having six nodal and five ordinary base-points or, symbolically, $(6, 2^6, 1^5)$. Its quadrisecants lie on the cubic form, which has ten nodes at ordinary points of the surface, there being no improper nodes.

(4) This is the projection of a Bordiga septimic and it lies on a cubic and a quartic form, the residual intersection being a rational quintic scroll; it is, of course, represented by means of quartic curves $(4, 1^9)$. The cubic form contains its quadrisecants; it has seven nodes, three of which are at the improper nodes of the surface.

(5) This surface is rational† and it is represented by means of sextics $(6, 2^7, 1)$. The cubic form containing its quadrisecants is ten-nodal, so that, from the formulae for l and q, the quadrisecants obviously consist of two simple systems and the lines of a plane. The surface is obtained as the intersection of the cubic form and a quartic form, residual to the plane, which meets it in a quartic curve, and a rational quartic scroll which meets the plane in a generator.

(6) The surface is of the Castelnuovo type, represented by means of curves $(4, 2, 1^5)$, and is contained in two quartic forms whose residual intersection is a certain novenic surface with linear sections of genus 6.

(7) This is of the Del Pezzo type, represented by means of curves $(3, 1^2)$, and it lies on two quartic forms.

* See Segre (quoted in the introduction).

† It is considered by Castelnuovo (*loc. cit.*, § 16).

(8) The surface is the intersection of a cubic and a quartic form, residual to a rational quartic scroll. The cubic form is seven-nodal, one node being at the improper node of the surface, and one of its simple systems consists of the quadrisecants.

(9) The surface is contained in two quartic forms and is self-residual. It is rational and is represented by means of curves $(6, 2^5, 1^8)$.

(10) This also is self-residual.

(11) The surface is contained in two quartic forms and is rational; it is represented by means of curves $(7, 3, 2^7, 1^4)$.

(12) This also is contained in two quartic forms and is rational; it is represented by means of curves $(7, 2^{10}, 1)$.

(13) This is self-residual and rational, being represented by means of curves $(5, 2^2, 1^9)$.

(14) The surface is rational and represented by means of curves $(6, 2^7)$.

(15) This is the octavic scroll of genus 2; it is the intersection of two quartic forms residual to a Castelnuovo octavic; the characters may be found from the general case given in § 17.

(16) This is a surface, normal in [4], which is referable to an elliptic scroll; it has been considered in a recent paper*, to which reference may be made.

* Roth, *Proc. Camb. Phil. Soc.*, 29 (1933), 184.

ON THE SUMMABILITY OF FOURIER SERIES AND THE BOUNDED VARIATION OF POWER SERIES

By B. N. PRASAD.

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1. *Introduction.*

1.1. Let
$$f(x) = \sum_0^\infty a_n x^n, \tag{1.11}$$
the series being supposed convergent in $(0 \leqslant x < 1)$. If the Abel limit
$$\lim_{x \to 1-0} f(x) \tag{1.12}$$
exists, the series Σa_n is *summable* (A). If $f(x)$ is of *bounded variation* in (0, 1), the Abel limit will *necessarily* exist, and we may say that in that case the series Σa_n is *absolutely summable* (A), or the limit (1.12) "exists absolutely"*. Thus a series which is *absolutely summable* (A) is also *summable* (A), but the converse is not always true. Moreover, just as every convergent series is summable (A), so every absolutely convergent series is absolutely summable (A). In this and in certain other respects, the absolute summability (A) is analogous to the absolute summability by Borel's integral method of summation, or absolute summability (B)†.

1.2. Suppose that
$$c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n + \ldots \tag{1.21}$$

* *Cf. Nature*, 126 (25 Oct., 1930), 643.

† See E. Borel, *Leçons sur les séries divergentes* (Paris, 1901), ch. 3; G. H. Hardy, "Researches in the theory of divergent series", *Quart. J. of Math.*, 35 (1904), 25, 28, and 43-44.

is a power series for which the radius of the circle of convergence is unity. If we put

$$c_n = a_n - ib_n,$$

$$z = re^{i\theta},$$

we get

$$\sum_{n=0}^{\infty} c_n z^n = \sum_{0}^{\infty} r^n(a_n \cos n\theta + b_n \sin n\theta) - i\sum_{0}^{\infty} r^n(b_n \cos n\theta - a_n \sin n\theta) \quad (1.22)$$

$$= P(r, \theta) + iQ(r, \theta).$$

The question is: When is the power series $\Sigma c_n z^n$ a function of bounded variation on a radius vector? That the convergence of the series at a point on the circle of convergence is *not* sufficient for it to be of bounded variation on the corresponding radius vector, is seen by the example*

$$z(1-z)^i \left\{\log \frac{1}{1-z}\right\}^{-1} = \sum_{n=0}^{\infty} c_n z^n,$$

which, although convergent on the circle of convergence of radius unity for every θ, is not of bounded variation on the radius vector $\theta = 0$. It is proposed to investigate conditions under which $\Sigma c_n z^n$ is of bounded variation on a radius vector. This depends upon the simultaneous, absolute summability (A) of the two trigonometric series

$$\sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \quad (1.23)$$

and

$$\sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta), \quad (1.24)$$

the constants having been omitted for the sake of simplicity. If there is an integrable function $f(u)$ such that

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\cos nu\,du, \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(u)\sin nu\,du,$$

the integration being performed in the sense of Lebesgue, then the series (1.23) is a Lebesgue-Fourier series and the series (1.24) is the series *allied* or *conjugate* to the Fourier series (1.23). It is now well known that the allied series is not necessarily a Fourier series. With a

* See J. M. Whittaker, "The absolute summability of Fourier series", *Proc. Edinburgh Math. Soc.* (2), 2 (1930), 1–5 (4).

view to secure greater generality in the results regarding power series, the series (1.23) and (1.24) are taken not as Fourier series, but rather as trigonometric series of the type of restricted Fourier series* considered by Young. In fact, it is necessary to take into consideration only the interval or intervals of restriction, this being compatible with the property of absolute summability (A), which depends upon the nature of the associated function in the immediate neighbourhood of the point under consideration.

1.3. In a previous paper†, I have investigated conditions of the absolute summability (A) of Fourier series; in § 2 I prove that the series will be absolutely summable (A) if $f(\theta)$ is of bounded variation—a condition corresponding to that of Jordan's test for the convergence of Fourier series. It is interesting to note that a Fourier series may be convergent without being absolutely summable (A), or it may be absolutely summable (A) without being convergent in the ordinary sense‡.

In § 3 we consider the absolute summability (A) of the allied series. In § 4 the *almost everywhere* absolute summability (A) of the Fourier series and of its allied series is investigated. J. M. Whittaker has proved§ that the series

$$\sum_1^\infty \frac{a_n \cos n\theta + b_n \sin n\theta}{n^\alpha},$$

obtained by multiplying the Fourier series by the factors $n^{-\alpha}$, $(\alpha > 0)$, is absolutely summable (A) almost everywhere. It is proved here that, even if the Fourier series or the allied series is multiplied by factors, which we may term *summable factors*,

$$\frac{1}{(\log n)^\beta}, \quad \frac{1}{\log n (\log\log n)^\beta}, \quad \ldots, \quad \frac{1}{\log n \log\log n \ldots \log\log \ldots \log_{p-1} n (\log\log \ldots \log_p n)^\beta},$$

* See W. H. Young, "On the convergence of the derived series of Fourier series", *Proc. London Math. Soc.* (2), 17 (1918), 195–236; also "On restricted Fourier series and the convergence of power series", *ibid.*, 353–366; and "On the ordinary convergence of restricted Fourier series", *Proc. Royal Soc.* (A), 93 (1917), 276–292. See also *Bull. de la Soc. Mat. de France*, 52 (1924), 585.

† B. N. Prasad, "The absolute summability (A) of Fourier series", *Proc. Edinburgh Math. Soc.* (2), 2 (1931), 129–134.

‡ *Ibid.*, 134.

§ I am indebted to Mr. J. M. Whittaker for a letter in which he gave me his proof.

where $\beta > 1$, the resulting series is absolutely summable (A) almost everywhere. It is an interesting analogue of the property of almost everywhere ordinary convergence of Fourier series when they are multiplied by such convergent factors*.

In § 5 all these results are extended to the case of restricted Fourier series.

In § 6 the results obtained in the preceding articles are combined to yield conditions for $\Sigma c_n z^n$ to be of bounded variation on a radius vector. A natural question from this point of view is : Is the power series of bounded variation on the radius OC, if it is of bounded variation on AB which is an arc of the circle of convergence surrounding the point C? That the answer to this question is in the affirmative is shown in this article.

Some of the theorems dealing with Fourier series and the allied series are essentially concerned with harmonic and analytic functions, and I state them in these terms towards the end of the paper.

I wish to express my thanks to Prof. E. C. Titchmarsh for his kind interest and advice during the preparation of this work.

In what follows, we shall put

$$2\phi(t) = f(\theta+2t)+f(\theta-2t)-2f(\theta),$$

$$\psi(t) = f(\theta+t)-f(\theta-t),$$

$$\Phi(t) = \int_0^t \phi(u)\,du, \quad \text{and} \quad \Psi(t) = \int_0^t \psi(u)\,du.$$

2. *The absolute summability* (A) *of Fourier series.*

2.1. In my paper already referred to the following results are established:

The Fourier series is absolutely summable (A) *if* $\phi(t)$ *is absolutely continuous in* $(0, \delta)$. (A)

The Fourier series is absolutely summable (A) *if the integral*

$$\int_0^\delta \left|\frac{\Phi(t)}{t^2}\right| dt \qquad (2.11)$$

exists. (B)

* See E. W. Hobson, *Theory of functions of a real variable*, 2 (1926), 623.

The result obtained by J. M. Whittaker is*:

The Fourier series is absolutely summable (A) if the integral

$$\int_0^\delta \left| \frac{\phi(t)}{t} \right| dt \qquad (2.12)$$

exists. (C)

It should be noticed that (C) is included in (B).

2.2. THEOREM 1. *The Fourier series is absolutely summable (A) at a point θ if a neighbourhood of θ can be found such that $f(\theta)$ is of bounded variation in it.*

From
$$P(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} x^n(a_n \cos n\theta + b_n \sin n\theta)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\alpha)\,\frac{1-x^2}{1-2x\cos(\theta-\alpha)+x^2}\,d\alpha,$$

we get

$$\tfrac{1}{2}\pi Q(x) = \tfrac{1}{2}\pi\{P(x)-f(\theta)\} = \int_0^{\frac{1}{2}\pi} \phi(t)\,\frac{1-x^2}{1-2x\cos 2t+x^2}\,dt$$
$$= \left(\int_0^\delta + \int_\delta^{\frac{1}{2}\pi}\right) \phi(t)\,\frac{1-x^2}{1-2x\cos 2t+x^2}\,dt$$
$$= Q_1(x)+Q_2(x),$$

where δ is a constant such that $0 < \delta < \frac{1}{2}\pi$.

Since $Q_2(x)$ is a function of bounded variation in (0, 1), it is only necessary to show that $Q_1(x)$ is also a function of bounded variation in (0, 1).

We suppose that θ is a point of continuity or simple discontinuity such that $\phi(t)$ tends to zero with t. For, if $\phi(t)$ does not tend to zero but to a constant K as t tends to zero, we put

$$Q_1(x) = \int_0^\delta \{\phi(t)-K\}\,\frac{1-x^2}{1-2x\cos 2t+x^2}\,dt + K\int_0^\delta \frac{1-x^2}{1-2x\cos 2t+x^2}\,dt$$
$$= Q_3(x)+Q_4(x).$$

Here $Q_4(x)$ is a function of bounded variation in (0, 1) and the method of this section will be applicable to $Q_3(x)$.

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* J. M. Whittaker, *loc. cit.*, 2.

Now, since $f(\theta)$ is of bounded variation, a neighbourhood of $t = 0$ also exists in which $\phi(t)$ is of bounded variation; let this neighbourhood be $(0, \delta)$. In $(0, \delta)$, $\phi(t)$ can be expressed as the difference

$$\phi_1(t) - \phi_2(t)$$

of two functions ϕ_1 and ϕ_2, each positive and monotone non-decreasing such that

$$\lim_{t\to 0} \phi_1(t) = 0 = \lim_{t\to 0} \phi_2(t).$$

Let
$$J(x) = \int_0^\delta \phi_1(t) \frac{1-x^2}{1-2x\cos 2t+x^2}\, dt$$

and
$$K(x) = \int_0^\delta \phi_2(t) \frac{1-x^2}{1-2x\cos 2t+x^2}\, dt,$$

so that
$$Q_1(x) = J(x) - K(x). \qquad (2.21)$$

Now, by the theorem of integration by parts for the Lebesgue-Stieltjes integral, we get*

$$\begin{aligned} J(x) &= \left[\phi_1(t) \tan^{-1}\left\{\frac{1+x}{1-x}\tan t\right\}\right]_0^\delta - \int_0^\delta \tan^{-1}\left\{\frac{1+x}{1-x}\tan t\right\} d\phi_1(t) \\ &= \phi_1(\delta) \tan^{-1}\left\{\frac{1+x}{1-x}\tan\delta\right\} - \int_0^\delta \tan^{-1}\left\{\frac{1+x}{1-x}\tan t\right\} d\phi_1(t) \\ &= G_1(x) - G_2(x). \end{aligned}$$

Here $G_1(x)$ is a function of bounded variation of x in $(0, 1)$.

Regarding $G_2(x)$, we note that, if $x' > x$,

$$\begin{aligned} G_2(x') - G_2(x) &= \int_0^\delta \left[\tan^{-1}\left\{\frac{1+x'}{1-x'}\tan t\right\} - \tan^{-1}\left\{\frac{1+x}{1-x}\tan t\right\}\right] d\phi_1(t) \\ &\geqslant 0, \end{aligned}$$

and hence $G_2(x)$ is monotonic increasing. Moreover, if

$$K = \tfrac{1}{2}\pi \int_0^\delta d\phi_1(t),$$

then
$$K - G_2(x) = \int_0^\delta \left[\tfrac{1}{2}\pi - \tan^{-1}\left\{\frac{1+x}{1-x}\tan t\right\}\right] d\phi_1(t) \geqslant 0,$$

and this shows that $G_2(x)$ is also bounded.

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* See E. W. Hobson, *loc. cit.*, 1 (1927), 665.

Hence $G_2(x)$ is a function of bounded variation and consequently $J(x)$ is a function of bounded variation in (0, 1). A similar result can be proved for $K(x)$. This proves the theorem.

3. *The absolute summability (A) of the allied series.*

3.1. THEOREM 2. *The allied series* (1.24) *is absolutely summable* (*A*) *if the integral*

$$\int_0^\delta \left| \frac{\psi(t)}{t} \right| dt \tag{3.11}$$

exists.

The existence of the integral (3.11) is a sufficient condition for the ordinary convergence of the allied series.

We have, for $0 \leqslant x < 1$,

$$\begin{aligned} I(x) &= \sum_{n=1}^{\infty} (b_n \cos n\theta - a_n \sin n\theta) x^n \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\alpha) \frac{x \sin(\alpha-\theta)}{1-2x\cos(\alpha-\theta)+x^2} d\alpha \\ &= \frac{1}{\pi} \int_0^\delta \psi(t) \frac{x \sin t}{1-2x\cos t+x^2} dt + \frac{1}{\pi} \int_\delta^\pi \psi(t) \frac{x \sin t}{1-2x\cos t+x^2} dt, \end{aligned}$$

where $0 < \delta < \pi$. We write this

$$\pi I(x) = I_1(x) + I_2(x). \tag{3.12}$$

Since $I_2'(x)$ is bounded, $I_2(x)$ is of bounded variation in (0, 1).

The total variation of $I_1(x)$ in $(0, x_1)$, where $0 < x_1 < 1$, is

$$\begin{aligned} \int_0^{x_1} |I_1'(x)|\, dx &= \int_0^{x_1} \left| \int_0^\delta \psi(t) \frac{d}{dx} \left\{ \frac{x \sin t}{1-2x\cos t+x^2} \right\} dt \right| dx \\ &\leqslant \int_0^{x_1} dx \int_0^\delta |\psi(t)| \left| \frac{(1-x^2)\sin t}{(1-2x\cos t+x^2)^2} \right| dt \\ &= \int_0^\delta dt\, |\psi(t)| \int_0^{x_1} \left| \frac{(1-x^2)\sin t}{(1-2x\cos t+x^2)^2} \right| dx, \end{aligned}$$

by inverting the order of integration. Now

$$\int_0^{x_1} \left| \frac{(1-x^2)\sin t}{(1-2x\cos t+x^2)^2} \right| dx = \frac{x_1 \sin t}{1-2x_1\cos t+x_1^2} \leqslant \tfrac{1}{2}\pi \frac{1}{t}.$$

Therefore

$$\int_0^{x_1} |I'(x)|\, dx \leqslant \tfrac{1}{2}\pi \int_0^\delta \left| \frac{\psi(t)}{t} \right| dt.$$

Hence $I_1(x)$ and consequently $I(x)$ from (3.12) is a function of bounded variation in (0, 1), provided that the integral (3.11) exists. Thus the theorem is proved.

3.2. We want to prove now a more general theorem for the absolute summability (A) of the allied series.

THEOREM 3. *The allied series* (1.24) *is absolutely summable* (A) *if the integral*

$$\int_0^{\delta} \left| \frac{\Psi(t)}{t^2} \right| dt \tag{3.21}$$

exists.

Let $$U(x, t) = \frac{x \sin t}{1-2x \cos t+x^2}.$$

Taking $\delta = \frac{1}{2}\pi$, we have

$$\begin{aligned}\int_0^{x_1} |I_1'(x)|\, dx &= \int_0^{x_1} \left| \int_0^{\frac{1}{2}\pi} \psi(t) \frac{\partial U(x, t)}{\partial x}\, dt \right| dx \\ &= \int_0^{x_1} \left| \left[\Psi(t) \frac{\partial U(x, t)}{\partial x} \right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \Psi(t) \frac{\partial^2 U(x, t)}{\partial t\, \partial x}\, dt \right| dx \\ &\leqslant A + \int_0^{x_1} dx \int_0^{\frac{1}{2}\pi} |\Psi(t)| \left| \frac{\partial^2 U(x, t)}{\partial t\, \partial x} \right| dt,\end{aligned}$$

where A is a constant. Inverting the order of integration, we get

$$\int_0^{x_1} |I_1'(x)|\, dx \leqslant A + \int_0^{\frac{1}{2}\pi} |\Psi(t)| \left\{ \int_0^{x_1} \left| \frac{\partial^2 U(x, t)}{\partial t\, \partial x} \right| dx \right\} dt.$$

Now

$$\begin{aligned}\int_0^{x_1} \left| \frac{\partial^2 U(x, t)}{\partial t\, \partial x} \right| dx &= \int_0^{x_1} \left| (1-x^2) \frac{\cos t(1+x^2) - 2x(1+\sin^2 t)}{(1-2x \cos t+x^2)^3} \right| dx \\ &= \int_0^{x_1} |V(x, t)|\, dx,\end{aligned}$$

where $$V(x, t) = (1-x^2) \frac{(1+x^2) \cos t - 2x(1+\sin^2 t)}{(1-2x \cos t+x^2)^3}.$$

Now, let t_1 $(0 < t_1 < \frac{1}{2}\pi)$ be given by

$$\frac{\cos t_1}{1+\sin^2 t_1} = \frac{2x_1}{1+x_1^2}.$$

Then, if $0 \leqslant t \leqslant t_1$, we have

$$\begin{aligned}\int_0^{x_1}\left|\frac{\partial^2 U(x,t)}{\partial t\,\partial x}\right|dx &= \int_0^{x_1} V(x,t)\,dx \\ &= \frac{(x_1+x_1^3)\cos t - 2x_1^2}{(1-2x_1\cos t+x_1^2)^2} \\ &\leqslant \frac{x_1(1-x_1)^2}{(1-x_1)^4+8x_1(1-x_1)^2\sin^2\frac{1}{2}t+16x_1^2\sin^4\frac{1}{2}t} \\ &< \frac{\pi^2}{t^2}.\end{aligned}$$

If $t_1 \leqslant t \leqslant \frac{1}{2}\pi$ and x', such that $0 < x' < 1$, is given by

$$\frac{2x'}{1+x'^2} = \frac{\cos t}{1+\sin^2 t},$$

we have

$$\begin{aligned}\int_0^{x_1}\left|\frac{\partial^2 U(x,t)}{\partial t\,\partial x}\right|dx &\leqslant \int_0^{1}\left|\frac{\partial^2 U(x,t)}{\partial t\,\partial x}\right|dx \\ &= \int_0^{x'} V(x,t)\,dx - \int_{x'}^{1} V(x,t)\,dx \\ &= 2\,\frac{(x'+x'^3)\cos t - 2x'^2}{(1-2x'\cos t+x'^2)^2} + \frac{1}{2(1-\cos t)} \\ &\leqslant \frac{2\pi^2}{t^2}.\end{aligned}$$

Thus we get

$$\begin{aligned}\int_0^{x_1}|I_1'(x)|\,dx &< A+\left(\int_0^{t_1}+\int_{t_1}^{\frac{1}{2}\pi}\right)|\Psi(t)|\left\{\int_0^{x_1}\left|\frac{\partial^2 U(x,t)}{\partial t\,\partial x}\right|dx\right\}dt \\ &\leqslant A+\pi^2\int_0^{t_1}\frac{|\Psi(t)|}{t^2}\,dt+2\pi^2\int_{t_1}^{\frac{1}{2}\pi}\frac{|\Psi(t)|}{t^2}\,dt \\ &< A+B\int_0^{\frac{1}{2}\pi}\frac{|\Psi(t)|}{t^2}\,dt,\end{aligned}$$

where B is a constant. The proof of Theorem 3 is thus completed. It may be noted that Theorem 3 includes in itself Theorem 2.

3.3. *The allied series of a Fourier series may be absolutely summable (A) at a point without being convergent there in the ordinary sense.*

If we take the function $f(x)$ defined by

$$f(2x) = \tfrac{5}{4}x^{\frac{1}{2}}\sin\frac{1}{x} - \frac{1}{x^{\frac{3}{4}}}\cos\frac{1}{x} \quad \text{for} \quad x > 0; \quad f(2x) = 0 \quad \text{for} \quad x \leqslant 0,$$

we observe that this function $f(x)$ is of the Lebesgue class $L^p(p>1)$. Hence we can regard the Fourier series corresponding to $f(x)$ as the allied series of the Fourier series corresponding to the conjugate function

$$g(x)=\frac{1}{2\pi}\int_0^\pi \{f(x+t)-f(x-t)\}\cot\tfrac{1}{2}t\,dt,$$

which is integrable*. But I have proved† that the Fourier series corresponding to $f(x)$ thus defined is absolutely summable (A) at the origin, but that it is not convergent in the ordinary sense. This proves the statement made in §1.3‡.

4. *The almost everywhere absolute summability (A) of the Fourier series and of its allied series.*

4.1. Let

$$f(\theta)\sim \tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

be the Fourier series corresponding to the function $f(\theta)$ which is integrable (L) in $(-\pi, \pi)$. Then we shall prove the following theorem:

THEOREM 4. *The series*

$$\sum_{n=2}^{\infty}\frac{a_n\cos n\theta+b_n\sin n\theta}{\log n(\log\log n)^\gamma}, \tag{4.11}$$

where $\gamma>1$, is absolutely summable (A) for almost every value of θ in $(-\pi, \pi)$.

It is known that the series (4.11) is a Fourier series which converges almost everywhere§. We shall prove the following lemma‖:

LEMMA I. *If*

$$P(x)=\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)x^n,$$

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* See M. Riesz, *Math. Zeitschrift*, 27 (1927), 218–244.

† B. N. Prasad, *loc. cit.*, 133–134.

‡ If we take $\Psi(t)=t^{\frac{1}{2}}\sin(1/t)$, then, with the help of a result of Kuniyeda, *Quart. J. of Math.*, 48 (1920), 135, it can be proved *directly* that the corresponding allied series is absolutely summable (A), but not convergent at the origin.

§ W. H. Young, *Comptes rendus*, 155 (1912), 480; also *Proc. Royal Soc.*, 88 (1912), 179.

‖ J. M. Whittaker has proved this lemma with the help of a certain integral *inequality* which he has established. The proof given here is a form of his proof so modified as to be independent of that inequality.

and $0 < \xi < 1$, *then*

$$\int_0^\xi |P'(x)|\,dx < K \log \frac{1}{1-\xi}$$

almost everywhere, where K *is independent of* ξ.

We have

$$\int_0^\xi |P'(x)|\,dx \leqslant \frac{2}{\pi}\int_0^\epsilon dt\,|\phi(t)| \int_0^\xi \left| 2\,\frac{(1+x^2)\cos 2t - 2x}{(1-2x\cos 2t+x^2)^2}\right| dx$$
$$+\frac{2}{\pi}\int_\epsilon^{\frac{1}{4}\pi} dt\,|\phi(t)| \int_0^\xi \left| 2\,\frac{(1+x^2)\cos 2t - 2x}{(1-2x\cos 2t+x^2)^2}\right| dx,$$

where
$$\epsilon = \sin^{-1}\frac{1-\xi}{\sqrt{2}\sqrt{(1+\xi^2)}}.$$

If $0 \leqslant t \leqslant \epsilon$, we have

$$\int_0^\xi \left| 2\,\frac{(1+x^2)\cos 2t - 2x}{(1-2x\cos 2t+x^2)^2}\right| dx \leqslant \frac{1-\xi^2}{1-2\xi\cos 2t+\xi^2}$$
$$< \frac{A}{\epsilon},$$

where A is a constant.

If $\epsilon \leqslant t \leqslant \frac{1}{4}\pi$, then

$$\int_0^\xi \left| 2\,\frac{(1+x^2)\cos 2t - 2x}{(1-2x\cos 2t+x^2)^2}\right| dx \leqslant \frac{\pi}{2t}.$$

Hence

$$\int_0^\xi |P'(x)|\,dx \leqslant \frac{2}{\pi} A\,\frac{1}{\epsilon}\int_0^\epsilon |\phi(t)|\,dt + \int_\epsilon^{\frac{1}{4}\pi}\left|\frac{\phi(t)}{t}\right| dt + \frac{2}{\pi} D \int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi} |\phi(t)|\,dt$$
$$< B\left[\frac{1}{\epsilon}\int_0^\epsilon |\phi(t)|\,dt + \int_\epsilon^{\frac{1}{4}\pi}\left|\frac{\phi(t)}{t}\right| dt\right] + C$$
$$= B\,\frac{4}{\pi}\,\chi(\tfrac{1}{4}\pi) + B\int_\epsilon^{\frac{1}{4}\pi}\frac{\chi(t)}{t^2}\,dt + C,$$

where D, B, and C are constants and

$$\chi(t) = \int_0^t |\phi(u)|\,du.$$

Now $\chi(t) = o(t)$ in the Lebesgue set, that is almost everywhere. Therefore

$$\int_0^\xi |P'(x)|\,dx = o\left(\log\frac{1}{\epsilon}\right) = o\left(\log\frac{1}{1-\xi}\right)$$

almost everywhere. Hence the required lemma.

If $$Q(x)=\sum_{n=2}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)x^n,$$

then it follows from the lemma that

$$\int_0^{\xi}|Q'(x)|\,dx<K\log\frac{1}{1-\xi}.$$

Also, as $x\to 0$, $$Q'(x)=O(x^2).$$

Hence $$\int_0^{\xi}|Q'(x)|\,dx=O(\xi^3),$$

as $\xi\to 0$; and consequently

$$\int_0^{\xi}|Q'(x)|\,dx<K\left(\log\frac{1}{1-\xi}-\xi-\frac{\xi^2}{2}\right)$$

for $0<\xi<1$,

Now

$$P_\gamma(x)=\sum_{n=3}^{\infty}\frac{(a_n\cos n\theta+b_n\sin n\theta)x^n}{\log n(\log\log n)^\gamma}$$

$$=\frac{1}{\Gamma(\gamma)}\int_0^{\infty}d\beta\,\frac{\beta^{\gamma-1}}{\Gamma(\beta+1)}\int_0^{\infty}d\alpha\,\frac{\alpha^\beta}{\Gamma(\alpha)}\int_0^1 Q(xu)\left(\log\frac{1}{u}\right)^{\alpha-1}\frac{du}{u}.$$

Then the total variation of $P_\gamma(x)$ in $(0,\ \xi)$ is

$$\int_0^{\xi}|P_\gamma'(x)|\,dx=\int_0^{\xi}\left|\frac{1}{\Gamma(\gamma)}\int_0^{\infty}d\beta\,\frac{\beta^{\gamma-1}}{\Gamma(\beta+1)}\int_0^{\infty}d\alpha\,\frac{\alpha^\beta}{\Gamma(\alpha)}\int_0^1 Q'(xu)\left(\log\frac{1}{u}\right)^{\alpha-1}du\right|dx$$

$$\leqslant\int_0^{\xi}dx\,\frac{1}{\Gamma(\gamma)}\int_0^{\infty}d\beta\,\frac{\beta^{\gamma-1}}{\Gamma(\beta+1)}\int_0^{\infty}d\alpha\,\frac{\alpha^\beta}{\Gamma(\alpha)}\int_0^1|Q'(xu)|\left(\log\frac{1}{u}\right)^{\alpha-1}du$$

$$=\frac{1}{\Gamma(\gamma)}\int_0^{\infty}d\beta\,\frac{\beta^{\gamma-1}}{\Gamma(\beta+1)}\int_0^{\infty}d\alpha\,\frac{\alpha^\beta}{\Gamma(\alpha)}\int_0^1 du\,\frac{(\log 1/u)^{\alpha-1}}{u}\int_0^{\xi}|Q'(xu)|\,u\,dx.$$

Now

$$\int_0^1 du\,\frac{(\log 1/u)^{\alpha-1}}{u}\int_0^{\xi}|Q'(xu)|\,u\,dx=\int_0^1 du\left(\log\frac{1}{u}\right)^{\alpha-1}\int_0^{\xi u}|Q'(v)|\,dv$$

$$<K\int_0^1 du\,\frac{(\log 1/u)^{\alpha-1}}{u}\left\{\log\frac{1}{1-\xi u}-\xi u-\frac{\xi^2u^2}{2}\right\}$$

$$=K\int_0^1\sum_3^{\infty}\frac{\xi^n}{n}\,u^{n-1}\left(\log\frac{1}{u}\right)^{\alpha-1}du.$$

Hence we have

$$\int_0^\xi |P_\gamma'(x)|\,dx < \frac{K}{\Gamma(\gamma)}\int_0^\infty d\beta\,\frac{\beta^{\gamma-1}}{\Gamma(\beta+1)}\int_0^\infty da\,\frac{a^\beta}{\Gamma(a)}\sum_3^\infty \frac{\xi^n}{n}\,\frac{\Gamma(a)}{n^a}$$

$$= \frac{K}{\Gamma(\gamma)}\int_0^\infty d\beta\,\frac{\beta^{\gamma-1}}{\Gamma(\beta+1)}\int_0^\infty da\sum_3^\infty \frac{\xi^n}{n}\,e^{-a\log n}\,a^\beta$$

$$= \frac{K}{\Gamma(\gamma)}\int_0^\infty d\beta\sum_3^\infty \frac{\xi^n}{n\log n}\,\frac{\beta^{\gamma-1}}{(\log n)^\beta}$$

$$= \frac{K}{\Gamma(\gamma)}\int_0^\infty d\beta\sum_3^\infty \frac{\xi^n}{n\log n}\,e^{-\beta(\log\log n)}\,\beta^{\gamma-1}$$

$$= \frac{K}{\Gamma(\gamma)}\sum_3^\infty \frac{\xi^n}{n\log n}\,\frac{\Gamma(\gamma)}{(\log\log n)^\gamma}$$

$$< K\sum_3^\infty \frac{1}{n\log n(\log\log n)^\gamma},$$

and this is finite since $\gamma > 1$.

The method is perfectly general for such summable factors. In fact, for $\nu > 1$,

$$\sum_r^\infty \frac{(a_n\cos n\theta + b_n\sin n\theta)\,x^n}{\log n\log\log n\ldots(\log\log\ldots\log_p n)^\nu}$$

$$= \frac{1}{\Gamma(\nu)}\int_0^\infty dx_p\,\frac{x_p^{\nu-1}}{\Gamma(x_p+1)}\int_0^\infty dx_{p-1}\,\frac{x_{p-1}^{x_p}}{\Gamma(x_{p-1}+1)}\int_0^\infty \ldots\int_0^\infty dx_1\,\frac{x_1^{x_2}}{\Gamma(a)}\int_0^1 Q_r(xu)\left(\log\frac{1}{u}\right)^{a-1}\frac{du}{u},$$

and, as shown above, the theorem can be proved. The proof will be simpler if we take the multiplying factor to be

$$(\log n)^{-a} \quad (a > 1).$$

4.2. For the allied series we require the following lemma:

LEMMA II. *If*

$$I(x) = \sum_{n=1}^\infty (b_n\cos n\theta - a_n\sin n\theta)\,x^n$$

and $0 < \xi < 1$, *then*

$$\int_0^\xi |I'(x)|\,dx < K\log\frac{1}{1-\xi}$$

almost everywhere, where K *is independent of* ξ.

With the help of § 3 . 1, the proof can be given on the lines of Lemma I. Reasoning as in § 4 . 1, we get the following theorem for the allied series:

THEOREM 5. *The series*

$$\Sigma \frac{b_n \cos n\theta - a_n \sin n\theta}{\log n \log\log n \ldots (\log\log \ldots \log_p n)^{\beta}},$$

where $\beta > 1$, *is absolutely summable* (A) *for almost every* θ *in* $(-\pi, \pi)$.

5. *The absolute summability* (A) *of restricted Fourier series.*

5 . 1. We now want to extend the results obtained in the preceding articles to the case when

$$\sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \qquad (5.11)$$

is not a Fourier series, but a restricted Fourier series of the second class, the interval of restriction being (α, β). Let us suppose that a_n, b_n are bounded, or in particular tend to zero, as $n \to \infty$, and that the second integrated series (which is evidently a Fourier series), with its sign changed,

$$\sum_{n=1}^{\infty} \left(\frac{b_n}{n^2} \sin n\theta + \frac{a_n}{n^2} \cos n\theta\right), \qquad (5.12)$$

converges to $F(\theta)$, which is, by definition, a second integral in (α, β). Then the function associated with the RF-series is $f(\theta) = d^2 F(\theta)/d\theta^2$; and this is, of course, defined only almost everywhere in the open interval (α, β).

Now, taking

$$\frac{a_n}{n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} F(u) \cos nu \, du \quad \text{and} \quad \frac{b_n}{n^2} = \frac{1}{\pi} \int_{-\pi}^{\pi} F(u) \sin nu \, du,$$

we have

$$\begin{aligned} R(x) &= \sum_{n=1}^{\infty} x^n (a_n \cos n\theta + b_n \sin n\theta) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(u) \sum_{n=1}^{\infty} x^n n^2 \cos n(u-\theta) \, du \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} F(u) \, 2x(1-x^2) \frac{(1+x^2)\cos(u-\theta) - 2x\{1+\sin^2(u-\theta)\}}{\{1-2x\cos(u-\theta)+x^2\}^3} \, du \\ &= \frac{1}{\pi} \int_{0}^{\pi} \{F(\theta+t) + F(\theta-t)\} \, 2x(1-x^2) \frac{(1+x^2)\cos t - 2x(1+\sin^2 t)}{(1-2x\cos t + x^2)^3} \, dt. \end{aligned}$$

Thus we get

$$\tfrac{1}{2}\pi R(x) = \left(\int_0^\delta + \int_\delta^\pi\right) \chi(t)\, x(1-x^2)\, \frac{(1+x^2)\cos t - 2x(1+\sin^2 t)}{(1-2x\cos t+x^2)^3}\, dt$$

$$= R_1(x) + R_2(x), \tag{5.13}$$

where $\chi(t) = F(\theta+t)+F(\theta-t)$, and $0 < \delta < \pi$.

Now $R_2(x)$ is obviously a function of bounded variation in (0, 1). Integrating by parts twice, we get

$$R_1(x) = \left[\chi(t)\, \frac{x(1-x^2)\sin t}{(1-2x\cos t+x^2)^2}\right]_0^\delta + \tfrac{1}{2}\left[\frac{d\chi(t)}{dt}\, \frac{1-x^2}{1-2x\cos t+x^2}\right]_0^\delta$$

$$- \tfrac{1}{2}\int_0^\delta \frac{d^2\chi(t)}{dt^2}\, \frac{1-x^2}{1-2x\cos t+x^2}\, dt$$

$$= \chi(\delta)\, \frac{x(1-x^2)\sin\delta}{(1-2x\cos\delta+x^2)^2} + \tfrac{1}{2}\chi'(\delta)\, \frac{1-x^2}{1-2x\cos\delta+x^2}$$

$$- \tfrac{1}{2}\int_0^\delta \frac{d^2\chi(t)}{dt^2}\, \frac{1-x^2}{1-2x\cos t+x^2}\, dt$$

$$= T_1(x) + T_2(x) - T_3(x). \tag{5.14}$$

Since each of $T_1(x)$ and $T_2(x)$ is a function of bounded variation of x in (0, 1), $R_1(x)$ is of bounded variation, if $T_3(x)$ is so. In my paper already referred to, in obtaining the result (B) of § 2.1, it has been shown that

$$V(x) = \int_0^\delta g(t)\, \frac{1-x^2}{1-2x\cos t+x^2}\, dt$$

is a function of bounded variation of x in (0, 1) if the integral

$$\int_0^\delta \left| \int_0^t g(u)\, du \right| t^{-2}\, dt$$

exists, where $g(t)$ is integrable (L). Hence $T_3(x)$ is of bounded variation if the integral

$$\int_0^\delta \left| \frac{d\chi(t)}{dt} \right| t^{-2}\, dt \tag{5.15}$$

exists.

Now suppose that $\dfrac{d^2\chi(t)}{dt^2}$

is of bounded variation in the interval of restriction. Then, by the working of the proof of Theorem 1, it is clear that $T_3(x)$ is of bounded

variation. *Thus it is shown that the RF-series* (5.11) is *absolutely summable* (A) *in the interval* $(\alpha < \theta < \beta)$, *if either the integral* (5.15) *exists or the second differential coefficient of* $\chi(t)$ *is of bounded variation in* (α, β).

5.2. From (5.13), (5.14), and the working of Lemma I, it is clear that

$$\int_0^{\xi} |R'(x)|\,dx < K \log \frac{1}{1-\xi},$$

where K is independent of ξ. Then a proof similar to that of the Theorem 4 will show that the RF-series (5.11), with its general term multiplied by factor of the type

$$\frac{1}{\log n \log\log n \ldots (\log\log \ldots \log_p n)^{\beta}},$$

where $\beta > 1$, is absolutely summable (A) almost everywhere in (α, β).

6. *Bounded variation of power series.*

6.1. The various theorems obtained in the preceding articles, when interpreted in the language of the complex variable theory, will give conditions under which power series will be of bounded variation in the case when the coefficients are bounded, or, in particular, tend towards zero. We shall assume the radius of the circle of convergence to be unity. We thus get the following theorems.

THEOREM 6. *If a power series* $\sum_1^{\infty} c_n z^n$ *with bounded coefficients is such that its second integrated series, which necessarily converges uniformly on the circle of convergence, has for sum a function of* θ *whose second differential coefficient is a function of bounded variation for all points of an arc, then the power series is a function of bounded variation on every radius vector in that arc.*

If $c_n = 0(1)$, then, by a theorem* due to Young, we know that, under the conditions of our theorem, the power series also converges uniformly throughout the arc, if the second differential coefficient is also continuous.

THEOREM 7. *If a power series* $\sum_1^{\infty} c_n z^n$ *with* c_n *bounded is such that its second integrated series, which necessarily converges uniformly on the circle of convergence, has for sum a function* $F(\theta)$ *whose differential coefficient is*

* See W. H. Young, *Proc. London Math. Soc.* (2), 17 (1918), 364.

absolutely continuous for the points of an arc, then the power series is a function of bounded variation on the radius vector for every θ in that arc for which the integral*

$$\int_0^\delta \frac{|F'(\theta+t)-F'(\theta-t)|}{t^2}\,dt$$

exists.

THEOREM 8. *If a power series* $\sum_1^\infty c_n z^n$ *with bounded coefficients is such that its second integrated series, which necessarily converges uniformly on the circle of convergence, has for sum a function of* θ *whose differential coefficient is absolutely continuous for the points of an arc, then the power series with its general term multiplied by factor of the type*

$$\frac{1}{\log n \log\log n \ldots (\log\log\ldots\log_p n)^\beta},$$

where $\beta > 1$, *is a function of bounded variation on almost every radius vector in that arc.*

6.2. Now suppose that $\sum_1^\infty c_n z^n$, where c_n is bounded, has a unique limiting value $f(\theta)$ at each point θ of an arc (α, β) of the circle of convergence. Then, if

$$f(r, \theta) = \sum_{n=1}^\infty c_n r^n e^{in\theta} \quad (r<1),$$

we get†, for $\alpha < \theta < \beta$,

$$-\sum_{n=1}^\infty n^{-2} c_n e^{in\theta} = \int_\alpha^\theta d\theta \int_\alpha^\theta f(\theta)\,d\theta,$$

whence, by Theorem 6, we get at once

THEOREM 9. *If the sum-function of a power series, with bounded coefficients has a unique limit* $f(\theta)$ *at each point* θ *of an arc* (α, β) *of the circle of convergence, and if* $f(\theta)$ *is a function of bounded variation, then the power series is a function of bounded variation on every radius vector in that arc.*

If $c_n = o(1)$, then, again by a theorem† due to Young, we know that the power series also converges uniformly on that arc.

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* That is, both $F_1(\theta)$ and $F_2(\theta)$ are absolutely continuous. where $F'(\theta) = F_1(\theta) + iF_2(\theta)$.

† See W. H. Young, *Proc. London Math. Soc.* (2), 17 (1918), 365.

7. *Harmonic and analytic functions**.

7.1. It will be observed that some of our theorems, although given for Fourier series and its allied series, may be at once stated in terms of *harmonic* and *analytic* functions and as such may prove of wider applicability. Thus, from the working of Theorem 1, we can state the following theorem:

If a harmonic function is the Poisson integral of an integrable boundary function which is of bounded variation on an arc round $\theta = \alpha$, then it is of bounded variation on the radius to $r = 1$, $\theta = \alpha$.

If we were dealing with analytic functions, we could say:

If an analytic function $f(z)$ belongs to the class L, and if the boundary function $f(e^{i\theta})$ is of bounded variation on an arc round $\theta = \alpha$, then it is of bounded variation on the radius to $r = 1$, $\theta = \alpha$.

Again, Theorems (C) and (B) of § 2.1 and Theorems 2 and 3 can be similarly stated, giving relations between conjugate harmonic functions.

* Written at the kind suggestion of a Referee.

SOME GENERATING FUNCTIONS

By ALFRED YOUNG.

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The primary object of the following is to obtain the generating function in two variables whose coefficients are the number of concomitants of degree δ, of the weights indicated by the indices of the variables, of a single ternary n-ic. The method used is an application of quantitative substitutional analysis as developed in my third paper on that subject*.

It is proved that the generating function may be obtained from that for gradients by multiplication by factors of the form

$$(1-y)(1-z)\left(1-\frac{z}{y}\right).$$

The second part of the paper deals with generating functions of covariants of a system of binary forms of the same order; the generating function obtained is for covariants of the type

$$T_{a_1 a_2 \dots a_h} C,$$

where the permutations interchange quantics as such. As an immediate application the generating function for combinants is given.

The third part of the paper extends the result for ternary forms to forms with any number of variables, and gives the generating function for seminvariants in terms of that for gradients.

I. *Ternary forms.*

1. Consider a gradient X of a ternary quantic

$$\Sigma \binom{n}{r,\ s} A_{r,s} x_1^{n-r-s} x_2^r x_3^s.$$

* *Proc. London Math. Soc.* (2), 28 (1928), 255–292.

We shall describe the coefficient $A_{r,s}$ as having three weights $n-r-s$, r, s, the first, second, and third. Thus the gradient has three weights, and the sum of the three weights is $n\delta$, where δ is the degree of X. Gradients in which the weights are in descending order of magnitude will be considered. In the symbolical notation the quantic is written a_x^n, where

$$a_x = ax_1 + a'x_2 + a''x_3,$$

using dashes for convenience as the suffixes are used to distinguish different symbolical letters. The coefficient $A_{r,s}$ is then $a^{n-r-s}a'^r a''^s$. The gradient

$$X = \Pi A_{r,s}^{\lambda_{r,s}} = \Pi a_t^{n-r_t-s_t} a_t'^{r_t} a_t''^{s_t}.$$

In order to apply the substitutional analysis we polarize this to make it linear in every symbolical letter, and so replace a_t by the n letters $a_{t1}a_{t2}\dots a_{tn}$, introducing a substitutional operator

$$\frac{1}{n!}\{a_{t1}a_{t2}\dots a_{tn}\},$$

because the letters are all equivalent.

Thus

$$X = \Gamma\Pi a_{t1}a_{t2}\dots a_{tn-r_t-s_t}a'_{tn-r_t-s_t+1}\dots a'_{tn-s_2}a''_{tn-s_2+1}\dots a''_{t,n};$$

where Γ is the product of the operators mentioned, it may be replaced by a Σ. The effect of a permutation $(\beta\gamma)$ on a product $\beta\gamma$, $\beta'\gamma'$, or $\beta''\gamma''$ is to make no change, but it interchanges the pairs of products $\beta\gamma'$, $\beta'\gamma$; $\beta\gamma''$, $\beta''\gamma$; $\beta'\gamma''$, $\beta''\gamma'$.

Thus the product above is unaltered by any permutation of the symmetric group of the w_1 undashed letters, or that of the w_2 letters with a single dash, or that of the w_3 letters with a double dash. That is, it is of the form

$$\frac{1}{w_1!\ w_2!\ w_3!}\, G_{w_1} G_{w_2} G_{w_3} Y = Y,$$

where G_r is a positive symmetric group of degree r. Then

$$T_{\alpha_1\alpha_2\dots\alpha_h} Y = 0,$$

when $h > 3$, or when $\alpha_1 < w_1$, or $\alpha_1+\alpha_2 < w_1+w_2$. Indeed, the latest TY which is not zero is $T_{w_1w_2w_3}Y$. The product $G_{w_1}G_{w_2}G_{w_3}$ is the product of the positive symmetric groups of the letters in each row of a tableau for this T. We may ensure that this tableau is standard by taking the

letters in such a sequence that (i) no two letters of any set are separated in the sequence by a letter of another set, and (ii) those sets come first whose earliest weight is greatest. Let F_ρ be this tableau, so that

$$G_{w_1} G_{w_2} G_{w_3} = P_\rho.$$

Then
$$T_{w_1 w_2 w_3} = \sum_\sigma P_\sigma N_\sigma M_\sigma$$

and*
$$T_{w_1 w_2 w_3} Y = P_\rho N_\rho M_\rho P_\rho \frac{1}{w_1!\ w_2!\ w_3!} Y$$

$$= \frac{1}{f} \binom{n}{w_1 w_2 w_3} P_\rho N_\rho Y.$$

Then $N_\rho Y$ is the product of w_3 symbolical factors of the form (abc), of $w_2 - w_3$ factors of the form $(ab' - a'b) = (ab)_3$, and of $w_1 - w_2$ undashed letters, *i.e.* it is the coefficient of $u_3^{w_2 - w_3} x_1^{w_1 - w_2}$ in a concomitant

$$\Pi(abc)\, \Pi(deu)\, \Pi f_x.$$

This will be called the first term or seminvariant†.

The concomitant is completely defined by F_ρ, which in its turn is defined by the gradient X, and may be written

$$(X).$$

The argument applies equally to forms with any number of variables.

THEOREM I. *A gradient X, whose weights w_1, w_2, w_3 are in descending order of magnitude, when expressed in the form ΣTX, has for its term corresponding to $T_{w_1 w_2 w_3}$ the seminvariant*

$$(X)$$

and has no term corresponding to later T's.

2. When $T_{a_1 a_2 a_3}$ precedes $T_{w_1 w_2 w_3}$,

$$T_{a_1 a_2 a_3} X$$

is not zero, but it is a sum of terms which are the coefficients of concomitants of weights a_1, a_2, a_3. It will be more illuminating to replace

* Q.S.A., III, 264.

† It is important for our purpose to define this term thus, and not as the coefficient of $u_1^{w_2 - w_3} x_1^{(\sigma_1 - w_2}$.

the variable x by the matrix form (vu), where v and u are variables contragredient to x and cogredient with each other. Then the concomitants belonging to $T_{\alpha\beta\gamma}$ are of order $\alpha-\beta$ in (vu) and of order $\beta-\gamma$ in u. The seminvariants are the coefficients of

$$v_2^{\alpha-\beta}\, u_3^{\alpha-\gamma}.$$

In effect the second rows of the tableaux are filled up with $\alpha-\beta$ letters v, and the third rows with $\alpha-\gamma$ letters u, making them all of equal length.

We need the *independent* terms of a concomitant. In binary forms of order ρ with a seminvariant of weight w there is just one term of each weight from w to $w+\rho$.

In ternary forms, however, the coefficients of all the variable products

$$x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} u_1^{\mu_1} u_2^{\mu_2} u_3^{\mu_3},$$

$$\lambda_1+\lambda_2+\lambda_3 = \alpha-\beta, \quad \mu_1+\mu_2+\mu_3 = \beta-\gamma,$$

have to be considered. The weights of the coefficient of this term are

$$\text{(I)} \qquad \begin{cases} w_1 = \alpha-\lambda_2-\lambda_3-\mu_1 = \beta+\lambda_1-\mu_1, \\ w_2 = \alpha-\lambda_3-\lambda_1-\mu_2 = \beta+\lambda_2-\mu_2, \\ w_3 = \alpha-\lambda_1-\lambda_2-\mu_3 = \beta+\lambda_3-\mu_3, \end{cases}$$

and these are not all different for the different coefficients.

But, in virtue of the equation

$$u_1 x_1+u_2 x_2+u_3 x_3 = 0,$$

there is a relation between the variable products; and there is an exactly correlative relation between the coefficients of these forms. We therefore consider no terms in which the product $u_2 x_2$ occurs; the rest will be independent with their coefficients. This is equivalent, but differently expressed, to the statement of the fact made by Grace*.

Thus to find the number of linearly independent coefficients of weights w_1, w_2, w_3 of a concomitant of type $T_{\alpha\beta\gamma}$ we consider the coefficients of those variable products for which either λ_2 or μ_2 is zero; *i.e.* either

$$\lambda_2 = 0, \quad \mu_2 = \beta-w_2,$$

or

$$\mu_2 = 0, \quad \lambda_2 = w_2-\beta.$$

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* *Journal London Math. Soc.*, 5 (1930), 65.

3. Consider the number of gradients X of degree δ and weights w_1, w_2, w_3 which are in descending order of magnitude. Each one is expressible as a linear function of the corresponding seminvariant (X) of its own proper weights, and of terms of these weights of concomitants represented by earlier T's. Let n_{w_1, w_2, w_3} be the number of gradients X, $N_{\alpha, \beta, \gamma}$ the number of linearly independent seminvariants of weights α, β, γ; and $\begin{pmatrix} w_1 & w_2 & w_3 \\ \alpha & \beta & \gamma \end{pmatrix}$ the number of independent coefficients of weights w_1, w_2, w_3 of a concomitant of the $T_{\alpha, \beta, \gamma}$ set. Then

$$n_{w_1, w_2, w_3} = \Sigma \begin{pmatrix} w_1 & w_2 & w_3 \\ \alpha & \beta & \gamma \end{pmatrix} N_{\alpha, \beta, \gamma}.$$

And thus we proceed to obtain a relation expressing N_{w_1, w_2, w_3} in terms of $n_{\alpha, \beta, \gamma}$. When $w_3 = 0$, γ must also be zero, and we have merely the binary case

$$n_{w_1, w_2, 0} = \sum_{\rho=0}^{w_2} N_{w_1+\rho, w_2-\rho, 0}.$$

When $w_3 = 1$, consider the number of terms supplied by a concomitant

$$T_{w_1+\rho, w_2-\rho+1, 0}.$$

Unless $\rho = 0$, $w_2 - \beta$ is positive or zero and thus

$$\mu_2 = 0, \quad \lambda_2 = \rho - 1.$$

The last equation of (I) gives

$$1 = \lambda_3 + \mu_1 + \mu_2,$$

whence $\lambda_3 = 0, \ \mu_1 = 1$ or $\lambda_3 = 1, \ \mu_1 = 0.$

These give two independent terms in all cases $0 < \rho < w_2 + 1$. For $\rho = 0$, we have $\lambda_2 = 0, \ \mu_2 = 1, \ \lambda_3 = 0, \ \mu_1 = 0$, and hence only one term; in the same way there is one term only for

$$T_{w_1+w_2+1, 0, 0}.$$

It is necessary to consider also forms

$$T_{w_1+\rho, w_2-\rho, 1}.$$

This may be most easily considered by noting that there is a symbolical factor (abc) of weights (1, 1, 1) multiplying all the terms; the removal of this (that is the first column from the tableau) leaves us to reckon the

number of independent terms of weights $(w_1-1, w_2-1, 0)$ of a form $T_{w_1-1+\rho, w_2-1-\rho, 0}$, which is of course one. Thus, collecting results and inserting the coefficients 1 or 2 according to the above several cases, we have

$$n_{w_1, w_2, 1} = N_{w_1, w_2, 1} + N_{w_1+1, w_2-1, 1} + \cdots + N_{w_1+w_2-1, 1, 1}$$
$$+ N_{w_1, w_2+1, 0} + 2N_{w_1+1, w_2, 0} + \cdots + 2N_{w_1+w_2, 1, 0} + N_{w_1+w_2+1, 0, 0}.$$

Hence

$$n_{w_1, w_2, 1} - n_{w_1+1, w_2-1, 1} = N_{w_1, w_2, 1} + N_{w_1, w_2+1, 0} + N_{w_1+1, w_2, 0}.$$

In the same way we find

$$n_{\alpha, \beta, 2} = N_{\alpha, \beta, 2} + N_{\alpha+1, \beta-1, 2} + \cdots + N_{\alpha+\beta-2, 2, 2} + N_{\alpha, \beta+1, 1}$$
$$+ 2N_{\alpha+1, \beta, 1} + 2N_{\alpha+2, \beta-1, 1} + \cdots + 2N_{\alpha+\beta-1, 2, 1} + N_{\alpha+\beta, 1, 1}$$
$$+ N_{\alpha, \beta+2, 0} + 2N_{\alpha+1, \beta+1, 0} + 3N_{\alpha+2, \beta, 0} + \cdots + 3N_{\alpha+\beta, 2, 0}$$
$$+ 2N_{\alpha+\beta+1, 1, 0} + N_{\alpha+\beta+2, 0, 0}.$$

Also

$$n_{\alpha, \beta, \gamma} = N_{\alpha, \beta, \gamma} + N_{\alpha+1, \beta-1, \gamma} + \cdots + N_{\alpha+\beta-\gamma, \gamma, \gamma}$$
$$+ N_{\alpha, \beta+1, \gamma-1} + 2N_{\alpha+1, \beta, \gamma-1} + \cdots + \cdots$$
$$+ N_{\alpha, \beta+r, \gamma-r} + 2N_{\alpha+1, \beta+r-1, \gamma-r} + \cdots + (r+1)N_{\alpha+r, \beta, \gamma-r} + \cdots$$
$$+ (r+1)N_{\alpha+\beta-\gamma+r, \gamma, \gamma-r} + rN_{\alpha+\beta-\gamma+r+1, \gamma-1, \gamma-r} + \cdots$$
$$+ N_{\alpha+\beta-\gamma+2r, \gamma-r, \gamma-r} + \cdots + \cdots.$$

Then

$$n_{\alpha, \beta, \gamma} - n_{\alpha+1, \beta-1, \gamma} = N_{\alpha, \beta, \gamma} + N_{\alpha, \beta+1, \gamma-1} + N_{\alpha+1, \beta, \gamma-1}$$
$$+ N_{\alpha, \beta+2, \gamma-2} + N_{\alpha+1, \beta+1, \gamma-2} + N_{\alpha+2, \beta, \gamma-2} + \cdots$$
$$+ N_{\alpha, \beta+r, \gamma-r} + N_{\alpha+1, \beta+r-1, \gamma-r} + \cdots + N_{\alpha+r, \beta, \gamma-r} + \cdots.$$

Hence

$$(n_{\alpha, \beta, \gamma} - n_{\alpha+1, \beta-1, \gamma}) - (n_{\alpha, \beta+1, \gamma-1} - n_{\alpha+1, \beta, \gamma-1})$$
$$- (n_{\alpha+1, \beta, \gamma-1} - n_{\alpha+2, \beta-1, \gamma-1}) + (n_{\alpha+1, \beta+1, \gamma-2} - n_{\alpha+2, \beta, \gamma-2}) = N_{\alpha, \beta, \gamma}.$$

Now let $$\Gamma_\delta = \Sigma\, n_{\alpha, \beta, \gamma} y^\beta z^\gamma,$$

where δ is the degree of the seminvariants of weights α, β, γ; that is Γ_δ is the generating function for gradients of degree δ.

Then $N_{\alpha,\beta,\gamma}$ is the coefficient of $y^\beta z^\gamma$ in

$$(1-y)(1-z)\left(1-\frac{z}{y}\right)\Gamma_\delta.$$

THEOREM II. *The generating function for seminvariants of a ternary quantic is obtained from that for gradients of the same degree by multiplication by*

$$(1-y)(1-z)\left(1-\frac{z}{y}\right),$$

where the indices of y and z indicate the second and third weights.

4. To find the generating function for gradients of degree δ, we use the fact that the sum of all homogeneous powers and products of powers of $z_1, z_2, \ldots, z_m$, the roots of

$$f(z) = z^m + p_1 z^{m-1} + \ldots = 0$$

of degree δ, is*

$$\sum_{r=1}^{m} \frac{z_r^{\delta+m-1}}{f'(z_r)} = \frac{1}{\Delta}\begin{vmatrix} z_1^{\delta+m-1} & z_2^{\delta+m-1} & \ldots & z_m^{\delta+m-1} \\ z_1^{m-2} & z_2^{m-2} & \ldots & z_m^{m-2} \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \ldots & 1 \end{vmatrix},$$

where

$$\Delta = \prod_{r<s} (z_r - z_s).$$

To obtain the generating function, instead of replacing the roots of $f(z)$ by the coefficients $A_{r,s}$ we replace them by the variable products $y^r z^s$. Then $z^{\delta+m-1}/f'(z)$ becomes

$$\frac{y^{r\delta} z^{s\delta}}{\prod_{\rho+\sigma=0}^{n}{}' (1 - x^{\rho-r} y^{\sigma-s})},$$

where the product symbol has a dash affixed to indicate that one factor is omitted, viz. that for which $\rho = r$, $\sigma = s$.

THEOREM III. *The generating function for seminvariants of degree δ of the ternary n-ic is*

$$(1-y)(1-z)\left(1-\frac{z}{y}\right)\sum_{r+s=0}^{n} \frac{y^{r\delta} z^{s\delta}}{\prod_{\rho+\sigma=0}^{n}{}' (1 - y^{\rho-r} z^{\sigma-s})}.$$

---

* A Referee has pointed out that this result is due to Jacobi; see his *Gesammelte Werke*, 3 (1884), 7.

It is evident that this form of generating function for gradients is perfectly general. Thus for quaternary forms we have

$$\sum_{r+s+t=0}^{n} \frac{y^{r\delta} z^{s\delta} u^{t\delta}}{\prod_{\rho+\sigma+\tau=0}^{n}{}' (1-y^{\rho-r} z^{\sigma-s} u^{\tau-t})}.$$

The result may be stated in the form

$$N_{w_1, w_2, w_3} = n_{w_1, w_2, w_3} + n_{w_1+2, w_2-1, w_3-1} + n_{w_1+1, w_2+1, w_3-2} - n_{w_1+1, w_2-1, w_3} - n_{w_1, w_2+1, w_3-1} - n_{w_1+2, w_2, w_3-2}.$$

The interchange of a pair of suffixes in the coefficients of a gradient merely interchanges the corresponding weights of the gradient, and hence

$$n_{w_1, w_2, w_3} = n_{w_1, w_3, w_2} = n_{w_2, w_1, w_3} = \ldots.$$

Thus, when the generating function G_δ for seminvariants is fully expanded, it will include terms for which the weights are not in descending order. But for this part of the expansion there is a relation to the first part of the same nature as in the case of generating functions for binary forms; but of an appropriately extended kind. In fact, from the above relations we obtain at once

$$\begin{aligned} N_{w_1, w_2, w_3} &= -N_{w_2-1, w_1+1, w_3} = -N_{w_1, w_3-1, w_2+1} \\ &= -N_{w_3-2, w_2, w_1+2} = N_{w_2-1, w_3-1, w_1+2} = N_{w_3-2, w_1+1, w_2+1}. \end{aligned}$$

And thus

$$G_\delta = \Sigma N_{w_1, w_2, w_3} [x_1^{w_1} x_2^{w_2} x_3^{w_3} + x_1^{w_2-1} x_2^{w_3-1} x_3^{w_1+2} + x_1^{w_3-2} x_2^{w_1+1} x_3^{w_2+1} - x_1^{w_1} x_2^{w_3-1} x_3^{w_2+1} - x_1^{w_3-2} x_2^{w_2} x_3^{w_1+2} - x_1^{w_2-1} x_2^{w_1+1} x_3^{w_3}],$$

where now the expansion is restricted to

$$w_1 \geqslant w_2 \geqslant w_3.$$

II. *Systems of binary forms.*

5. Consider a binary gradient type X of degree δ; that is one that includes just one coefficient from each of δ forms of order n. Then X may be obtained by polarization from a gradient of a single binary form

$$A_{r_1}^{\lambda_1} A_{r_2}^{\lambda_2} \ldots A_{r_k}^{\lambda_k} = X_1;$$

where X includes λ_1 coefficients with suffix r_1.

Permutations will be considered of the quantics themselves; then X will be unchanged by any permutation of the λ_1 quantics whose coefficient has a suffix r_1. Thus X may be regarded substitutionally as a function affected by an operator Γ which is the product of positive symmetric groups of degrees $\lambda_1, \lambda_2, \ldots, \lambda_k$, which are supposed to be in descending order of magnitude.

Operate with the series

$$1 = \Sigma T$$

on X. Then $T_{a_1 a_2 \ldots a_h} X$ is zero, unless $\alpha_1 \geqslant \lambda_1$, $\alpha_1 + \alpha_2 \geqslant \lambda_1 + \lambda_2$, and so on. Consider

$$\Gamma T_{a_1 a_2 \ldots a_h} = \Gamma \sum_{r=1}^{f} P_r N_r M_r,$$

where F_r is a standard tableau. The definition of "standard" presupposes a fixed sequence of the δ letters subject to permutation. We choose this so that the λ_1 letters of the first group of Γ come first, the λ_2 letters of the second group next, and so on. Then, when a pair of letters of the same group of Γ appears in the same column of F_r, we can prove that

$$\Gamma P_r N_r M_r = \Sigma \mu_s \Gamma_s P_s \sigma N_r M_r,$$

where F_s precedes F_r, σ is a permutation which transforms F_r into F_s, and μ_s is numerical. We may suppose that $b_1 b_2 \ldots b_\lambda$ are the letters of the group in question and that part of the F_r tableau is

$$\begin{array}{l} \ldots \quad \ldots \quad \ldots \quad \ldots b_1 b_2 \ldots b_p d_1 d_2 \ldots d_h \\ c_1 c_2 \ldots c_k b_{p+1} \ldots b_q \ldots, \end{array}$$

where the letters c precede the b's, which in turn precede the d's. Then operation with the positive symmetric group

$$\{c_1 c_2 \ldots c_k b_{p+1} \ldots b_q b_1 b_2 \ldots b_p d_1 d_2 \ldots d_h\}$$

produces zero, for it is of greater degree than the length of the upper line in the tableau. On expansion we first have terms in which the letters b only are permuted; these, owing to the presence of Γ to the left of the sum, are equal to a multiple of $\Gamma P_r N_r M_r$, and all the other terms are of the form $\Gamma P_s \sigma N_r M_r$. Thus when the letters of any one group of Γ *overlap* in the rows of F_r, we can express $\Gamma P_r N_r M_r X$ in terms of earlier forms. The function X is only affected by permutations operating on the left, the permutations of PNX are all effected by a multiplication by permutations σ,

$$\sigma PNX.$$

Thus the possible forms are completely defined by the tableau of P. There are thus just the same number of linearly independent forms TX as there are standard tableaux in which the letters of the various groups of Γ do not overlap.

6. It is proposed to find how many gradients of degree δ and weight w there are which are linear in the coefficients of each of δ binary quantics of order r, and are of the substitutional type $T_{\alpha_1 \alpha_2 \dots \alpha_h} X$. The permutations considered are of the quantics themselves.

The gradient X will be said to be of the type $(\lambda_0, \lambda_1, \dots, \lambda_r)$ when it contains λ_r coefficients with the suffix r for $r = 0, 1, \dots, n$. Then it is to be noticed that the fact that there must be no overlapping in the tableaux of the independent forms does not depend in any way on the sequence of magnitudes of the λ groups. Thus we shall take the letters of the group λ_0 first in the sequence of letters irrespective of the magnitude of λ_0. Moreover, the respective values of $\lambda_0, \lambda_1, \dots, \lambda_n$ are not fixed.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_h;\ n\}$ be the number of independent gradients $T_{\alpha_1 \alpha_2 \dots \alpha_h} X$, that is the number of standard tableaux F_r where the letters are divided into $n+1$ groups, of any respective magnitudes $\lambda_0, \lambda_1, \dots, \lambda_n$; the letters λ_0 come first and so on, and the letters of no group overlap. Then consider the effect of removing the letters of the last group from F_r. Since these are the last λ_n letters the result is a standard tableau F_r' of the same character but of n groups. Hence

$$\{\alpha_1, \alpha_2, \dots, \alpha_h;\ n\} = \Sigma \{\beta_1, \beta_2, \dots, \beta_h;\ n-1\},$$

where the sum on the right includes all values of $\beta_1, \beta_2, \dots, \beta_h$ such that

$$\beta_r \leqslant \alpha_r, \quad \beta_r \geqslant \alpha_{r+1},$$

for, if $\beta_r < \alpha_{r+1}$, it would mean that, on filling up the β tableau with letters of the last group, these letters would overlap.

In the same way,

$$\{\alpha_1, \alpha_2, \dots, \alpha_h - 1;\ n\} = \Sigma \{\beta_1', \beta_2', \dots, \beta_h';\ n-1\}.$$

The two series are the same except that in the first there are terms with $\beta_h = \alpha_h$ not included in the second, and in the second there are terms with $\beta_{h-1} = \alpha_h - 1$ not included in the first.

Let δ_r be an operation which diminishes the value of α_r by unity.

Then

$$\text{(II)} \qquad \left[\prod_{r=1}^{h} (1-\delta_r)\right] \{\alpha_1, \alpha_2, \dots, \alpha_h;\ n\} = \{\alpha_1, \alpha_2, \dots, \alpha_h;\ n-1\}.$$

For the term on the right appears in the sum obtained for the first term only on the left, and every other term in these sums occurs an even number of times half with a positive and half with a negative sign.

7. THEOREM IV. *The generating function for covariants of type*

$$T_{\alpha_1 \alpha_2 \dots \alpha_h},$$

of degree $\delta = \Sigma \alpha$, *which are linear in the coefficients of each of* δ *quantics of order* n, *is*

$$(1-z)\left|\phi\binom{\alpha_r+n-r+s}{n}\right|,$$

where

$$\phi\binom{\xi}{\eta} = \frac{(1-z^{\xi-\eta+1})(1-z^{\xi-\eta+2})\dots(1-z^{\xi})}{(1-z)(1-z^2)\dots(1-z^{\eta})}.$$

As usual for binary forms the generating function for covariants is obtained from that for gradients by multiplication by $1-z$; it is then only necessary to consider gradients. The well known property

$$\phi\binom{\xi+1}{\eta} - z^{\eta}\phi\binom{\xi}{\eta} = \phi\binom{\xi}{\eta-1} \tag{III}$$

is needed.

Let $\psi(\alpha_1, \alpha_2, \dots, \alpha_h;\ n)$ be the generating function for gradients of type $T_{\alpha_1, \alpha_2, \dots, \alpha_h}$. Then, as in the last paragraph,

$$\psi(\alpha_1, \alpha_2, \dots, \alpha_h;\ n) = \Sigma z^{n\Sigma(\alpha-\beta)}\psi(\beta_1, \beta_2, \dots, \beta_h;\ n-1).$$

Hence, as before in (II),

$$\psi(\alpha_1, \alpha_2, \dots, \alpha_h;\ n-1) = \left[\prod_{r=1}^{h}(1-z^n\delta_r)\right]\psi(\alpha_1, \alpha_2, \dots, \alpha_h;\ n).$$

In virtue of the property (III),

$$\Delta = \left|\phi\binom{\alpha_r+n-r+s}{n}\right|$$

(a determinant whose rows are given by $r = 1, 2, \dots, h$ and columns by $s = 1, 2, \dots, h$ satisfies this difference equation, for the operator $1-z^n\delta_r$ operates on the elements of the r-th row only of the determinant above. Hence, if the theorem is true for all earlier values of the arguments $\alpha_1, \alpha_2, \dots, \alpha_h, n$, it is true for these.

(i) When $\alpha_r = \alpha_{r+1}-1$ the determinant is zero since two rows become identical; this corresponds to the fact that such a T does not exist and therefore the generating function should be zero.

(ii) When $a_h = 0$ the last row of the determinant is 0, 0, ..., 0, 1, and we reduce it to the corresponding determinant of $h-1$ rows and columns.

(iii) When $a_1 = \delta$, $a_2 = 0$, ..., $a_h = 0$, we obtain the ordinary generating function for gradients of a single quantic; this is the correct result for T_δ.

Thus the truth of the theorem follows by induction.

8. The determinant Δ can be factorized thus. In the first place, the denominators of the elements give a factor D^{-h}, where

$$D = (1-z)(1-z^2)\dots(1-z^n).$$

The common factors of the r-th row,

$$(1-z^{a_r-r+h+1})(1-z^{a_r-r+h+2})\dots(1-z^{a_r-r+n+1}),$$

may then be removed. We are left with the determinant

$$\Delta' = |(1-z^{a_r-r+s+1})\dots(1-z^{a_r-r+h})(1-z^{a_r-r+n+2})\dots(1-z^{a_r+n-r+s})|.$$

On subtracting the s-th column from the $(s+1)$-th, the r-th element of the $(s+1)$-th column becomes

$$(1-z^{a_r-r+s+2})\dots(1-z^{a_r-r+h})(1-z^{a_r-r+n+2})\dots(1-z^{a_r+n-r+s})(1-z^n)z^{a_r-r+s+1}.$$

Then in Δ' we subtract the last column but one from the last, then the $(h-2)$-th from the $(h-1)$-th, and so on, finally the first from the second. The result is

$$\Delta' = (1-z^n)^{h-1} z^{\binom{h+2}{2}-1} \Delta_1',$$

where the first column of Δ_1' is the same as that of Δ', but the s-th column $s > 1$ has for its r-th element

$$z^{a_r-r}(1-z^{a_r-r+s+1})\dots(1-z^{a_r-r+h})(1-z^{a_r-r+n+2})\dots(1-z^{a_r-r+n+s-1}).$$

Next we subtract the $(h-1)$-th column of Δ_1' from the h-th, and so on, and finally the second from the third, obtaining

$$\Delta_1' = (1-z^{n-1})^{h-2} z^{\binom{h+2}{2}-3} \Delta_2'.$$

Proceeding thus, we finally obtain

$$\Delta' = (1-z^n)^{h-1}(1-z^{n-1})^{h-2}\dots(1-z^{n-h+2})\Delta'',$$

where $$\Delta'' = |z^{(s-1)(a_r-r+s)}(1-z^{a_r-r+s+1})\dots(1-z^{a_r-r+h})|.$$

Now multiply the $(s-1)$-th column of Δ'' by z^{s-2} and add to the s-th; the

r-th element of the s-th column becomes

$$z^{(s-2)(a_r-r+s)}(1-z^{a_r-r+s+1})\dots(1-z^{a_r-r+h}).$$

Next we make this transformation for the last column, then for the last but one, and so on. The result is a determinant Δ_1'', whose first column is that of Δ'', and for $s>1$ the element is that just given. Then

$$\Delta''=\Delta_1''.$$

Next, in Δ_1'' we multiply the $(s-1)$-th column by z^{s-3} and add to the s-th in regular sequence from the end, as far as $s=3$; and thus obtain

$$\Delta_1''=\Delta_2''.$$

Proceeding thus, we eventually obtain

$$\Delta''=\Delta'''=|(1-z^{a_r-r+s+1})\dots(1-z^{a_r-r+h})|,$$

the last column being made up of units. Now, by means of row subtractions, we find that

$$\Delta'''=z^n\Big[\prod_{r<t}(1-z^{a_r-a_t+t-r})\Big]\bar{\Delta}.$$

The first row of $\bar{\Delta}$ is

$$z^{2+3+\dots+h}\quad 0\quad 0\quad \dots\quad 0.$$

In the r-th row the last $h-r$ elements are zero and the r-th element is

$$z^{(r+1)+\dots+h},$$

the last element of the last row is unity.

Thus $$\bar{\Delta}=z^{\frac{1}{8}(h+1)h(h-1)}.$$

Hence, finally,

$$\text{(IV)}\quad \left|\phi\binom{a_r+n-r+s}{n}\right|=z^{\varpi}D^{-h}(1-z^n)^{h-1}(1-z^{n-1})^{h-2}\dots(1-z^{n-h+2})$$
$$\times\prod_{r=1}^{h}(1-z^{a_r-r+h+1})\dots(1-z^{a_r-r+n+1})\prod_{r<t}(1-z^{a_r-a_t+t-r}),$$

where $$\varpi=(h-1)a_h+(h-2)a_{h-1}+\dots+a_2.$$

By writing $a_1=a_2=\dots=a_h=a$, we obtain the case of combinants of what may be called the h-th class.

THEOREM V. *The generating function for combinants of the h-th class and degree ah of the binary n-ic is*

$$z^{a\binom{h}{2}}(1-z)\,D^{-h}(1-z^n)^{h-1}(1-z^{n-1})^{h-2}\dots(1-z^{n-h+2})$$
$$\times(1-z)^{h-1}(1-z^2)^{h-2}\dots(1-z^{h-1})$$
$$\times(1-z^{a+1})(1-z^{a+2})^2\dots(1-z^{a+h})^h(1-z^{a+h+1})^h$$
$$\dots(1-z^{a+n-h+1})^h(1-z^{a+n-h+2})^{h-1}\dots(1-z^{a+n}),$$

when $n \geqslant 2h-1$.

9. Let $z^{a\binom{h}{2}}\psi(a, h)$ be the generating function for the combinants of class h and degree ah. Then we shall prove that

$$\psi(a, h) = \psi(a, n+1-h).$$

Let $$n+1-h > h.$$

There are three sets of factors in the numerator of ψ; two of these do not contain a in the index of z, and we consider them first. Let these factors in $\psi(a, h)$ be P_h and let the remaining factors which have a in the index be Q_h. Then

$$P_h = (1-z^n)^{h-1}(1-z^{n-1})^{h-2}\dots(1-z^{n-h+2})\,(1-z)^{h-1}(1-z^2)^{h-2}\dots(1-z^{h-1}),$$

$$P_{n+1-h} = D^{n+1-2h}\,P_h,$$

$$P_h D^{-h} = P_{n+1-h}D^{-(n+1-h)}.$$

The factors Q_h may be written in a rectangle, the first row being those for $r=1$, the next those for $r=2$, and so on; the rectangle for Q_{n+1-h} is obtained from that for Q_h by changing columns into rows; and hence

$$Q_{n+1-h} = Q_h.$$

THEOREM VI. *The number of combinants of class h, degree ah, and weight* $w+a\binom{h}{2}$ *is the same as the number of combinants of class* $n+1-h$, *degree* $a(n+1-h)$, *and weight*

$$w+a\binom{n+1-h}{2}.$$

10. The gradient generating function (IV) should be unchanged when h is changed to $h+1$, and $a_{h+1}=0$. This can easily be verified to be

the case. In fact, ϖ is unaltered, in the first row there is an additional factor in the denominator

$$(1-z)(1-z^2)\dots(1-z^{n-h}),$$

but this appears in the numerator of the second row for $r=h+1$. The second row loses the factors of the first Π

$$\prod_{r=1}^{h}(1-z^{a_r-r+h+1}),$$

and these appear in the last Π for $t=h+1$. Thus we may take in general $h=n+1$, where it is understood that some of the a's may be zero; all the second set of factors then disappear, and the generating function takes a simpler form

$$\text{(V)} \qquad (1-z)z^{\varpi}\prod_{r=1}^{n}(1-z^r)^{r-n-1}\prod_{r<t}(1-z^{a_r-a_t+t-r}).$$

Let us now consider the generating function for the gradients

$$T_{\beta_1,\beta_2,\dots,\beta_{n+1}}X,$$

where $\beta_r=\lambda-a_{n+2-r}$, where λ is not less than a_1. It is exactly the same as (V), except for the initial factor $z^{\varpi'}$, where

$$\varpi'=\sum_{r=1}^{n+1}(r-1)\beta_r=\varpi+\lambda\binom{n+1}{2}-n\delta.$$

These two sets T_a, T_β may be called conjugate.

THEOREM VII. *The number of covariant types $T_{a_1\dots a_{n+1}}$, of degree δ and order ρ, is the same as the number of covariant types of the conjugate set of degree $(n+1)\lambda-\delta$ and of order ρ.*

11. A similar process may be applied to obtain the generating function of gradients or seminvariants of the nature

$$T_{a_1a_2\dots a_h}X$$

for forms with more than two variables. To do this we consider sets of N things, which may be represented by algebraic symbols, and such that in each set the things may be identified algebraically by what may be called their weights. These weights may be numerically additive or not. We shall first suppose that they are entirely distinct and represented by different symbols $z_1, z_2, \dots, z_N$. We consider first a homogeneous product $\bar{X}$ of symbols representing the things themselves; $\bar{X}$ being such that

there is just one thing from each set. If the symbols representing the things be now replaced by the symbols representing their weight, the index of z_r is the number of things selected in $\bar{X}$ of this particular weight; let $\bar{X}$ be called X after this change.

Let now $$\phi\binom{\delta}{N}$$

be the sum of all the products of powers of $z_1, z_2, \ldots, z_N$ of total degree δ. This is, in effect, the generating function for $\bar{X}$ when all the sets are alike. Then

(V) $$\phi\binom{\delta+1}{N} - z_N \phi\binom{\delta}{N} = \phi\binom{\delta+1}{N-1}$$

is an obvious identity, of which indeed (III), § 7, is a particular case.

Let us now consider the expressions

$$T_{\alpha_1 \alpha_2 \ldots \alpha_h} \bar{X},$$

where the permutations interchange the different sets. Let the generating function (a function of $z_1, z_2, \ldots, z_N$) of this be

$$\psi(\alpha_1, \alpha_2, \ldots, \alpha_h;\ N).$$

Then, just as in §§ 6, 7, we obtain

$$\psi(\alpha_1, \alpha_2, \ldots, \alpha_h;\ N) = \Sigma z_N^{\Sigma(\alpha-\beta)} \psi(\beta_1, \beta_2, \ldots, \beta_h;\ N-1),$$

and deduce

$$\psi(\alpha_1, \alpha_2, \ldots, \alpha_h;\ N-1) = \left[\prod_{r=1}^{h} (1 - z_N \delta_r)\right] \psi(\alpha_1, \alpha_2, \ldots, \alpha_h;\ N).$$

That $$\psi(\alpha_1, \alpha_2, \ldots, \alpha_h;\ N) = \left| \phi\binom{\alpha_s + s - r}{N} \right|$$

satisfies this difference equation follows at once from (V). And exactly the same considerations as were used in § 7 complete the proof by induction that this is the generating function required. It remains but to point out that, now that this form of generating function has been obtained in the general case, the result may be applied to any particular case; any relations may be introduced among the weights z_r. In fact, the sets may be the coefficients of p-ary q-ics, and the weights the appropriate sets of numbers.

THEOREM VIII. *When* $\phi\binom{\delta}{N}$ *is a generating function for gradients* X *of any quantic of degree* δ, *the quantics having* N *coefficients, the generating*

function for gradients $T_{a_1 a_2 \dots a_h} X$ *is*

$$\left| \phi \binom{a_r + s - r}{N} \right|.$$

III. *Quaternary and higher forms.*

12. The considerations used for ternary forms may be applied to higher forms, but the increase in the number of kinds of variables and their inter-relations makes the process cumbersome.

The difficulty may be overcome thus. A seminvariant of a form with h variables is given by a PN belonging to $T_{a_1 a_2 \dots a_h}$; and this is completely defined by the corresponding tableau F. Moreover, F is completely defined by a gradient

$$X = \Pi A_{r_1 r_2 \dots r_h}^{\lambda_{r_1 r_2 \dots r_h}},$$

where in F the number of sets of n equivalent letters which have r_1 in the first row, r_2 in the second, and so on is $\lambda_{r_1 r_2 \dots r_h}$.

Further, every seminvariant can be expressed linearly in terms of seminvariants thus defined. The weights of such a seminvariant are

$$w_1 = a_1, \quad w_2 = a_2, \quad \dots, \quad w_h = a_h.$$

A term of the concomitant led by this seminvariant has the same substitutional properties defined by the operator PN, but its weights are different from those of the seminvariant. Consider a term whose weights are $\varpi_1, \varpi_2, \dots, \varpi_h$. Then $w_1 \geqslant \varpi_1$, $w_1 + w_2 \geqslant \varpi_1 + \varpi_2$, and so on. Let δ be the degree of the concomitant, the number of letters in PN or F is then $n\delta$.

Let us use separate letters $a, b, c, \dots$ for the moment for each of these, and consider them as ordinary symbolical letters so that

$$a_x = a_1 x_1 + a_2 x_2 + \dots + a_h x_h$$

in the usual way.

Then of these $n\delta$ letters, in any gradient of the term considered, ϖ_1 have the suffix 1, and this gradient is unaffected by the permutations of all the letters with this suffix, *i.e.* it is unaffected by the permutations of a positive symmetric group of degree ϖ_1. Similarly, it is unaffected by the permutations of a positive symmetric group of degree ϖ_2, and so on. Thus, calling the letters with *symbolical suffix* r $\theta_{r1}, \theta_{r2}, \dots, \theta_{r\varpi_r}$, the positive symmetric group of these letters G_r, and the product of these groups G, PN may be rewritten and supposed multiplied on the left by G. Then, as

in § 5, GPN can be expressed in terms of forms $P'\sigma N$ derived from tableaux F' which are standard according to the sequence of θ sets, and in which there is no *overlapping*. Let us call the letters in the r-th row of F, which defined our concomitant,

$$B_{r1}B_{r2}\dots B_{r\alpha_r}.$$

The concomitant may be developed into actual coefficients from its PN form, by first expanding according to the symbolical letters B, and then using polarizing operators. As an example, consider the seminvariant of the Hessian of the binary cubic, defined by A_0A_2 or

$$F=\begin{pmatrix}a_1a_2a_3b_1\\ b_2b_3\end{pmatrix}.$$

It is $$(B_1B_2)^2B_1^2,$$

using B_1 for the first row letters and B_2 for the second.

We polarize with

$$\frac{1}{4!\ 2!}\left(a\frac{\partial}{\partial B_1}\right)^3\left(b\frac{\partial}{\partial B_1}\right)\left(b\frac{\partial}{\partial B_2}\right)^2,$$

and obtain $$\tfrac{1}{6}(ab)^2a_1b_1.$$

Consider any gradient Y of weights $\varpi_1, \varpi_2, \dots, \varpi_h$. Then

$$T_{a_1a_2\dots a_h}Y=\Sigma\lambda P'N',$$

where $P'N'$ is defined by the sets θ_r. There are then just the same number of linearly independent forms $P'N'$ as there are standard non-overlapping θ tableaux.

Next consider the correspondence between θ and B; no matter how this is arranged, $P'N'$ is unaffected by the permutations of

$$\{B_{11}B_{12}\dots B_{1a_1}\}.$$

Operation with this group places each of these letters in the first row of the tableau, completely filling it. Similarly the second row can be filled with $B_{21}\dots B_{2a_2}$. Thus for the terms of our concomitant each one of the kind $P'N'$ can be expressed in terms of the same one. Thus the number of independent terms of weights $\varpi_1, \varpi_2, \dots, \varpi_h$ is the number of standard non-overlapping tableaux F'. Let this number be

$$\begin{bmatrix}w_1w_2\dots w_h\\ \varpi_1\varpi_2\dots\varpi_h\end{bmatrix};$$

this is zero when $\varpi_1>w_1$ or $\varpi_1+\varpi_2>w_1+w_2, \dots$. Let $\{w_1w_2\dots w_h\}$ be

the number of seminvariants of these weights. Then the number of gradients of weights $\varpi_1, \varpi_2, \ldots, \varpi_h$ is

$$[\varpi_1, \varpi_2, \ldots, \varpi_h] = \sum_w \begin{bmatrix} w_1 w_2 \ldots w_h \\ \varpi_1 \varpi_2 \ldots \varpi_h \end{bmatrix} \{w_1 w_2 \ldots w_h\}.$$

Let δ_{rs} be an operator which increases the r-th row of a tableau by unity and decreases the s-th row by unity. Then $\begin{bmatrix} w_1 w_2 \ldots w_h \\ \varpi_1 \varpi_2 \ldots \varpi_h \end{bmatrix}$ is the number of terms in

$$\prod_{r<s} \delta_{rs}^{\lambda_{rs}} T_{\varpi_1 \varpi_2 \ldots \varpi_h},$$

where *overlapping* terms are excluded and the indices $\lambda_{r,s}$ are such as to make the final rows $w_1, w_2, \ldots, w_h$.

Thus, if D is the operator

$$D = \prod_{r<s} (1 - \delta_{r,s}),$$

as in equation I, § 2, Q.S.A., VI*,

$$D \begin{bmatrix} w_1 w_2 \ldots w_h \\ \varpi_1 \varpi_2 \ldots \varpi_h \end{bmatrix} = 0$$

(the operators affecting the ϖ's and not the w's), except when

$$w_1 = \varpi_1, \quad w_2 = \varpi_2, \quad \ldots, \quad w_h = \varpi_h;$$

in which case the right-hand side is unity. Hence

$$D[\varpi_1, \varpi_2, \ldots, \varpi_h] = \Sigma \{w_1, w_2, \ldots, w_h\} D \begin{bmatrix} w_1 w_2 \ldots w_h \\ \varpi_1 \varpi_2 \ldots \varpi_h \end{bmatrix}$$

$$= \{\varpi_1, \varpi_2, \ldots, \varpi_h\}.$$

Then, when G is the generating function for gradients, so that

$$G = \Sigma [\varpi_1 \varpi_2 \ldots \varpi_h] x_1^{\varpi_1} x_2^{\varpi_2} \ldots x_h^{\varpi_h},$$

and Γ that for seminvariants, so that

$$\Gamma = \Sigma \{w_1, w_2, \ldots, w_h\} x_1^{w_1} x_2^{w_2} \ldots x_h^{w_h},$$

we have $$\Gamma = \left[\prod_{r<s} \left(1 - \frac{x_r}{x_s}\right) \right] G = x_1^{-h+1} x_2^{-h+2} \ldots x_{h-1}^{-1} \Delta G,$$

* *Proc. London Math. Soc.* (2), 34 (1932), 199.

where Δ is the alternating function

$$\prod_{r<s} (x_r - x_s).$$

THEOREM IX. *The generating function for the seminvariants of the q-ary p-ics is obtained from that for gradients by multiplication by*

$$x_1^{-q+1} x_2^{-q+2} \dots x_{q-1}^{-1} \Delta,$$

where Δ is the alternating function

$$\prod_{r<s} (x_r - x_s).$$

It may be observed that the generating function for seminvariants of the q-ary p-ic (in the same way as that for binary and ternary forms) exhibits a peculiar symmetry. Thus it may be written

$$\Sigma\{w_1, w_2, \dots, w_q\} \left[\Sigma \pm \prod_{r=1}^{q} x_r^{w_s + r - s} \right],$$

where in each product $w_1, w_2, \dots, w_q$ appear in some sequence in the indices; and the sign is determined as positive when this sequence is obtained from that of the suffixes by a permutation belonging to the alternating group, and otherwise as negative. The whole sum is limited by the inequalities

$$w_1 \geqslant w_2 \geqslant \dots \geqslant w_q.$$

ON THE THEORY OF TRIGONOMETRIC SERIES (IV)

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1. The references in this paper are to the preceding one in this series. It will be referred to as III; and a knowledge of it will be assumed. We here establish the existence of perfect sets of unicity for summable trigonometric series

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx),$$

where a_n, b_n are not supposed to tend to zero as $n\to\infty$ through all the positive integers, but only through specified sequences S. The striking thing about our theorems is the freedom of choice which is left to S. We shall illustrate this by quoting Theorem XXI, § 10, in a form which does not require a knowledge of the terms used in the enunciation given at that point. Suppose that the trigonometric series is obtained by formal differentiation of a Denjoy-Fourier series, so that we may have $a_n=o(n^2)$, $b_n=o(n^2)$. Let σ be a closed set of type H with respect to an increasing sequence $(n_p)(p=1, 2, \ldots)$ of positive integers. The theorem referred to enables us to construct sequences S such that, if

$$\lim a_n=\lim b_n=0$$

as $n\to\infty$ in S, and if the trigonometric series is summable by Poisson's method to zero in the set complementary to σ, then $a_0=0$, and $a_n=b_n=0$ for all n. The sequences S are constructed by choosing a $\delta>0$ and forming the aggregate numbers

$$\{n_p,\ 2n_p,\ 3n_p,\ \ldots,\ [n_p\delta]n_p\}\quad(p=1, 2, \ldots).$$

Denote this aggregate by T. Then S may be any sequence of which T is a central sequence; *i.e.* given any positive integer m, there must be an $n(m)$ such that, if $n\geqslant n(m)$ belongs to T, then $n\pm m$ belong to S. Suppose,

for example, that (n_p) is the sequence $(2^{p!})$. Take $\delta = 1$, so that T is the aggregate

$$\{2^{p!},\ 2.2^{p!},\ 3.2^{p!},\ \ldots,\ 2^{p!}.2^{p!}\} \quad (p = 1, 2, \ldots).$$

Since $(2^{p!})^2 < 2^{(p+1)!}$, the numbers obtained by writing these groups of integers serially for $p = 1, 2, \ldots$ are in order of magnitude. Denote them by

$$\nu_1,\ \nu_2,\ \ldots,\ \nu_r,\ \ldots.$$

Let $\{\lambda_r\}$ be a non-diminishing sequence of positive integers with $\lambda_r \to \infty$. Form the aggregate

$$\{\nu_r - \lambda_r,\ \nu_r - \lambda_r + 1,\ \ldots,\ \nu_r,\ \nu_r + 1,\ \ldots,\ \nu_r + \lambda_r\} \quad (r = 1, 2, \ldots).$$

This aggregate is a required sequence S. We see from this example that, if λ_r increases slowly, then S is very sparsely distributed among the sequence of all positive integers. And this phenomenon is more accentuated the more rapidly (n_p) increases.

The methods and results of this paper present a distinct analogy to those of III. This is due to the convention which we adopt and which is explained at the beginning of § 2. One would not expect much to be gained from agreeing that, in the expression

$$\int_a^b g(x)\,df = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f(x)\,g'(x)\,dx,$$

the first member shall be defined as identical with the second member, whenever the second member has a meaning. It is only a matter of notation; but the value of a suggestive notation cannot be over-estimated. In III we were concerned with Lebesgue-Stieltjes integrals; the convention referred to indicates what theorems of III are likely to remain formally true when the function $f(x)$ is no longer of bounded variation.

In this paper we consider, besides sets of type H with respect to a sequence of integers, sets of type H with respect to a sequence of positive numbers. Though we shall not repeat the definitions of III, we consider it desirable to recall the definition and notation formerly used in connection with sets of type H. Let σ be a closed set in $(0, 2\pi)$. Let (μ_p) be a sequence of positive numbers with $\lim \mu_p = \infty$. Let $E(x)$ denote the excess of x over the greatest integer multiple of 2π which does not exceed x. As x describes σ, $E(\mu_p x)$ describes a set σ_p. If $\Sigma \sigma_p$ is not everywhere dense in $(0, 2\pi)$, we say that σ is of type H with respect to (μ_p).

In this paper we also consider trigonometric series with bounded coefficients. In this case the restrictions on the sequence S are lighter than in the case already mentioned.

2. Let $f(x)$ denote a function which is finite almost everywhere in the interval (a, b), which is integrable L in that interval, and such that $f(a), f(b)$ are finite. Let $g(x)$ denote a function defined in the closed interval (a, b), and such that $g'(x)$ is bounded in that interval. We define the expression

$$\int_a^b g(x)\,df \tag{1}$$

by means of the equation

$$\int_a^b g(x)\,df = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f(x)\,g'(x)\,dx. \tag{2}$$

This equation is consistent with the definition of the Lebesgue-Stieltjes integral; whenever $f(x)$ is of bounded variation in (a, b), then the expression (1), interpreted as a Lebesgue-Stieltjes integral, satisfies the equation (2).

Let $f(x)$ denote a function which is finite almost everywhere in $(0, 2\pi)$, which is L-integrable in $(0, 2\pi)$, and such that $f(0), f(2\pi)$ are finite. If

$$\int_0^{2\pi} e^{inx}\,df \tag{3}$$

is uniformly bounded in the integer n, we say that $f(x)$ belongs to class β. If there is an unbounded sequence S of positive integers, such that

$$\lim_{n\to\infty} \int_0^{2\pi} e^{inx}\,df = 0 \quad (n \subset S), \tag{4}$$

then we say that $f(x)$ is associated with S. Since

$$\int_0^{2\pi} e^{inx}\,df = f(2\pi) - f(0) - in \int_0^{2\pi} f(x)\,e^{inx}\,dx,$$

it follows that, if $f(x)$ belongs to class β, then

$$\left| n \int_0^{2\pi} f(x)\,e^{inx}\,dx \right| < K \quad (n = \pm 1, \pm 2, \ldots), \tag{5}$$

where K is a constant independent of n.

THEOREM I. *If $f(x)$ belongs to class β and is associated with the dense sequence S, and if*

$$t(x) = \tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + \beta_n \sin nx)$$

$$\left[a_n = o\left(\frac{1}{n \log n}\right), \quad \beta_n = o\left(\frac{1}{n \log n}\right)\right],$$

where $\Sigma|\alpha_n|+|\beta_n|$ *converges, then*

$$\lim_{p\to\infty}\int_0^{2\pi} e^{in_p x}\,d(ft)=0,$$

where (n_p) *is any central sequence of* S.

The theorem is true when $t(x)$ is a constant. When $t(x)$ is of the form e^{ikx}, where $k\neq 0$ is an integer, we have

$$\int_0^{2\pi} e^{in_p x}\,d(fe^{ikx})=f(2\pi)-f(0)-in_p\int_0^{2\pi} f(x)\,e^{i(n_p+k)x}\,dx. \tag{6}$$

In virtue of (5) we have

$$\lim_{p\to\infty} ik\int_0^{2\pi} f(x)\,e^{i(n_p+k)x}\,dx=0. \tag{7}$$

If we add the expression under the limiting sign in (7) to the second member of (6), we infer that

$$\begin{aligned}\lim_{p\to\infty}\int_0^{2\pi} e^{in_p x}\,d(fe^{ikx}) &= \lim_{p\to\infty}\left[f(2\pi)-f(0)-i(n_p+k)\int_0^{2\pi} f(x)\,e^{i(n_p+k)x}\,dx\right]\\ &= \lim_{p\to\infty}\int_0^{2\pi} e^{i(n_p+k)x}\,df\\ &= 0,\end{aligned}$$

since n_p+k belongs to S for $p\geqslant p(k)$. The theorem is therefore true when $t(x)$ is a trigonometric polynomial.

In the general case we write

$$t(x)=\tfrac{1}{2}\sum_{r=-\infty}^{\infty}(\alpha_r-i\beta_r)\,e^{irx},$$

where $\alpha_{-r}=\alpha_r$, $\beta_{-r}=-\beta_r$. Given $\epsilon>0$, there is an $m(\epsilon)$ such that, for $m\geqslant m(\epsilon)$, $|t(x)-\tau(x)|<\epsilon$ for all x, where

$$\tau(x)=\tfrac{1}{2}\sum_{r=-m}^{m}(\alpha_r-i\beta_r)\,e^{irx}.$$

Then
$$\int_0^{2\pi} e^{in_p x}\,d(ft)=\int_0^{2\pi} e^{in_p x}\,d(f\{t-\tau\})+\int_0^{2\pi} e^{in_p x}\,d(f\tau),$$

and so
$$\overline{\lim_{p\to\infty}}\left|\int_0^{2\pi} e^{in_p x}\,d(ft)\right|=\overline{\lim_{p\to\infty}}\left|\int_0^{2\pi} e^{in_p x}\,d(f\{t-\tau\})\right|. \tag{8}$$

Now

$$\int_0^{2\pi} e^{in_p x}\, d(f\{t-\tau\})$$

$$= f(2\pi)\{t(2\pi)-\tau(2\pi)\} - f(0)\{t(0)-\tau(0)\} - in_p \int_0^{2\pi} f(x)\{t(x)-\tau(x)\}\, e^{in_p x}\, dx,$$

and

$$n_p \int_0^{2\pi} f(x)\{t(x)-\tau(x)\}\, e^{in_p x}\, dx$$

$$= \sum_{r=m+1}^{\infty} (\alpha_r - i\beta_r)\frac{n_p}{2}\int_0^{2\pi} f(x)\, e^{i(n_p+r)x}\, dx + \sum_{r=m+1}^{\infty} (\alpha_r + i\beta_r)\frac{n_p}{2}\int_0^{2\pi} f(x)\, e^{i(n_p-r)x}\, dx.$$

By (5), we have

$$\left| (n_p \pm r)\int_0^{2\pi} f(x)\, e^{i(n_p \pm r)x}\, dx \right| < K$$

for all p and r. Hence the first term in the second member of the preceding equation is

$$O\Big[\sum_{r=m+1}^{\infty} |\alpha_r| + |\beta_r|\Big].$$

The second term of the second member is

$$\sum_{r=m+1}^{[n_p/2]} + \sum_{r=[n_p/2]+1}^{n_p-1} + (\alpha_{n_p} + i\beta_{n_p})\frac{n_p}{2}\int_0^{2\pi} f(x)\, dx + \sum_{r=n_p+1}^{[3n_p/2]} + \sum_{r=[3n_p/2]+1}^{\infty}$$

$$= J_1 + J_2 + o(1) + J_3 + J_4.$$

We have

$$J_1 = O\Big[\sum_{r=m+1}^{[n_p/2]} (|\alpha_r| + |\beta_r|)\frac{n_p}{n_p - r}\Big]$$

$$= O\Big[\sum_{r=m+1}^{\infty} |\alpha_r| + |\beta_r|\Big].$$

$$J_2 = o\Big[\sum_{r=[n_p/2]}^{n_p-1} \frac{1}{r\log r}\frac{n_p}{n_p - r}\Big]$$

$$= o\Big[\frac{1}{\log n_p}\sum_{r=[n_p/2]}^{n_p-1} \frac{1}{r} + \frac{1}{n_p - r}\Big]$$

$$= o(1).$$

$$J_3 = o\Big[\sum_{r=n_p+1}^{[3n_p/2]} \frac{1}{r\log r}\frac{n_p}{r - n_p}\Big]$$

$$= o\Big[\frac{1}{\log n_p}\sum_{r=n_p+1}^{[3n_p/2]} \frac{1}{r - n_p} - \frac{1}{r}\Big]$$

$$= o(1).$$

$$J_4 = O\Big[\sum_{r=[3n_p/2]}^{\infty} (|\alpha_r| + |\beta_r|)\frac{n_p}{r - n_p}\Big]$$

$$= o(1).$$

Hence

$$\overline{\lim_{p\to\infty}}\left|\int_0^{2\pi} e^{in_p x}\,d(f\{t-\tau\})\right| < \epsilon[|f(2\pi)|+|f(0)|]+K\sum_{r=m+1}^{\infty}|\alpha_r|+|\beta_r|.$$

By (8), the upper limit, which is independent of ϵ and m, must be zero.

A proof which is similar to and simpler than the above shows that

$$\int_0^{2\pi} e^{inx}\,d(ft)$$

is uniformly bounded in n. We thus have

THEOREM II. *If $f(x)$ is of class β, and associated with the dense sequence S, and if $t(x)$ is a function whose Fourier series converges absolutely and whose Fourier constants are $o(1/n\log n)$, then $f(x)t(x)$ is of class β, and is associated with any central sequence of S.*

3. *Definition* 1. An increasing sequence S of positive integers will be said to be complete if it contains a sequence of groups $\{S_r\}$ of positive integers with the properties

(i) S_r consists of consecutive integers $\alpha_r, \alpha_{r+1}, \ldots, \beta_r$ such that $\beta_r/\alpha_r > 1+\theta$, where $\theta > 0$, for $r = 1, 2, \ldots$.

(ii) Every member of S_{r+1} is greater than every member of S_r.

Given the complete sequence S, we can find a central sequence Σ of S which is itself a complete sequence. For, if we choose $\lambda > 0$ such that

$$\frac{(1+\theta)(1-\lambda)}{(1+\lambda)} > 1+\tfrac{1}{2}\theta,$$

we can take for Σ that sequence of groups of which the r-th is

$$\alpha_r+[\lambda\alpha_r],\quad \alpha_r+[\lambda\alpha_r]+1,\quad \ldots,\quad \beta_r-[\lambda\beta_r];$$

and we then have

$$\frac{\beta_r-[\lambda\beta_r]}{\alpha_r+[\lambda\alpha_r]} > \frac{\beta_r(1-\lambda)}{\alpha_r(1+\lambda)} > 1+\tfrac{1}{2}\theta.$$

A function $f(x)$, which is finite almost everywhere in $(0, 2\pi)$, which is integrable L in $(0, 2\pi)$, and which is such that $f(0)$, $f(2\pi)$ are finite, will be said to belong to class γ. If there is an unbounded sequence S such that

$$\lim_{n\to\infty}\int_0^{2\pi} e^{inx}\,df = 0 \quad (n \subset S),$$

we say that $f(x)$ is associated with S.

THEOREM III. *If $f(x)$ belongs to class γ and is associated with the complete sequence S, and if*

$$t(x) = \tfrac{1}{2}a_0 + \Sigma (a_n \cos nx + \beta_n \sin nx)$$

$$\left[a_n = O\left(\frac{1}{n^2}\right), \quad \beta_n = O\left(\frac{1}{n^2}\right)\right],$$

then

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, d(ft) = 0,$$

where (n_p) is a sequence with the property that there is a $\delta > 0$ such that all integers between $n_p(1-\delta)$ and $n_p(1+\delta)$ belong to S.

It is clear that a sequence such as (n_p) can always be found, and that (n_p) can be chosen so as to be itself a complete sequence. For example, (n_p) may be the sequence Σ constructed above; in that case, we can take $\delta = \frac{1}{2}\lambda$, since $(\beta_r - [\lambda\beta_r])(1+\frac{1}{2}\lambda) < \beta_r$, and $(a_r + [\lambda a_r])(1-\frac{1}{2}\lambda) > a_r$.

The proof that the theorem holds when $t(x)$ is a trigonometric polynomial is the same as for Theorem I. To complete the proof, we must show, with the notation previously used, that

$$\overline{\lim_{p\to\infty}} \left| \int_0^{2\pi} e^{in_p x}\, d(f\{t-\tau\}) \right|$$

can be made arbitrarily small by suitable choice of $\tau(x)$. Everything therefore depends on the evaluation of

$$\sum_{r=m+1}^{\infty} (a_r - i\beta_r) \frac{n_p}{2} \int_0^{2\pi} f(x)\, e^{i(n_p+r)x}\, dx + \sum_{r=m+1}^{\infty} (a_r + i\beta_r) \frac{n_p}{2} \int_0^{2\pi} f(x)\, e^{i(n_p-r)x}\, dx. \quad (9)$$

Since the numbers n_p, n_p+1, ..., $n_p+[n_p\delta]$ belong to S, the first term in (9) is

$$o\left[\sum_{r=m+1}^{[n_p\delta]} |a_r| + |\beta_r|\right] + o\left[n_p \sum_{r=[n_p\delta]+1}^{\infty} \frac{1}{r^2}\right], \quad (10)$$

where we have used the theorem that, since $f(x)$ is integrable L,

$$\int_0^{2\pi} f(x)\, e^{i(n_p+r)x}\, dx = o(1)$$

as $p\to\infty$. Further, since n_p, n_p-1, ..., $n_p-[n_p\delta]$ belong to S, the second term in (9) is less in absolute value than

$$o\left[\sum_{r=m+1}^{[n_p\delta]} |a_r| + |\beta_r|\right] + \sum_{r=[n_p\delta]+1}^{\infty} (|a_r| + |\beta_r|) \frac{n_p}{2} \left| \int_0^{2\pi} f(x)\, e^{i(n_p-r)x}\, dx \right|.$$

To evaluate the second term, let $0<\delta<\theta<1$. The term in question is then

$$\sum_{r=[n_p\delta]+1}^{[n_p\theta]} + \sum_{[n_p\theta]+1}^{n_p+N} + \sum_{n_p+N+1}^{\infty} = J_1+J_2+J_3,$$

where N denotes a positive integer. In the sum J_1, $n_p-r>n_p(1-\theta)$, so that

$$J_1 = o\left[\sum_{r=[n_p\delta]+1}^{[n_p\theta]} \frac{n_p}{r^2}\right]$$

$$= o(1).$$

In J_2 we have $$\int_0^{2\pi} f(x)\, e^{i(n_p-r)x}\, dx = O(1),$$

so that $$J_2 = O\left[n_p \sum_{[n_p\theta]+1}^{n_p+N} \frac{1}{r^2}\right]$$

$$= O\left[\frac{1-\theta}{\theta^2}\right].$$

In J_3 we have $n_p-r<-N-1$, so that

$$J_3 = O(\epsilon_N),$$

where $\epsilon_N\to\infty$ as $N\to\infty$. Letting $p\to\infty$, and then $\theta\to 1$, $N\to\infty$, we see that the limit of (9) is zero.

Definition 2. Given a complete sequence S, a sequence (n_p) of S with the property that there is a $\delta>0$ such that all integers in the interval $n_p-n_p\delta$, $n_p+n_p\delta$ belong to S will be said to be a principal sequence of S.

By what precedes, every complete sequence possesses a principal sequence which is complete. We have, from Theorem III,

THEOREM IV. *If $f(x)$ belongs to class γ and is associated with the complete sequence S, and if $t(x)$ is a function whose Fourier constants are $O(1/n^2)$, then $f(x)t(x)$ belongs to class γ and is associated with any principal sequence of S.*

4. THEOREM V. *If $f(x)$ belongs to class β or class γ, and is associated with a dense sequence or with a complete sequence respectively, and if the interval d is interior to an interval in which $f(x)$ is of bounded variation, then $f(x)$ can only have removable discontinuities in d.*

Let ξ be a point of discontinuity of $f(x)$ in d. Then there is an $\epsilon>0$ such that $f(x)$ is of bounded variation in an interval which contains

$(\xi-3\epsilon,\ \xi+3\epsilon)$. Define $t(x)$ by the formulae

$$t(x)=0 \quad (0\leqslant x\leqslant \xi-3\epsilon,\ \ \xi+3\epsilon\leqslant x\leqslant 2\pi)$$
$$=1 \quad (\xi-\epsilon\leqslant x\leqslant \xi+\epsilon),$$

and take $t(x)$ to be linear in the remaining two intervals, and continuous in $(0,\ 2\pi)$. Then the Fourier constants of $t(x)$ are $O(1/n^2)$, and so, by Theorem II or Theorem IV, $f(x)t(x)$ belongs to class β or to class γ and is in either case associated with a dense sequence T. Since $f(x)t(x)$ is of bounded variation in $(0,\ 2\pi)$, we can, in the relation

$$\lim_{n\to\infty}\int_0^{2\pi} e^{inx}\,d(ft)=0 \quad (n\subset T),$$

interpret the integral in the sense of Lebesgue-Stieltjes. Thus, with the nomenclature of III, $f(x)t(x)$ is a function of class B, associated with the dense sequence T. By III, Theorem 2, $f(x)t(x)$ is equivalent to a function which is continuous in $(0,\ 2\pi)$. Hence $f(x)$ is equivalent in $(\xi-\epsilon,\ \xi+\epsilon)$ to a continuous function, and so ξ must be a removable discontinuity.

It is of interest to examine the case in which $f(x)$, of class β or of class γ, possesses the property of being of bounded variation in two intervals of the form $(0,\ \delta)$, $(2\pi-\delta,\ 2\pi)$. Let $0<h<\delta$. We require the following lemma.

LEMMA 1. *Let $\phi(x)$ be of bounded variation in the closed intervals $(0,\ h)$, $(2\pi-h,\ 2\pi)$. Let M be a dense sequence of positive integers such that*

$$\lim_{n\to\infty}\int_0^h+\int_{2\pi-h}^{2\pi} e^{inx}\,d\phi=0 \quad (n\subset M).$$

Then, if $\lambda(x)$, defined in $(0,\ 2\pi)$, is bounded, B-measurable, and satisfies $\lambda(0)=\lambda(2\pi)$, we have

$$\lim_{q\to\infty}\int_0^h+\int_{2\pi-h}^{2\pi} e^{in_q x}\lambda(x)\,d\phi=0,$$

where (n_q) is a central sequence of M.

The proof is similar to that of III, Theorem I. We now wish to prove

THEOREM VI. *If $f(x)$ belongs to class β or to class γ, and is associated with a dense sequence or with a complete sequence respectively, and is of bounded variation in the intervals $(0,\ \delta)$, $(2\pi-\delta,\ 2\pi)$, then $f(x)$ is equivalent to a function $\psi(x)$ which is continuous and of bounded variation in*

$(0, h)$, $(2\pi-h, 2\pi)$, *and which is such that, for every integer* n,

$$\int_0^{2\pi} e^{inx}\,df = \int_0^{2\pi} e^{inx}\,d\psi.$$

Let $0<h<h'<\delta$. By Theorem V, $f(x)$ can have only removable discontinuities in $(0, h')$, $(2\pi-h', 2\pi)$. We have, by definition,

$$\int_0^{2\pi} e^{inx}\,df = \int_0^{h'} + \int_{2\pi-h'}^{2\pi} + \int_{h'}^{2\pi-h'} e^{inx}\,df,$$

and the first and second integrals on the right may be taken in the sense of Lebesgue-Stieltjes. By the argument of III, § 5, we may, without altering the value of the integrals, remove the discontinuities of $f(x)$ in the open intervals $0<x<h'$, $2\pi-h'<x<2\pi$, by altering the value of $f(x)$ at the relevant enumerable set. Denote the altered function by $\phi(x)$. Then $\phi(x)$ possesses all the properties of $f(x)$ in the enunciation. Consider the function $t(x)$ which is defined by

$$t(x) = 1 \quad (0 \leqslant x \leqslant h,\ \ 2\pi-h \leqslant x \leqslant 2\pi)$$

$$= 0 \quad \left(h+\frac{1}{m} \leqslant x \leqslant 2\pi-h-\frac{1}{m}\right),$$

where m is a large integer, and which is linear in the remaining two intervals and continuous in $(0, 2\pi)$. We choose m so that $h+1/m<h'$. The Fourier constants of $t(x)$ are $O(1/n^2)$, and hence

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\,d(\phi t) = 0,$$

where (n_p) is a sequence mentioned in Theorem I or Theorem III according as $f(x)$ is of class β or of class γ. In virtue of our definition of $t(x)$, the preceding equation is

$$\lim_{p\to\infty} \int_0^{h+1/m} + \int_{2\pi-h-1/m}^{2\pi} e^{in_p x}\,d(\phi t) = 0. \tag{11}$$

We have

$$\int_h^{h+1/m} e^{in_p x}\,d(\phi t) = \int_h^{h+1/m} e^{in_p x}\,t(x)\,d\phi + \int_h^{h+1/m} e^{in_p x}\,\phi(x)\,dt.$$

The first integral on the right does not exceed in absolute value the total variation of $\phi(x)$ in $(h, h+1/m)$, which is of the form ϵ_m, where $\epsilon_m \to 0$ as $m\to\infty$. The second integral tends to zero as $p\to\infty$, since $t(x)$ has a

bounded derivative. Treating

$$\int_{2\pi-h-1/m}^{2\pi-h} e^{in_p x}\, d(\phi t)$$

in a similar way, we infer from (11) that

$$\lim_{p\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x}\, d(\phi t) = 0.$$

By the definition of $t(x)$, this is

$$\lim_{p\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x}\, d\phi = 0.$$

Let $\lambda(0) = \lambda(2\pi) = 1$, and $\lambda(x) = 0$ for $0 < x < 2\pi$. Since (n_p) may be taken to be a dense sequence M, we have, by Lemma 1,

$$\lim_{q\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_q x}\lambda(x)\, d\phi = 0,$$

where (n_q) is a central sequence of M. By the definition of $\lambda(x)$, this gives

$$\phi(2\pi)-\phi(2\pi-0)+\phi(+0)-\phi(0) = 0,$$

which, by the definition of $\phi(x)$, is the same as

$$f(2\pi)-f(2\pi-0)+f(+0)+f(0) = 0. \tag{12}$$

If now $\psi(x)$ denotes the function defined by

$$\psi(0) = f(+0), \quad \psi(2\pi) = f(2\pi-0), \quad \psi(x) = \phi(x) \quad (0 < x < 2\pi),$$

then

$$\begin{aligned}\int_0^{2\pi} e^{inx}\, d\psi &= \psi(2\pi)-\psi(0)-in\int_0^{2\pi}\psi(x)\, e^{inx}\, dx \\ &= f(2\pi)-f(0)-in\int_0^{2\pi} f(x)\, e^{inx}\, dx \\ &= \int_0^{2\pi} e^{inx}\, df.\end{aligned}$$

5. We wish to obtain propositions which are analogous to Theorems I and III, but in which $t(x)$ is replaced by a sequence $\{t_p(x)\}$ of functions. It is unnecessary for our purpose to make any serious attempt at generality in the enunciations. We enunciate the results as lemmas.

LEMMA 2. *If $f(x)$ belongs to class β and is associated with the dense sequence S, and if*

$$t_p(x) = \tfrac{1}{2}a_0^{(p)} + \Sigma(a_n^{(p)}\cos nx + \beta_n^{(p)}\sin nx), \tag{13}$$

where

$$|a_0^{(p)}| < K, \quad |a_n^{(p)}| < K/n(\log n)^2, \quad |\beta_n^{(p)}| < K/n(\log n)^2, \tag{14}$$

where K is an absolute constant, then

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, d(ft_p) = 0, \tag{15}$$

where (n_p) is any central sequence of S. If, further, the functions $\{t_p'(x)\}$ are uniformly bounded and equi-continuous, then

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x} t_p(x)\, df = 0. \tag{16}$$

The first part is seen to be true on examining the proof of Theorem I. For the second part we have

$$\begin{aligned}
&\int_0^{2\pi} e^{in_p x} t_p(x)\, df \\
&\quad = t_p(2\pi) f(2\pi) - t_p(0) f(0) - \int_0^{2\pi} f(x)\, d\{e^{in_p x} t_p(x)\} \\
&\quad = t_p(2\pi) f(2\pi) - t_p(0) f(0) - in_p \int_0^{2\pi} f(x)\, t_p(x)\, e^{in_p x}\, dx - \int_0^{2\pi} f(x)\, t_p'(x)\, e^{in_p x}\, dx \\
&\quad = \int_0^{2\pi} e^{in_p x}\, d(ft_p) - \int_0^{2\pi} f(x)\, t_p'(x)\, e^{in_p x}\, dx.
\end{aligned}$$

By (15) it remains to prove that

$$\lim_{p\to\infty} \int_0^{2\pi} f(x)\, t_p'(x)\, e^{in_p x}\, dx = 0. \tag{17}$$

Write $$s_p^{(m)}(x) = \frac{i}{2} \sum_{r=-m}^{m} r(a_r^{(p)} - i\beta_r^{(p)}) \left(1 - \frac{|r|}{m}\right) e^{irx}.$$

This is the m-th Cesàro mean of order 1 of the Fourier series of $t_p'(x)$. Since $f(x)$ is integrable L, we have, by (14),

$$\lim_{p\to\infty} \int_0^{2\pi} f(x)\, s_p^{(m)}(x)\, e^{in_p x}\, dx = 0. \tag{18}$$

Finally, since the $\{t_p'(x)\}$ are uniformly bounded and equi-continuous, we have $|t_p'(x) - s_p^{(m)}(x)| < \epsilon_m$ for all x, where ϵ_m is independent of p and tends to zero as $m \to \infty$. From (18) it follows that

$$\begin{aligned}
\lim_{p\to\infty} \int_0^{2\pi} f(x)\, t_p'(x)\, e^{in_p x}\, dx &= \lim_{m\to\infty} \lim_{p\to\infty} \int_0^{2\pi} f(x) \{t_p'(x) - s_p^{(m)}(x)\} e^{in_p x}\, dx \\
&= 0.
\end{aligned}$$

LEMMA 3. *If $f(x)$ belongs to class γ and is associated with the complete sequence S, and if*

$$t_p(x) = \tfrac{1}{2}a_0^{(p)} + \Sigma(a_n^{(p)} \cos nx + \beta_n^{(p)} \sin nx),$$

where $$|a_0^{(p)}| < K, \quad |a_n^{(p)}| < K/n^2, \quad |\beta_n^{(p)}| < K/n^2,$$

where K is an absolute constant, then

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, d(ft_p) = 0,$$

where (n_p) is any principal sequence of S. If, further, the functions $\{t_p'(x)\}$ are uniformly bounded and equi-continuous, then

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, t_p(x)\, df = 0.$$

The first part is seen to be true on examining the proof of Theorem III; and the second part is proved in the same way as the second part of Lemma 2.

We further require the following extension of Lemma 1.

LEMMA 4. *Let $\phi(x)$ be continuous and of bounded variation in the closed intervals $(0, h)$, $(2\pi-h, 2\pi)$. Let M be a dense sequence such that*

$$\lim_{n\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{inx}\, d\phi = 0 \quad (n \subset M).$$

If $\{\lambda_p(x)\}$ is a uniformly bounded semi-convergent series of B-measurable functions, then

$$\lim_{p\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x} \lambda_p(x)\, d\phi = 0,$$

where (n_p) is any central sequence of M.

We first observe that in Lemma 1, if $\phi(x)$ is continuous in the two intervals, we have

$$\lim_{n\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x} \lambda(x)\, d\phi = 0, \tag{19}$$

where $\lambda(x)$ is bounded and B-measurable. The condition $\lambda(0) = \lambda(2\pi)$ may be neglected since, in virtue of the continuity of $\phi(x)$, we can alter the value of $\lambda(x)$ at $x = 2\pi$. If the result of the lemma was false, we could select a subsequence (p_s) such that

$$\lim_{s\to\infty} \left| \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_{p_s} x} \lambda_{p_s}(x)\, d\phi \right| = \eta,$$

where $\eta > 0$. Since $\{\lambda_p(x)\}$ is semi-convergent, we can select from (p_s) a subsequence ν such that $\lambda_\nu(x) \to \lambda(x)$. The preceding relation gives

$$\lim_{\nu\to\infty} \left| \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_\nu x} \lambda_\nu(x)\, d\phi \right| = \eta. \tag{20}$$

But $$\left| \int_0^h e^{in_\nu x} \{\lambda_\nu(x) - \lambda(x)\} d\phi \right| \leqslant \int_0^h |\lambda_\nu(x) - \lambda(x)|\, dv(x),$$

where $v(x)$ is the total variation of $\phi(x)$ in $(0, x)$. The last expression tends to zero as $\nu \to \infty$. Treating the integral between the limits $2\pi - h$, 2π in a similar way, we see that (20) implies that

$$\lim_{\nu\to\infty} \left| \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_\nu x} \lambda(x)\, d\phi \right| = \eta,$$

which contradicts (19).

LEMMA 5. *Let $f(x)$ be of class β and associated with the dense sequence S. Let $f(x)$ be continuous and of bounded variation in two intervals of the form $(0, \delta)$, $(2\pi - \delta, 2\pi)$. Let $0 < h < \delta$. If $\{\lambda_p(x)\}$ is a uniformly bounded semi-convergent sequence of B-measurable functions, then*

$$\lim_{p\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x} \lambda_p(x)\, df = 0,$$

where (n_p) is any central sequence of S.

We first observe that (n_p) is a central sequence of a dense sequence T which is itself a central sequence of S. An examination of the proof of Theorem VI shows that

$$\lim_{n\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{inx}\, df = 0 \quad (n \subset T); \tag{21}$$

so that the result follows from Lemma 4.

LEMMA 6. *Let $f(x)$ be of class γ and associated with the complete sequence S. Let $f(x)$ be continuous and of bounded variation in two intervals of the form $(0, \delta)$, $(2\pi - \delta, 2\pi)$. Let $0 < h < \delta$. If $\{\lambda_p(x)\}$ is a uniformly bounded semi-convergent sequence of B-measurable functions, then*

$$\lim_{p\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x} \lambda_p(x)\, df = 0,$$

where (n_p) is a central sequence of any principal sequence T of S.

An examination of the proof of Theorem VI shows that (21) holds. The statement that (n_p) is a central sequence of T implies that T is dense, so that the result follows from Lemma 4.

6. *Definition* 3. A function $f(x)$ of class β or of class γ which is continuous and of bounded variation in two intervals of the form $(0, \delta)$, $(2\pi-\delta, 2\pi)$ will be said to be of class β^* or of class γ^* respectively.

LEMMA 7. *Let $f(x)$ belong to class β^* and be associated with the dense sequence S. If $\{\theta_p\}$ is a sequence of real numbers satisfying $|\theta_p| \leqslant 1$, then*

$$\lim_{p\to\infty} \int_0^{2\pi} e^{i(n_p+\theta_p)x}\, df = 0,$$

where (n_p) is any central sequence of S.

Let $0 < h < \delta$, where δ is the number which enters into Definition 3. Define the function $t_p(x)$ by the formulae

$$t_p(x) = \cos\theta_p x \qquad (h \leqslant x \leqslant 2\pi-h)$$

$$= \frac{x^2}{h^2}\cos\theta_p x + \frac{2}{h^3}\cos\theta_p h\,.\,(h-x)\,x^2 \quad (0 \leqslant x \leqslant h)$$

$$= \frac{(2\pi-x)^2}{h^2}\cos\theta_p x + \frac{2}{h^3}\cos\theta_p(2\pi-h)\,.\,(2\pi-h-x)(2\pi-x)^2 \qquad (2\pi-h \leqslant x \leqslant 2\pi).$$

Then $\{t_p'(x)\}$ forms a uniformly bounded sequence of equi-continuous functions. The conditions of Lemma 2 are certainly satisfied, since the functions $t_p'(x)$ are uniformly of bounded variation in $(0, 2\pi)$. Hence

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, t_p(x)\, df = 0. \tag{22}$$

Define $\lambda_p(x)$ by the formula

$$\lambda_p(x) = \cos\theta_p x - t_p(x) \quad (0 \leqslant x \leqslant 2\pi).$$

The sequence $\{\lambda_p(x)\}$ is uniformly bounded and equi-continuous. It is therefore semi-convergent. By Lemma 5,

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, \lambda_p(x)\, df = 0. \tag{23}$$

From (22) and (23) we have, by addition,

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x} \cos\theta_p x\, df = 0. \tag{24}$$

In a similar way we can show that

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x} \sin\theta_p x\, df = 0. \tag{25}$$

Referring to the proof of Lemma 2, we see that, under the conditions of that lemma,

$$\int_0^{2\pi} e^{inx} t_p(x)\, df$$

is uniformly bounded in p and all integer n. Further, in Lemma 5,

$$\int_0^h + \int_{2\pi-h}^{h} e^{inx} \lambda_p(x)\, df$$

is uniformly bounded in p and all integer n. It follows that, in the present lemma,

$$\int_0^{2\pi} e^{i\mu x}\, df$$

is bounded for all real μ, since any infinite sequence $\mu_p \to \infty$ can be expressed as $n+\theta_p$, where $|\theta_p| \leqslant 1$ and n runs through a sequence of integers.

If in Lemma 7 we take (n_p) to be a dense central sequence of S, then (n_p) determines a compact set R consisting of intervals all the integer points of which belong to (n_p). The result of Lemma 7 is then equivalent to

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x}\, df = 0 \quad (\mu \subset R).$$

We thus have

THEOREM VII. *Let $f(x)$ be of class β^* and associated with the dense sequence S. Let R be a compact set determined by a dense central sequence of S. Then*

$$\lim_{\mu\to\infty} \int_0^{2\pi} e^{i\mu x}\, df = 0 \quad (\mu \subset R);$$

and

$$\int_0^{2\pi} e^{i\mu x}\, df$$

is uniformly bounded in μ.

We further have

LEMMA 8. *Let $f(x)$ belong to class γ^* and be associated with the complete sequence S. If $\{\theta_p\}$ is a sequence of real numbers which satisfy $|\theta_p| \leqslant 1$, then*

$$\lim_{p\to\infty} \int_0^{2\pi} e^{i(n_p+\theta_p)x}\, df = 0,$$

where (n_p) is any central sequence of a principal sequence T of S.

We define the functions $\{t_p(x)\}$ as in Lemma 7. They satisfy the conditions of Lemma 3, so that (22) holds. By Lemma 6, (23) holds. This gives (24), and (25) is obtained similarly. We thus have

THEOREM VIII. *Let $f(x)$ be of class γ^* and associated with the complete sequence S. Let R be a compact set determined by a dense central sequence of a principal sequence T of S. Then*

$$\lim_{\mu\to\infty}\int_0^{2\pi} e^{i\mu x}\,df = 0 \quad (\mu \subset R).$$

7. THEOREM IX. *Let $f(x)$ be of class β and associated with the sequence S. Let*

$$t(x) = \tfrac{1}{2}\alpha_0 + \Sigma(\alpha_n \cos nx + \beta_n \sin nx)$$

possess a bounded differential coefficient. Let (n_p) be a regular sequence of S. Then

$$\lim_{p\to\infty}\int_0^{2\pi} t(n_p x)\,df = \frac{f(2\pi)-f(0)}{2\pi}\int_0^{2\pi} t(x)\,dx.$$

We have, by definition,

$$\int_0^{2\pi} t(n_p x)\,df = t(2\pi)f(2\pi) - t(0)f(0) - n_p\int_0^{2\pi} f(x)\,t'(n_p x)\,dx. \tag{26}$$

Further,
$$t'(x) \sim \frac{i}{2}\sum_{r=-\infty}^{\infty} r(\alpha_r - i\beta_r)\,e^{irx},$$

so that

$$\begin{aligned}
&-n_p\int_0^{2\pi} f(x)\,t'(n_p x)\,dx \\
&\quad = -\frac{n_p i}{2}\sum_{r=-\infty}^{\infty} r(\alpha_r - i\beta_r)\int_0^{2\pi} f(x)\,e^{irn_p x}\,dx \quad (C,\,1) \\
&\quad = -\frac{n_p i}{2}\Big[\sum_{r=1}^{\infty} r(\alpha_r - i\beta_r)\int_0^{2\pi} f(x)\,e^{irn_p x}\,dx - \sum_{r=1}^{\infty} r(\alpha_r + i\beta_r) \\
&\qquad\qquad \times \int_0^{2\pi} f(x)\,e^{-irn_p x}\,dx\Big] \quad (C,\,1).
\end{aligned}$$

Since (n_p) is a regular sequence of S, we have, for a fixed r, $rn_p \subset S$ for $p \geqslant p(r)$, so that

$$\lim_{p\to\infty}\int_0^{2\pi} e^{irn_p x}\,df = 0,$$

or
$$\lim_{p\to\infty} irn_p\int_0^{2\pi} f(x)\,e^{irn_p x}\,dx = f(2\pi) - f(0).$$

We thus have, if m is a positive integer,

$$\lim_{p\to\infty} -n_p \int_0^{2\pi} f(x)\, t'(n_p x)\, dx$$

$$= -\tfrac{1}{2} \sum_{r=-m}^{m}{}' (\alpha_r - i\beta_r)\{f(2\pi)-f(0)\}$$

$$- \lim_{p\to\infty} \frac{n_p i}{2}\Big[\sum_{r=m+1}^{\infty} r(\alpha_r - i\beta_r) \int_0^{2\pi} f(x)\, e^{irn_p x}\, dx - \sum_{r=m+1}^{\infty} r(\alpha_r + i\beta_r) \times \int_0^{2\pi} f(x)\, e^{-irn_p x}\, dx\Big],$$

where the accent denotes that the value $r=0$ is omitted, and the sum $(C, 1)$ of the series in square brackets is to be taken. Since $t'(x)$ is bounded, $\Sigma(|\alpha_r|+|\beta_r|)$ converges; since, further,

$$rn_p \int_0^{2\pi} f(x)\, e^{\pm irn_p x}\, dx$$

is uniformly bounded in r and p, it follows that the limit last written can be made arbitrarily small by suitable choice of m. Hence

$$\lim_{p\to\infty} -n_p \int_0^{2\pi} f(x)\, t'(n_p x)\, dx = -\tfrac{1}{2} \sum_{r=-\infty}^{\infty}{}' (\alpha_r - i\beta_r)\{f(2\pi)-f(0)\}$$

$$= -\{t(0)-\tfrac{1}{2}\alpha_0\}\{f(2\pi)-f(0)\}.$$

By (26), since $t(2\pi)=t(0)$,

$$\lim_{p\to\infty} \int_0^{2\pi} t(n_p x)\, df = \tfrac{1}{2}\alpha_0\{f(2\pi)-f(0)\}.$$

THEOREM X. *Let $f(x)$ be of class β and associated with the sequence S. Let σ be a closed set of type H with respect to (n_p), where (n_p) is a regular sequence of S. Let $f(x)$ be constant in each of the open intervals contiguous to σ. Then*

$$f(2\pi)=f(0).$$

By definition, $G=\Sigma\sigma_p$ is not everywhere dense in $(0, 2\pi)$. We can find an interval δ within $(0, 2\pi)$ which is free from points of G. We define a function $t(x)$ in $(0, 2\pi)$ by

$$t(x)=0 \quad (x \subset C\delta), \qquad t(x)>0 \quad (x \subset \delta),$$

such that $t(x)$ has a bounded differential coefficient, and we define $t(x)$ outside $(0, 2\pi)$ by the condition of periodicity. Then, by Theorem IX,

$$\lim_{p\to\infty} \int_0^{2\pi} t(n_p x)\, df = \frac{f(2\pi)-f(0)}{2\pi} \int_0^{2\pi} t(x)\, dx.$$

We shall prove the theorem by showing that the value of the first member is zero. Consider the function $t(n_p x)$ for a fixed value of p. When $x \subset \sigma$, $E(n_p x) \subset C\delta$, $t(n_p x) = 0$. Further, $t(n_p x) \geqslant 0$ for all x, and has a bounded differential coefficient. We can therefore write $t(n_p x) = \tau(x)$, where $\tau(x) = 0$ for $x \subset \sigma$, and $\tau(0) = \tau(2\pi) = 0$, and $\tau(x)$ has a bounded differential coefficient. Then

$$\begin{aligned}\int_0^{2\pi} t(n_p x)\, df &= \int_0^{2\pi} \tau(x)\, df \\ &= \tau(2\pi) f(2\pi) - \tau(0) f(0) - \int_0^{2\pi} f(x)\, \tau'(x)\, dx \\ &= -\int_0^{2\pi} f(x)\, \tau'(x)\, dx.\end{aligned}$$

Since $m\sigma = 0$, we have, if $\{\delta_i\}$ denote the intervals contiguous to σ, and c_i the value of $f(x)$ in δ_i,

$$\begin{aligned}\int_0^{2\pi} f(x)\, \tau'(x)\, dx &= \sum_{(i)} \int_{\delta_i} f(x)\, \tau'(x)\, dx \\ &= \sum_{(i)} c_i \int_{\delta_i} \tau'(x)\, dx \\ &= \sum_{(i)} c_i \{\tau(\beta_i) - \tau(\alpha_i)\} \\ &= 0,\end{aligned}$$

where $\delta_i \equiv (\alpha_i, \beta_i)$, since $\tau(\beta_i) = \tau(\alpha_i) = 0$ because β_i, α_i belong to σ.

8. LEMMA 9. *If the series*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \tag{27}$$

is obtained by two formal differentiations of a series whose periodic part is the Fourier series of a continuous function, and if the series (27) *is summable by Poisson's method to zero in an interval* δ, *then it is uniformly summable by Poisson's method to zero in every interval strictly interior to* δ, *and*

$$\tfrac{1}{2}a_0 x + \Sigma \frac{a_n \sin nx - b_n \cos nx}{n} \tag{28}$$

is uniformly summable by Poisson's method to a constant in every interval strictly interior to δ.

It is well known that the continuous function $F(x)$ given by

$$F(x) - \tfrac{1}{4}a_0 x^2 \sim -\Sigma \frac{a_n \cos nx + b_n \sin nx}{n^2}$$

satisfies $\bar{D}^2 F(x) \geqslant 0 \geqslant \underline{D}^2 F(x)$ for $x \subset \delta$. Thus $F(x)$ is linear in δ. The lemma is proved by writing down Poisson's integral for the Fourier series of $F(x) - \frac{1}{4}a_0 x^2$ and differentiating once or twice with respect to x.

THEOREM XI. *Let*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \tag{29}$$

be a trigonometric series with the properties

$$\text{(i)} \quad a_n = O(1), \quad b_n = O(1),$$

$$\text{(ii)} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \quad (n \subset S),$$

where S is a sequence. Let (n_p) be a regular sequence of S. Let the series (29) be summable by Poisson's method to zero in the set complementary to a closed set σ of type H with respect to (n_p). Then

$$a_0 = 0.$$

The series (28) converges to a function which is constant in each of the intervals contiguous to σ. If the points $x = 0, 2\pi$ do not belong to σ, we define a function $f(x)$ almost everywhere in $(0, 2\pi)$ as the sum of (28) wherever it exists. In this case, $f(0), f(2\pi)$ have respectively the values $-\Sigma b_n/n$, $\pi a_0 - \Sigma b_n/n$. If, however, the points $x = 0, 2\pi$ belong to σ, we define a function $f(x)$ by

$$f(0) = 0, \quad f(2\pi) = \pi a_0,$$

and $f(x)$ equal for $0 < x < 2\pi$ to the sum of (28) wherever it exists. In either case, $f(2\pi) - f(0) = \pi a_0$, $f(x)$ is constant in each interval contiguous to σ, and is integrable L. Further,

$$\begin{aligned}\int_0^{2\pi} \cos nx \, df &= f(2\pi) - f(0) + n\int_0^{2\pi} \sin nx f(x)\, dx \\ &= \pi a_0 + n\int_0^{2\pi} \sin nx \left\{\tfrac{1}{2}a_0 x + \frac{a_n}{n}\sin nx\right\} dx \\ &= \pi a_n,\end{aligned}$$

which tends to zero as $n \to \infty$ in S. And

$$\begin{aligned}\int_0^{2\pi} \sin nx \, df &= -n\int_0^{2\pi} f(x)\cos nx\, dx \\ &= -n\int_0^{2\pi} \cos nx \left\{\tfrac{1}{2}a_0 x - \frac{b_n}{n}\cos nx\right\} dx \\ &= \pi b_n,\end{aligned}$$

which tends to zero as $n \to \infty$ in S. Finally,

$$\int_0^{2\pi} e^{inx}\, df$$

is bounded in n by the hypothesis (i). Thus $f(x)$ is of class β and is associated with S. It satisfies the conditions of Theorem X, so that $\pi a_0 = f(2\pi) - f(0) = 0$.

THEOREM XII. *Let*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \tag{30}$$

be a trigonometric series with the properties

$$\text{(i)} \quad a_n = O(1), \quad b_n = O(1),$$

$$\text{(ii)} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \quad (n \subset S),$$

where S is a dense sequence. Let (n_p) be a regular sequence of a central sequence T of S. Let the series (30) *be summable by Poisson's method to zero in the set complementary to a closed set σ of type H with respect to (n_p). Then*

$$a_0 = 0, \quad a_n = 0, \quad b_n = 0 \quad (n = 1, 2, \ldots).$$

By Theorem XI, $a_0 = 0$. Let (α, β) denote an interval interior to a contiguous interval δ_i of σ. Then

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r^n$$

tends to zero as $r \to 1$ uniformly in (α, β). So then does

$$\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \cos mx r^n,$$

where $m \geqslant 1$ is a fixed positive integer. Hence, if $\alpha < x < \beta$, we have

$$\lim_{r\to 1} \int_\alpha^x \left[\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \cos mt r^n \right] dt = 0,$$

where the limit is uniform in x. Integrating term by term, and integrating each term by parts, we obtain

$$\lim_{r\to 1} \left\{ \left[\sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nt - \frac{b_n}{n} \cos nt \right) \cos mt r^n \right]_\alpha^x \right.$$

$$\left. + \int_\alpha^x \left[\sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nt - \frac{b_n}{n} \cos nt \right) m \sin mt r^n \right] dt \right\} = 0.$$

We can choose a so that the series

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin nx-\frac{b_n}{n}\cos nx\right)$$

whose coefficients are $O(1/n)$ is convergent at $x=a$; we can then replace Poisson summation by convergence in the preceding relation, and we thus infer that

$$\sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin nx-\frac{b_n}{n}\cos nx\right)\cos mx + \int_0^x\left[\sum_{n=1}^{\infty}\frac{a_n}{n}\sin nt-\frac{b_n}{n}\cos nt\right]m\sin mt\,dt \quad (31)$$

converges to a constant in each contiguous interval of σ. If $x=0$, $x=2\pi$ do not belong to σ, we define a function $\phi(x)$ almost everywhere in $(0, 2\pi)$ as the sum of (31) wherever it exists. Then

$$\phi(2\pi)-\phi(0)=\int_0^{2\pi}\left[\sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin nt-\frac{b_n}{n}\cos nt\right)m\sin mt\right]dt$$
$$=\pi a_m. \quad (32)$$

If, however, $x=0$, 2π belong to σ, we define a function $\phi(x)$ by

$$\phi(0)=0, \quad \phi(2\pi)=\pi a_m,$$

and $\phi(x)$ equal in $(0<x<2\pi)$ to the sum of (31) wherever it exists. In either case, (32) holds; $\phi(x)$ is integrable L in $(0, 2\pi)$; and $\phi(x)$ is constant in each interval contiguous to σ. If we show that $\phi(x)$ is of class β and associated with T, it will follow from Theorem X that $a_m=0$.

We have, for $x \subset C\sigma$,

$$\phi(x)=\sum_{n=1}^{\infty}\left(\frac{a_n}{n}\sin nx-\frac{b_n}{n}\cos nx\right)\cos mx$$
$$+\frac{m}{2}\left[\sum_{n=1}^{m-1}\frac{a_n}{n}\frac{\sin(n-m)x}{n-m}+\frac{a_m}{m}x+\sum_{n=m+1}^{\infty}\frac{a_n}{n}\frac{\sin(n+m)x}{(n+m)}\right.$$
$$-\sum_{n=1}^{\infty}\frac{b_n}{n}\left(\frac{1-\cos(n+m)x}{n+m}\right)+\sum_{n=1}^{m-1}\frac{b_n}{n}\left(\frac{1-\cos(n-m)x}{n-m}\right)$$
$$\left.+\sum_{n=m+1}^{\infty}\frac{b_n}{n}\left(\frac{1-\cos(n-m)x}{n-m}\right)\right]. \quad (33)$$

Let $\nu>m$ denote a positive integer. We have

$$\int_0^{2\pi}\cos\nu x\,d\phi=\phi(2\pi)-\phi(0)+\nu\int_0^{2\pi}\sin\nu x\phi(x)\,dx. \quad (34)$$

To evaluate
$$\int_0^{2\pi} \sin \nu x \phi(x)\, dx, \tag{35}$$

we observe that the equation (33) may be multiplied by $\sin \nu x$ and integrated term by term. We thus obtain

$$\nu \int_0^{2\pi} \sin \nu x\, \phi(x)\, dx = \frac{\nu}{2} \frac{a_{\nu-m}}{\nu-m} \int_0^{2\pi} \sin^2 \nu x\, dx + \frac{\nu}{2} \frac{a_{\nu+m}}{\nu+m} \int_0^{2\pi} \sin^2 \nu x\, dx$$
$$+ \frac{\nu}{2} a_m \int_0^{2\pi} x \sin \nu x\, dx + \frac{m}{2} \frac{a_{\nu-m}}{\nu-m} \int_0^{2\pi} \sin^2 \nu x\, dx. \tag{36}$$

It follows from (34) and (36) that

$$\int_0^{2\pi} \cos nx\, d\phi$$

is bounded in n. Also

$$\int_0^{2\pi} \sin \nu x\, d\phi = -\nu \int_0^{2\pi} \phi(x) \cos \nu x\, dx, \tag{37}$$

and

$$\nu \int_0^{2\pi} \cos \nu x\, \phi(x)\, dx = -\frac{\nu}{2} \frac{b_{\nu-m}}{\nu-m} \int_0^{2\pi} \cos^2 \nu x\, dx - \frac{\nu}{2} \frac{b_{\nu+m}}{\nu+m} \int_0^{2\pi} \cos^2 \nu x\, dx$$
$$+ \frac{m}{2} \frac{b_{\nu-m}}{\nu-m} \int_0^{2\pi} \cos^2 \nu x\, dx - \frac{m}{2} \frac{b_{\nu+m}}{\nu+m} \int_0^{2\pi} \cos^2 \nu x\, dx. \tag{38}$$

From (37) and (38) it follows that

$$\int_0^{2\pi} \sin nx\, d\phi$$

is bounded in n. Thus $\phi(x)$ is of class β. We must show that

$$\lim_{n\to\infty} \int_0^{2\pi} e^{inx}\, d\phi = 0 \quad (n \subset T).$$

Since T is a central sequence of S, we have for $n \geqslant n(m)$ and $n \subset T$ that $n \pm m$ belong to S. Hence

$$\lim_{n\to\infty} a_{n\pm m} = \lim_{n\to\infty} b_{n\pm m} = 0 \quad (n \subset T).$$

In (36) put $\nu = n$, where $n \subset T$. We get

$$n \int_0^{2\pi} \sin nx\, \phi(x)\, dx = -\pi a_m + o(1).$$

Substituting in (34), we find by (32) that

$$\int_0^{2\pi} \cos nx \, d\phi = o(1) \quad (n \subset T).$$

In (38) put $\nu = n$, where $n \subset T$. We get

$$n \int_0^{2\pi} \cos nx \, \phi(x) \, dx = o(1),$$

so that, by (37),

$$\int_0^{2\pi} \sin nx \, d\phi = o(1) \quad (n \subset T).$$

This proves that $a_m = 0$. Replacing $\cos mx$ by $\sin mx$ in the above proof, we show that $b_m = 0$.

9. We shall now give the investigations which are analogous to those of §§ 7, 8, but in which the class β is replaced by the class β^*.

THEOREM XIII. *Let $f(x)$ be of class β^* and associated with the dense sequence S. Let R be a compact set determined by a dense central sequence of S. Let (μ_p) be a regular sequence of R. Let*

$$t(x) = \tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + \beta_n \sin nx)$$

possess a bounded differential coefficient. Then

$$\lim_{p \to \infty} \int_0^{2\pi} t(\mu_p x) \, df = \frac{f(2\pi) - f(0)}{2\pi} \int_0^{2\pi} t(x) \, dx.$$

The proof follows the method used for Theorem IX; (μ_p) replaces (n_p), and Theorem VII is used.

THEOREM XIV. *Let $f(x)$ be of class β^* and associated with the dense sequence S. Let R be a compact set determined by a dense central sequence of S. Let (μ_p) be a regular sequence of R. If σ is a closed set of type H with respect to (μ_p), and $f(x)$ is constant in each of the intervals contiguous to σ, then $f(2\pi) = f(0)$.*

The proof follows the method used for Theorem X; Theorem XIII is used instead of Theorem IX.

In order to obtain the analogue of Theorem XI, we need the following lemma.

LEMMA 10. *Let (μ_p) be a sequence of positive numbers such that $\lim \mu_{p+1}/\mu_p = \infty$. Given any ξ satisfying $0 < \xi < 2\pi$, there is an every-*

where dense set $G(\xi)$ of numbers in $(0, 2\pi)$ such that, if $x \subset G(\xi)$, then $\lim_{p\to\infty} E(\mu_p x) = \xi$.

We must show that any assigned interval (a, b) in $(0, 2\pi)$ contains a point of $G(\xi)$. Let (ϵ_p) be a sequence which tends to zero. The points x of $(0, 2\pi)$, for which

$$\xi - \epsilon_p \leqslant E(\mu_p x) \leqslant \xi + \epsilon_p,$$

consist of those points of the series of intervals

$$\left\{\left(\frac{\xi-\epsilon_p+2r\pi}{\mu_p}, \frac{\xi+\epsilon_p+2r\pi}{\mu_p}\right)\right\} \quad (r = 0, \pm 1, \pm 2, \ldots)$$

which lie in $(0, 2\pi)$. Each interval is of length $2\epsilon_p/\mu_p$, and consecutive intervals are separated by a distance less than $2\pi/\mu_p$. If, then, we choose $\epsilon_p = 6\pi\mu_p/\mu_{p+1}$, we have

$$\frac{2\pi}{\mu_{p+1}} = \tfrac{1}{6}\frac{2\epsilon_p}{\mu_p},$$

and therefore there is an interval of the series

$$\left\{\left(\frac{\xi-\epsilon_{p+1}+2r\pi}{\mu_{p+1}}, \frac{\xi+\epsilon_{p+1}+2r\pi}{\mu_{p+1}}\right)\right\} \quad (r = 0, \pm 1, \pm 2, \ldots)$$

in every interval of the preceding series. We can find a value p_0 of p such that (a, b) contains an interval δ_0 of the series corresponding to p_0. There is an interval $\delta_1 \subset \delta_0$ of the series corresponding to p_0+1, and so on. The limit point of these intervals is a point of $G(\xi)$.

THEOREM XV. *Let*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \tag{39}$$

be a trigonometric series with the properties

$$\text{(i)} \quad a_n = O(1), \quad b_n = O(1),$$

$$\text{(ii)} \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \qquad (n \subset S),$$

where S is a dense sequence. Let R be a compact set determined by a central sequence of S. Let (μ_p) be a regular sequence of R, and let σ be a closed set of type H with respect to (μ_p). If (39) is summable by Poisson's method to zero in the set complementary to σ, then $a_0 = 0$.

We may suppose that (μ_p) satisfies the relation $\mu_{p+1}/\mu_p \to \infty$. For we can find a subsequence of (μ_p) which satisfies this relation, and the subsequence

is a regular sequence of R, and σ is of type H with respect to it. Let the set σ be continued outside $(0, 2\pi)$ periodically. Since σ is closed and non-dense we can find an η, arbitrarily small, belonging to the set $G(\xi)$ of Lemma 10, where ξ is any assigned number within $(0, 2\pi)$, such that, if the periodic set σ is displaced a distance η to the right, then the points $x = 0, 2\pi$ will be interior to contiguous intervals of the displaced periodic set. If we denote that part of the displaced periodic set which lies in $(0, 2\pi)$ by σ', then ξ and η can be chosen so that σ' is of type H with respect to (μ_p) for $p \geqslant p_0$. For, since $\Sigma\sigma_p$ is not everywhere dense in $(0, 2\pi)$, there is an interval (α, β) interior to $(0, 2\pi)$ which is free from points of $\Sigma\sigma_p$. We choose ξ to satisfy $0 < \xi < (\beta-\alpha)/4$. Then, since $\lim E(\mu_p \eta) = \xi$, the periodic set σ_p' consists of the periodic set σ_p displaced a distance $\xi+\epsilon_p$ to the right, where $\epsilon_p \to 0$ as $p \to \infty$; so that, from and after a certain value p_0 of p, all the sets σ_p' leave the centre fifth of (α, β) uncovered. It follows that σ' is of type H with respect to (μ_p) for $p \geqslant p_0$. We may clearly neglect values of $p < p_0$ for the purpose of the theorem.

The series

$$\tfrac{1}{2}a_0+\Sigma\{a_n \cos n(x-\xi)+b_n \sin n(x-\xi)\}$$

is summable by Poisson's method to zero in the intervals complementary to σ'. We can write the preceding series as

$$\tfrac{1}{2}a_0+\Sigma(a_n' \cos nx+b_n' \sin nx),$$

where a_n', b_n' satisfy the same conditions as a_n, b_n. Since $x = 0, 2\pi$ lie within intervals complementary to σ', the series

$$\tfrac{1}{2}a_0 x+\Sigma \frac{a_n' \sin nx-b_n' \cos nx}{n} \tag{40}$$

converges almost everywhere in $(0, 2\pi)$ to a function $\psi(x)$, for which $\psi(2\pi)-\psi(0) = \pi a_0$, which is constant in the intervals contiguous to σ', and which is of class β^* and associated with S. By Theorem XIV, $a_0 = 0$.

THEOREM XVI. *Let*

$$\tfrac{1}{2}a_0+\Sigma(a_n \cos nx+b_n \sin nx) \tag{41}$$

be a trigonometric series with the properties

(i) $a_n = O(1), \quad b_n = O(1),$

(ii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \quad (n \subset S),$

where S is a dense sequence. Let T be a central sequence of S. Let R be a

compact set determined by a central sequence of T. Let σ be a closed set of type H with respect to (μ_p), where (μ_p) is a regular sequence of R. Let the series (41) *be summable by Poisson's method to zero in the intervals complementary to σ. Then*

$$a_0 = 0, \quad a_n = 0, \quad b_n = 0 \quad (n = 1, 2, \ldots).$$

By Theorem XV, $a_0 = 0$. As in that theorem, we choose the number ξ so that the series

$$\Sigma(a_n' \cos nx + b_n' \sin nx),$$

where

$$a_n' = a_n \cos n\xi - b_n \sin n\xi, \quad b_n' = b_n \cos n\xi + a_n \sin n\xi,$$

is summable by Poisson's method to zero in the set complementary to σ', where $x = 0, 2\pi$ do not belong to σ'. We now follow the proof of Theorem XII, but in the present case, the series which corresponds to (31), namely

$$\sum_{n=1}^{\infty} \left(\frac{a_n'}{n} \sin nx - \frac{b_n'}{n} \cos nx\right) \cos mx + \int_0^x \sum_{n=1}^{\infty} \left(\frac{a_n'}{n} \sin nt - \frac{b_n'}{n} \cos nt\right) m \sin mt \, dt, \quad (42)$$

converges almost everywhere in $(0, 2\pi)$ to a function $\psi(x)$ such that $\psi(2\pi) - \psi(0) = \pi a_m'$; such that $\psi(x)$ is constant in the intervals contiguous to σ'; and such that $\psi(x)$ is of class β^* and associated with T. By Theorem XIV, $a_m' = 0$. Similarly, $b_m' = 0$. If we choose ξ so that $\sin 2m\xi > 0$, we infer $a_m = b_m = 0$.

10. We shall now give the investigations which are analogous to those of §§ 7, 8, but in which the class β is replaced by an extension of the class γ; an extension obtained by replacing the condition of L-integrability by a wider one.

Definition 4. Let S be a sequence of positive integers. A sequence (n_p) $(p = 1, 2, \ldots)$, with $\lim n = \infty$, will be said to be a multiplicative sequence of S if there is a $\delta > 0$ such that, for all p, the numbers $n_p, 2n_p, 3n_p, \ldots, [n_p \delta] n_p$, belong to S.

THEOREM XVII. *Let $f(x)$ be of class γ and associated with S. Let*

$$t(x) = \tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + \beta_n \sin nx)$$

$$[a_n = O(1/n^3), \quad \beta_n = O(1/n^3)].$$

Let (n_p) be a multiplicative sequence of S. Then

$$\lim_{p\to\infty}\int_0^{2\pi} t(n_p x)\,df = \frac{f(2\pi)-f(0)}{2\pi}\int_0^{2\pi} t(x)\,dx.$$

We have

$$\int_0^{2\pi} t(n_p x)\,df = t(2\pi)f(2\pi)-t(0)f(0)-n_p\int_0^{2\pi} f(x)\,t'(n_p x)\,dx; \qquad (43)$$

and

$$t'(x) = \frac{i}{2}\sum_{r=-\infty}^{\infty} r(a_r-i\beta_r)\,e^{irx},$$

so that

$$-n_p\int_0^{2\pi} f(x)\,t'(n_p x)\,dx$$

$$= -\frac{n_p i}{2}\left[\sum_{r=1}^{\infty} r(a_r-i\beta_r)\int_0^{2\pi} f(x)\,e^{irn_p x}\,dx - \sum_{r=1}^{\infty} r(a_r+i\beta_r)\int_0^{2\pi} f(x)\,e^{-irn_p x}\,dx\right].$$

Since (n_p) is a multiplicative sequence of S, we have $rn_p \subset S$ for $r = 1, 2, \ldots, [n_p\delta]$. Hence

$$\lim_{p\to\infty}\int_0^{2\pi} e^{\pm irn_p x}\,df = 0$$

uniformly for $r = 1, 2, \ldots, [n_p\delta]$; or

$$\lim_{p\to\infty} \pm irn_p\int_0^{2\pi} f(x)\,e^{irn_p x}\,dx = f(2\pi)-f(0)$$

uniformly for $r = 1, 2, \ldots, [n_p\delta]$. Hence

$$\lim_{p\to\infty} -n_p\int_0^{2\pi} f(x)\,t'(n_p x)\,dx$$

$$= \lim_{p\to\infty} -\tfrac{1}{2}\sum_{r=-[n_p\delta]}^{[n_p\delta]}{}' (a_r-i\beta_r)\{f(2\pi)-f(0)\}$$

$$-\lim_{p\to\infty}\frac{n_p i}{2}\left[\sum_{r=[n_p\delta]+1}^{\infty} r(a_r-i\beta_r)\int_0^{2\pi} f(x)\,e^{irn_p x}\,dx\right.$$

$$\left. -\sum_{r=[n_p\delta]+1}^{\infty} r(a_r+i\beta_r)\int_0^{2\pi} f(x)\,e^{-irn_p x}\,dx\right]. \qquad (44)$$

Since $f(x)$ is integrable L, we have, in the last expression,

$$\int_0^{2\pi} f(x)\,e^{\pm irn_p x}\,dx = o(1) \qquad (45)$$

as $p \to \infty$, for relevant r. Hence the second term on the right in (44) is

$$o\left[n_p \sum_{r=[n_p\delta]}^{\infty} \frac{1}{r^2}\right],$$

and thus

$$\lim_{p\to\infty} -n_p \int_0^{2\pi} f(x)\, t'(n_p x)\, dx = -\tfrac{1}{2} \sum_{-\infty}^{\infty}{}' (\alpha_r - i\beta_r)\{f(2\pi)-f(0)\}$$

$$= -\{t(0)-\tfrac{1}{2}a_0\}\{f(2\pi)-f(0)\}.$$

By (43), $$\lim_{p\to\infty} \int_0^{2\pi} t(n_p x)\, df = \tfrac{1}{2}a_0\{f(2\pi)-f(0)\}.$$

For the purpose of applications to the theory of trigonometric series the function $t(x)$ in the preceding theorem may be subjected to additional restrictions; and, in that case, the class of functions $f(x)$, for which the preceding theorem is true, may be extended. We call "the extended class γ" that class obtained by replacing integrability in the sense of Lebesgue by integrability in the general Denjoy sense. We assign a corresponding meaning to "the extended class γ^*". In that case the symbol

$$\int_0^{2\pi} g(x)\, df$$

will have a meaning if $g'(x)$ is of bounded variation. If we apply the preceding proof to the extended class γ, the relation (45) is now replaced by

$$\int_0^{2\pi} f(x)\, e^{\pm irn_p x}\, dx = o(rn_p),$$

as $p \to \infty$, uniformly for $r > [n_p\delta]$. If we assume that $\alpha_n = O(1/n^5)$, $\beta_n = O(1/n^5)$ the conclusion will hold. We can thus enunciate the following theorem, where "integrable D" means "integrable in the general Denjoy sense".

THEOREM XVIII. *Let $f(x)$ be integrable D, of the extended class γ, and associated with S. Let*

$$t(x) = \tfrac{1}{2}a_0 + \Sigma(\alpha_n \cos nx + \beta_n \sin nx)$$

$$[\alpha_n = O(1/n^5),\ \beta_n = O(1/n^5)].$$

Let (n_p) be a multiplicative sequence of S. Then

$$\lim_{p\to\infty} \int_0^{2\pi} t(n_p x)\, df = \frac{f(2\pi)-f(0)}{2\pi} \int_0^{2\pi} t(x)\, dx. \tag{46}$$

THEOREM XIX. *Let $f(x)$ be of the extended class γ and associated with S. Let σ be a closed set of type H with respect to (n_p), where (n_p) is a multiplicative*

sequence of S. If $f(x)$ is constant in each of the intervals contiguous to σ, then $f(2\pi) = f(0)$.

By hypothesis, the set $G = \Sigma \sigma_p$ is not everywhere dense in $(0, 2\pi)$. We can therefore find an interval δ interior to $(0, 2\pi)$ which is free from points of G. We can define a function $t(x)$ in $(0, 2\pi)$ by

$$t(x) = 0 \quad (x \subset C\delta), \qquad t(x) > 0 \quad (x \subset \delta),$$

such that $t(x)$ has five continuous derivatives; and we define $t(x)$ outside $(0, 2\pi)$ by the condition of periodicity. Then (46) holds. For a fixed p, $t(n_p x) = 0$ for $x \subset \sigma$, so that, if we write $t(n_p x) = \tau(x)$, we have $\tau(x) = 0$ for $x \subset \sigma$, $\tau(0) = \tau(2\pi) = 0$, and $\tau(x)$ has five continuous derivatives. Then

$$\int_0^{2\pi} t(n_p x)\, df = \int_0^{2\pi} \tau(x)\, df$$

$$= -\int_0^{2\pi} f(x)\, \tau'(x)\, dx.$$

The set σ is of measure zero. Let $\{\delta_i\} \equiv \{(\alpha_i, \beta_i)\}$ denote its contiguous intervals. Then

$$\int_{\alpha_i}^{\beta_i} f(x)\, \tau'(x)\, dx = c_i\{\tau(\beta_i) - \tau(\alpha_i)\} = 0,$$

where c_i is the value of $f(x)$ in δ_i, since $\tau(\beta_i) = \tau(\alpha_i) = 0$ because β_i, α_i belong to σ. Now we are given that $f(x)\tau'(x)$ is integrable D in $(0, 2\pi)$. Hence, since

$$\sum_{(i)} \int_{\alpha_i}^{\beta_i} f(x)\, \tau'(x)\, dx$$

certainly forms an absolutely convergent series, and the Lebesgue integral

$$\int_\sigma f(x)\, \tau'(x)\, dx$$

is zero because $m\sigma = 0$, we have

$$\int_0^{2\pi} f(x)\, \tau'(x)\, dx = \sum_{(i)} \int_{\alpha_i}^{\beta_i} f(x)\, \tau'(x)\, dx$$

$$= 0.$$

Thus the second member of (46) is zero; this implies that $f(2\pi) = f(0)$.

THEOREM XX. *Let*

$$\tfrac{1}{2}a_0+\Sigma(a_n\cos nx+b_n\sin nx) \tag{47}$$

be the series obtained by formal differentiation of a series whose periodic part is a Denjoy-Fourier series. Let

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \quad (n \subset S).$$

Let (n_p) be a multiplicative sequence of S, and let σ be a closed set of type H with respect to (n_p). Let (47) be summable by Poisson's method to zero in the set complementary to σ. Then $a_0 = 0$.

The series
$$\tfrac{1}{2}a_0 x+\Sigma\left(\frac{a_n}{n}\sin nx-\frac{b_n}{n}\cos nx\right) \tag{48}$$

is uniformly summable by Poisson's method to a constant, in an interval interior to a contiguous interval of σ. If the points $x = 0, 2\pi$ do not belong to σ, we define a function $\phi(x)$, which is integrable D, as the Poisson sum of (48) wherever it exists. If $x = 0, 2\pi$ belong to σ, we define $\phi(x)$ by

$$\phi(0) = 0, \quad \phi(2\pi) = \pi a_0,$$

and $\phi(x)$ for $0 < x < 2\pi$ equal to the Poisson sum of (48) wherever it exists. In either case, $\phi(2\pi)-\phi(0) = \pi a_0$. By Theorem XIX, it remains to prove that $\phi(x)$ is associated with S. The proof is the same as in Theorem XI.

THEOREM XXI. *Let*

$$\tfrac{1}{2}a_0+\Sigma(a_n\cos nx+b_n\sin nx) \tag{49}$$

be the series obtained by formal differentiation of a series whose periodic part is a Denjoy-Fourier series. Let

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \quad (n \subset S).$$

Let (n_p) be a multiplicative sequence of a central sequence T of S, and let σ be a closed set of type H with respect to (n_p). Let the series (49) be summable by Poisson's method to zero in the set complementary to σ. Then

$$a_0 = 0, \ a_n = 0, \ b_n = 0 \quad (n = 1, 2, \ldots).$$

By Theorem XX, $a_0 = 0$. The series

$$\Sigma(a_n\cos nx+b_n\sin nx) \tag{50}$$

is summable to zero by Poisson's method in $C\sigma$; so then is

$$\Sigma(a_n\cos nx+b_n\sin nx)\cos mx, \tag{51}$$

where m is a fixed positive integer. Further, if δ denotes an interval contiguous to σ and (α, β) an interval interior to δ, the summability of (50) and (51) is uniform in (α, β). Hence for $\alpha < x < \beta$ we have

$$\lim_{r\to 1} \Sigma \int_\alpha^x (a_n \cos nt + b_n \sin nt) \cos mt\, r^n\, dt = 0,$$

or

$$\lim_{r\to 1} \left\{ \left[\sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nt - \frac{b_n}{n} \cos nt \right) \cos mt\, r^n \right]_\alpha^x \right.$$

$$\left. + \int_\alpha^x \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nt - \frac{b_n}{n} \cos nt \right) m \sin mt\, r^n dt \right\} = 0. \quad (52)$$

We write

$$\psi(x) \sim \Sigma \left(\frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx \right), \quad (53)$$

where, by hypothesis, $\psi(x)$ is integrable D; and

$$\psi(r, t) = \Sigma \left(\frac{a_n}{n} \sin nt - \frac{b_n}{n} \cos nt \right) r^n, \quad (54)$$

$$\Psi(x) = \int_0^x \psi(t)\, dt;$$

and we use $\Psi(r, t)$ to denote the harmonic function associated with the Fourier series of $\Psi(t)$. We write

$$P_1(r, x) = \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx \right) \cos mx\, r^n, \quad (55)$$

so that, since (53) is summable by Poisson's method to $\psi(x)$ for $x \subset C\sigma$, we have

$$\lim_{r\to 1} P_1(r, x) = \psi(x) \cos mx \quad (x \subset C\sigma). \quad (56)$$

We further write

$$P_2(r, x) = \int_0^x \psi(r, t)\, m \sin mt\, dt \quad (57)$$

$$= m \sin mx\, \Psi(r, x) - m^2 \int_0^x \Psi(r, t) \cos mt\, dt.$$

Since $\Psi(x)$ is continuous, we have

$$\lim_{r\to 1} P_2(r, x) = m \sin mx\, \Psi(x) - m^2 \int_0^x \Psi(t) \cos mt\, dt$$

$$= \int_0^x \psi(t)\, m \sin mt\, dt, \quad (58)$$

for all x. The relation (52) is

$$\lim_{r\to 1}\Big[P_1(r,x)+P_2(r,x)\Big]_a^x = 0,$$

so that, by (56) and (58), the function

$$\psi(x)\cos mx+\int_0^x \psi(t)\, m\sin mt\, dt \tag{59}$$

is constant in each of the intervals contiguous to σ. If $x=0,\ 2\pi$ do not belong to σ, we define a function $\phi(x)$ equal to (59) in $C\sigma$; then

$$\phi(2\pi)-\phi(0) = \int_0^{2\pi}\psi(t)\, m\sin mt\, dt$$

$$= \pi a_m,$$

by (53). If $x=0,\ 2\pi$ belong to σ, we define a function $\phi(x)$ by

$$\phi(0)=0, \quad \phi(2\pi)=\pi a_m,$$

and $\phi(x)$ for $C\sigma(0<x<2\pi)$ equal to (59). In either case,

$$\phi(2\pi)-\phi(0)=\pi a_m,$$

$\phi(x)$ is integrable D, and constant in each interval contiguous to σ. To infer, by Theorem XIX, that $a_m=0$, we must show that $\phi(x)$ is associated with T. Let $\nu>m$ belong to T. Then

$$\int_0^{2\pi}\cos\nu x\, d\phi = \phi(2\pi)-\phi(0)+\nu\int_0^{2\pi}\sin\nu x\,\phi(x)\,dx$$

$$= \pi a_m+\nu\int_0^{2\pi}\sin\nu x\,\phi(x)\,dx.$$

Further,

$$\nu\int_0^{2\pi}\sin\nu x\,\phi(x)\,dx = \nu\int_0^{2\pi}\sin\nu x\Big[\psi(x)\cos mx+\int_0^x\psi(t)\, m\sin mt\,dt\Big]dx$$

$$= \frac{\nu}{2}\int_0^{2\pi}\psi(x)\{\sin(\nu+m)x+\sin(\nu-m)x\}\,dx$$

$$-\int_0^{2\pi}\frac{d}{dx}(\cos\nu x).\int_0^x\psi(t)\, m\sin mt\,dt\,.\,dx$$

$$= J_1-J_2.$$

From (53),

$$J_1 = \frac{\nu}{2}\Big[\frac{a_{\nu+m}}{\nu+m}+\frac{a_{\nu-m}}{\nu-m}\Big].$$

Since T is a central sequence of S, we have, as $\nu \to \infty$ in T, $\nu \pm m \subset S$, $a_{\nu \pm m} = o(1)$, $J_1 = o(1)$. Further,

$$\begin{aligned} J_2 &= -\int_0^{2\pi} \psi(t)\, m \sin mt\, dt + \int_0^{2\pi} \psi(x)\, m \sin mx \cos \nu x\, dx \\ &= -\pi a_m + \frac{m}{2} \int_0^{2\pi} \psi(x) \{\sin(\nu+m)x - \sin(\nu-m)x\}\, dx \\ &= -\pi a_m + \frac{m}{2} \left[\frac{a_{\nu-m}}{\nu+m} - \frac{a_{\nu-m}}{\nu-m} \right] \\ &= -\pi a_m + o(1). \end{aligned}$$

Hence
$$\int_0^{2\pi} \cos \nu x\, d\phi = o(1),$$

as $\nu \to \infty$ in T. Similarly,

$$\int_0^{2\pi} \sin \nu x\, d\phi = o(1),$$

as $\nu \to \infty$ in T. Thus $a_m = 0$; and, by replacing $\cos mx$ by $\sin mx$, we show that $b_m = 0$.

11. We now give the investigations which correspond to those of § 10 when the extended class γ is replaced by the extended class γ^*. We must first show that Theorem VIII remains true if $f(x)$ belongs to the extended class γ^*.

LEMMA 11. *If $f(x)$ belongs to the extended class γ and is associated with the complete sequence S, and if*

$$t(x) = \tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + \beta_n \sin nx)$$

$$[a_n = O(1/n^3),\ \beta_n = O(1/n^3)],$$

then
$$\lim_{p \to \infty} \int_0^{2\pi} e^{in_p x}\, d(ft) = 0,$$

where (n_p) is a principal sequence of S.

The proof follows the method of Theorem III; we require the stronger restriction on the Fourier constants of $t(x)$ because we can only assert that

$$\int_0^{2\pi} f(x)\, e^{inx}\, dx = o(n)$$

as $n \to \infty$.

LEMMA 12. *If $f(x)$ is of the extended class γ^* and is associated with the complete sequence S, and if $0 < h < \delta$, then*

$$\lim_{p\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x}\, df = 0,$$

where (n_p) is any principal sequence of S.

We define a function $t(x)$ by

$$t(x) = 0 \quad (h+1/m \leqslant x \leqslant 2\pi-h-1/m)$$
$$= 1 \quad (0 \leqslant x \leqslant h,\ 2\pi-h \leqslant x \leqslant 2\pi),$$

and by the conditions that $t(x)$ is monotone in

$$(h,\ h+1/m),\ (2\pi-h-1/m,\ 2\pi-h),$$

and possesses three continuous derivatives. Applying Lemma 11, we get

$$\lim_{p\to\infty} \int_0^{h+1/m} + \int_{2\pi-h-1/m}^{2\pi} e^{in_p x}\, d(ft) = 0.$$

This relation is then treated as in the proof of Theorem VI. From Lemmas 12 and 4 we infer

LEMMA 13. *Let $f(x)$ be of the extended class γ^* and associated with the complete sequence S. Let $0 < h < \delta$. If $\{\lambda_p(x)\}$ is a uniformly bounded semi-convergent sequence of B-measurable functions, then*

$$\lim_{p\to\infty} \int_0^h + \int_{2\pi-h}^{2\pi} e^{in_p x}\, \lambda_p(x)\, df = 0,$$

where (n_p) is any central sequence of a principal sequence T of S.

By examining the proof of Lemma 11, we infer the first part of

LEMMA 14. *If $f(x)$ belongs to the extended class γ and is associated with the complete sequence S, and if*

$$t_p(x) = \tfrac{1}{2}a_0^{(p)} + \Sigma(a_n^{(p)} \cos nx + \beta_n^{(p)} \sin nx),$$

where

$$|a_0^{(p)}| < K, \quad |a_n^{(p)}| < K/n^4, \quad |\beta_n^{(p)}| < K/n^4,$$

then

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, d(ft_p) = 0,$$

and

$$\lim_{p\to\infty} \int_0^{2\pi} e^{in_p x}\, t_p(x)\, df = 0,$$

where (n_p) is any principal sequence of S.

To prove the second part, we observe that

$$\int_0^{2\pi} e^{in_p x}\, t_p(x)\, df = \int_0^{2\pi} e^{in_p x}\, d(ft_p) - \int_0^{2\pi} f(x)\, t_p'(x)\, e^{in_p x}\, dx.$$

Since (n_p) is a principal sequence of S, there is an $\eta > 0$ such that all the integers of the interval $n_p - n_p\eta$, $n_p + n_p\eta$ belong to S. Writing

$$\sigma_p(x) = \frac{i}{2} \sum_{r=-[n_p\eta]}^{[n_p\eta]} r(\alpha_r^{(p)} - i\beta_r^{(p)})\, e^{irx},$$

we have

$$\int_0^{2\pi} f(x)\, \sigma_p(x)\, e^{in_p x}\, dx = \frac{i}{2} \sum_{r=-[n_p\eta]}^{[n_p\eta]} r(\alpha_r^{(p)} - i\beta_r^{(p)}) \int_0^{2\pi} f(x)\, e^{i(n_p+r)x}\, dx$$

$$= o\left[\sum_{-[n_p\eta]}^{[n_p\eta]}{}' \frac{1}{r^3(n_p+r)} \right],$$

the dash denoting that the value $r = 0$ is omitted. We have, of course, taken $\eta < 1$. Hence

$$\lim_{p\to\infty} \int_0^{2\pi} f(x)\, t_p'(x)\, e^{in_p x}\, dx = \lim_{p\to\infty} \int_0^{2\pi} f(x) \{t_p'(x) - \sigma_p(x)\}\, e^{in_p x}\, dx;$$

and the expression under the second limiting sign is

$$\frac{i}{2} \sum_{[n_p\eta]+1}^{\infty} r(\alpha_r^{(p)} - i\beta_r^{(p)}) \int_0^{2\pi} f(x)\, e^{i(n_p+r)x}\, dx$$

$$- \frac{i}{2} \sum_{[n_p\eta]+1}^{\infty} r(\alpha_r^{(h)} + i\beta_r^{(h)}) \int_0^{2\pi} f(x)\, e^{i(n_p-r)x}\, dx,$$

which is

$$o\left[\sum_{[n_p\eta]}^{\infty} \frac{n_p+r}{r^3} \right] + o\left[\sum_{[n_p\eta]}^{\infty} \frac{|n_p-r|}{r^3} \right] = o(1).$$

LEMMA 15. *If $f(x)$ belongs to the extended class γ^* and is associated with the complete sequence S, and if $\{\theta_p\}$ is a sequence of real numbers which satisfy $|\theta_p| \leqslant 1$, we have*

$$\lim_{p\to\infty} \int_0^{2\pi} e^{i(n_p+\theta_p)x}\, df = 0,$$

where (n_p) is any central sequence of a principal sequence T of S.

If we examine the proof of Lemma 7, we see that the problem of proving Lemma 15 is to define a sequence $\{t_p(x)\}$ of functions which satisfies the conditions of Lemma 14, and such that $t_p(x)$ is equal to $\cos\theta_p x (\sin\theta_p x)$ in $(h, 2\pi - h)$. We write

$$t_p(x) = \sum_{r=5}^{9} \alpha_r^{(p)} x^r \quad (0 \leqslant x \leqslant h).$$

Then $t_p(x)$ and its first four derivatives vanish at $x = 0$. We require that, at $x = h$, $t_p(x)$ and its first four derivatives shall be equal to $\cos \theta_p x$ and its first four derivatives. This gives us a system of five linear equations for the five unknowns $\alpha_r^{(p)}$. Each $\alpha_r^{(p)}$ is the quotient of two determinants; the numerator is bounded in p, and the denominator is a non-zero constant. If, further, we write

$$t_p(x) = \sum_{r=5}^{9} \beta_r^{(p)} (2\pi - x)^r \quad (2\pi - h \leqslant x \leqslant 2\pi)$$

and subject this expression to corresponding conditions, we see that $\{t_p(x)\}$ possesses the desired properties. In virtue of Lemma 15, we can enunciate the analogue of Theorem VIII.

THEOREM XXII. *Let $f(x)$ be of the extended class γ^* and associated with the complete sequence S. If R is a compact set determined by a central sequence of a principal sequence T of S, then*

$$\lim_{\mu \to \infty} \int_0^{2\pi} e^{i\mu x} df = 0 \quad (\mu \subset R).$$

12. We now extend the definition of a " multiplicative sequence " to a sequence of real numbers. The sequence (μ_p) with $\mu_p \to \infty$ is a multiplicative sequence of the compact set R if there is a $\delta > 0$ such that the numbers μ_p, $2\mu_p$, $3\mu_p$, ..., $[\mu_p \delta]\mu_p$ belong to R for all p.

THEOREM XXIII. *Let $f(x)$ be of the extended class γ^* and associated with the complete sequence S. Let*

$$t(x) = \tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + \beta_n \sin nx)$$

$$[a_n = O(1/n^5), \ \beta_n = O(1/n^5)].$$

Let R be a compact set determined by a central sequence of a principal sequence T of S; and let (μ_p) be a multiplicative sequence of R. Then

$$\lim_{p \to \infty} \int_0^{2\pi} t(\mu_p x)\, df = \frac{f(2\pi) - f(0)}{2\pi} \int_0^{2\pi} t(x)\, dx.$$

The proof follows the method of Theorem XVIII. We merely observe that

$$\int_0^{2\pi} f(x) e^{i\mu x} dx = o(\mu)$$

as $\mu \to \infty$, and that

$$\lim_{p \to \infty} \int_0^{2\pi} e^{\pm i r \mu_p x} df = 0$$

uniformly for $r = 1, 2, \ldots, [\mu_p \delta]$, by Theorem XXII.

THEOREM XXIV. *Let $f(x)$ be of the extended class γ^* and associated with the complete sequence S. Let R be a compact set determined by a central sequence of a principal sequence T of S. Let σ be a closed set of type H with respect to (μ_p), where (μ_p) is a multiplicative sequence of R. If $f(x)$ is constant in each of the intervals contiguous to σ, then $f(2\pi) = f(0)$.*

The proof follows the method of Theorem XIX; in this case, Theorem XXIII is used instead of Theorem XVIII.

THEOREM XXV. *Let*

$$\tfrac{1}{2}a_0+\Sigma(a_n \cos nx+b_n \sin nx) \tag{60}$$

be the series obtained by formal differentiation of a series whose periodic part is a Denjoy-Fourier series. Let

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \quad (n \subset S).$$

Let R be a compact set determined by a central sequence of a principal sequence T of S. Let (μ_p) be a multiplicative sequence of R, and let σ be a closed set of type H with respect to (μ_p). Let the series (60) be summable by Poisson's method to zero in the intervals complementary to σ. Then $a_0 = 0$.

As in the proof of Theorem XV, we can find $\xi > 0$ such that the series

$$\tfrac{1}{2}a_0+\Sigma\{a_n \cos n(x-\xi)+b_n \sin n(x-\xi)\}$$

is summable by Poisson's method to zero in the intervals complementary to a set σ' which is of type H with respect to (μ_p), and such that $x = 0,\ 2\pi$ do not belong to σ'. In this case we define a function $\psi(x)$ almost everywhere in $(0,\ 2\pi)$ as the Poisson sum of

$$\tfrac{1}{2}a_0 x+\Sigma \frac{a_n' \sin nx-b_n' \cos nx}{n}, \tag{61}$$

where

$$a_n' = a_n \cos n\xi-b_n \sin n\xi, \quad b_n' = b_n \cos n\xi+a_n \sin n\xi.$$

Then $\psi(x)$ is integrable D and is of the extended class γ^*. To prove that it is associated with S presents no new difficulty. Finally, since $\psi(2\pi)-\psi(0) = \pi a_0$, we have $a_0 = 0$ by Theorem XXIV.

THEOREM XXVI. *Let*

$$\tfrac{1}{2}a_0+\Sigma(a_n \cos nx+b_n \sin nx) \tag{62}$$

be the series obtained by formal differentiation of a series whose periodic part

is a Denjoy-Fourier series. Let

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0 \quad (n \subset S),$$

where S is a complete sequence. Let S' be a complete central sequence of S. Let R be the compact set determined by a central sequence of a principal sequence T of S'. Let (μ_p) be a multiplicative sequence of R and σ a closed set of type H with respect to (μ_p). Let the series (62) *be summable by Poisson's method to zero in the set complementary to σ. Then*

$$a_0 = 0, \quad a_n = 0, \quad b_n = 0 \quad (n = 1, 2, \ldots).$$

By Theorem XXV, $a_0 = 0$. Let m be a fixed positive integer. We can find $\xi > 0$ with $\sin 2m\xi > 0$, and such that

$$\Sigma(a_n' \cos nx + b_n' \sin nx) \tag{63}$$

is summable by Poisson's method to zero in the intervals complementary to a closed set σ' which is of type H with respect to (μ_p), where $x = 0, 2\pi$ do not belong to σ'. We now follow the proof of Theorem XXI, replacing σ by σ', a_n by a_n', and b_n by b_n'. We observe that (63) is obtained by the formal differentiation of a Denjoy-Fourier series. The function $\phi(x)$ is integrable D and is of the extended class γ^*. It is constant in each interval contiguous to σ'. It is associated with S', because, as $\nu \to \infty$ in S', $\nu \pm m$ belong to S. It then follows by Theorem XXIV, in which S is replaced by S', that $a_m' = 0$. Similarly, $b_m' = 0$. Since $\sin 2m\xi > 0$, this gives $a_m = b_m = 0$.

13. Finally we wish to show that there exist sets of unicity for trigonometric series which are of a less restrictive nature than those considered above, provided, however, that certain sequences of coefficients tend to zero with a specified rapidity. We shall make no attempt at generality in the enunciations, and shall confine ourselves to trigonometric series whose coefficients are bounded.

Given a closed set σ in $(0, 2\pi)$ and a sequence (n_p) of positive integers with $n_p \to \infty$, we denote by δ_p the upper bound of the lengths of intervals which are free from points of σ_p. Let k denote a positive integer.

THEOREM XXVII. *Let $f(x)$ be a function of class β. Let σ be a closed set of zero measure, and let $f(x)$ be constant in each of the intervals contiguous to σ. Let there be a sequence $\lambda_p \to \infty$ such that*

$$\operatorname*{Max}_{(k)} \left| \int_0^{2\pi} e^{ikn_p x}\, df \right| \Big/ \delta_p^2 \quad \left(1 \leqslant k \leqslant \left[\frac{\lambda_p}{\delta_p^2}\right]\right)$$

tends to zero as $p \to \infty$. Then $f(2\pi) = f(0)$.

By the definition of δ_p there is an interval (α_p, β_p) within $(0, 2\pi)$ and of length $d_p = \frac{1}{2}\delta_p$ which is free from points of σ_p. Let $t_p(x)$ be defined in $(0, 2\pi)$ by

$$\begin{aligned} t_p(x) &= 1 && (\alpha_p + \tfrac{1}{3}d_p \leqslant x \leqslant \beta_p - \tfrac{1}{3}d_p) \\ &= \sin[\tfrac{3}{2}\pi(x-\alpha_p)/d_p] && (\alpha_p \leqslant x \leqslant \alpha_p + \tfrac{1}{3}d_p) \\ &= -\cos[\tfrac{3}{2}\pi(x-\alpha_p)/d_p] && (\alpha_p + \tfrac{2}{3}d_p \leqslant x \leqslant \beta_p) \\ &= 0 && (0 \leqslant x \leqslant \alpha_p,\ \beta_p \leqslant x \leqslant 2\pi), \end{aligned}$$

and let $t_p(x)$ be defined outside $(0, 2\pi)$ by the condition of periodicity. Then the total variation of $t_p'(x)$ in $(0, 2\pi)$ does not exceed C/d_p, where C is an absolute constant. If we write

$$t_p(x) = \tfrac{1}{2}\sum_{r=-\infty}^{\infty}(\alpha_r^{(p)} - i\beta_r^{(p)})\, e^{irx},$$

we have

$$|\alpha_r^{(p)} - i\beta_r^{(p)}| < \frac{4C}{\pi r^2 d_p}.$$

Now

$$\begin{aligned} \int_0^{2\pi} t_p(n_p x)\, df &= -n_p \int_0^{2\pi} f(x)\, t_p'(n_p x)\, dx \\ &= -\frac{n_p i}{2}\sum_{r=-\infty}^{\infty} r(\alpha_r^{(p)} - i\beta_r^{(p)}) \int_0^{2\pi} f(x)\, e^{irn_p x}\, dx \\ &= -\frac{in_p}{2}\sum_{r=-m}^{m} r(\alpha_r^{(p)} - i\beta_r^{(p)}) \int_0^{2\pi} f(x)\, e^{irn_p x}\, dx \\ &\quad -\frac{in_p}{2}\Big[\sum_{r=m+1}^{\infty} r(\alpha_r^{(p)} - i\beta_r^{(p)}) \int_0^{2\pi} f(x)\, e^{irn_p x}\, dx \\ &\qquad - \sum_{r=m+1}^{\infty} r(\alpha_r^{(p)} + i\beta_r^{(p)}) \int_0^{2\pi} f(x)\, e^{-irn_p x}\, dx\Big]. \end{aligned}$$

Since

$$\int_0^{2\pi} e^{irn_p x}\, df = f(2\pi) - f(0) - irn_p \int_0^{2\pi} f(x)\, e^{irn_p x}\, dx,$$

we infer that

$$\begin{aligned} \int_0^{2\pi} t_p(n_p x)\, df &= \tfrac{1}{2}\sum_{r=-m}^{m}{}' (\alpha_r^{(p)} - i\beta_r^{(p)}) \Big\{\int_0^{2\pi} e^{irn_p x}\, df - \{f(2\pi) - f(0)\}\Big\} \\ &\quad -\frac{in_p}{2}\Big[\sum_{r=m+1}^{\infty} r(\alpha_r - i\beta_r) \int_0^{2\pi} f(x)\, e^{irn_p x}\, dx \\ &\qquad - \sum_{r=m+1}^{\infty} r(\alpha_r + i\beta_r) \int_0^{2\pi} f(x)\, e^{-irn_p x}\, dx\Big]. \end{aligned}$$

But
$$\tfrac{1}{2}\sum_{r=-m}^{m}{}'(\alpha_r^{(p)}-i\beta_r^{(p)}) = \sum_{r=1}^{m}\alpha_r^{(p)}$$
$$= -\tfrac{1}{2}a_0 - \sum_{r=m+1}^{\infty}\alpha_r^{(p)},$$

since
$$\tfrac{1}{2}a_0 + \sum_{r=1}^{\infty}\alpha_r^{(p)} = t_p(0) = 0.$$

Hence

$$\left|\int_0^{2\pi} t_p(n_p x)\,df - \frac{f(2\pi)-f(0)}{2\pi}\int_0^{2\pi} t_p(x)\,dx\right|$$
$$\leqslant \sum_{r=m+1}^{\infty}|\alpha_r^{(p)}| + \left|\tfrac{1}{2}\sum_{r=-m}^{m}{}'(\alpha_r^{(p)}-i\beta_r^{(p)})\int_0^{2\pi} e^{irn_p x}\,df\right|$$
$$+\left|\frac{in_p}{2}\sum_{r=m+1}^{\infty} r(\alpha_r - i\beta_r)\int_0^{2\pi} f(x)\,e^{irn_p x}\,dx - r(\alpha_r+i\beta_r)\int_0^{2\pi} f(x)\,e^{-irn_p x}\,dx\right|.$$

The first member is independent of m. We can therefore let $m\to\infty$ in the second member. We thus find that the first member does not exceed

$$\sum_{r=1}^{\infty}(|\alpha_r^{(p)}|+|\beta_r^{(p)}|)\left|\int_0^{2\pi} e^{irn_p x}\,df\right|$$
$$= o\left[\sum_{r=1}^{[\lambda_p/\delta_p{}^2]}\frac{4C}{\pi r^2 d_p}\,d_p{}^2\right] + O\left[\sum_{r=[\lambda_p/\delta_p{}^2]+1}^{\infty}\frac{4C}{\pi r^2 d_p}\right], \quad (64)$$

since
$$\left|\int_0^{2\pi} e^{inx}\,df\right| < K$$

for all n. The second member in (64) is

$$o(d_p) + O\left(\frac{d_p}{\lambda_p}\right) = o(d_p).$$

Hence
$$\left|\int_0^{2\pi} t_p(n_p x)\,df - \frac{f(2\pi)-f(0)}{2\pi}\int_0^{2\pi} t_p(x)\,dx\right| = o(d_p). \quad (65)$$

Consider the function $t_p(n_p x)$ for a fixed p. If $x\subset\sigma$, $E(n_p x)$ does not belong to (α_p, β_p), $t_p(n_p x) = 0$. We can therefore write $t_p(n_p x) = \tau(x)$, where $\tau(0) = \tau(2\pi) = 0$, $\tau(x) = 0$ for $x\subset\sigma$, and $\tau(x)$ has a bounded derivative. Then

$$\int_0^{2\pi} t_p(n_p x)\,df = \int_0^{2\pi}\tau(x)\,df$$
$$= -\int_0^{2\pi} f(x)\,\tau'(x)\,dx$$
$$= 0,$$

since $m\sigma = 0$, by an argument which we have already used. But

$$\int_0^{2\pi} t_p(x)\,dx > \tfrac{1}{3}d_p.$$

From (65) it follows that $f(2\pi) = f(0)$.

THEOREM XXVIII. *Let*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \tag{66}$$

be a trigonometric series with bounded coefficients. Let (66) *be summable by Poisson's method to zero in the intervals complementary to a closed set* σ *of zero measure. Let there be a sequence* $\lambda_p \to \infty$ *such that*

$$\operatorname*{Max}_{(k)} (|a_{kn_p}| + |b_{kn_p}|)/\delta_p{}^2 \quad (1 \leqslant k \leqslant [\lambda_p/\delta_p{}^2])$$

tends to zero as $p \to \infty$. *Then* $a_0 = 0$.

The series
$$\tfrac{1}{2}a_0 x + \Sigma\left(\frac{a_n}{n}\sin nx - \frac{b_n}{n}\cos nx\right) \tag{67}$$

converges to a constant in each interval contiguous to σ. If $x = 0,\ 2\pi$ do not belong to σ, we define a function $f(x)$ almost everywhere in $(0,\ 2\pi)$ as the sum of (67). If $x = 0,\ 2\pi$ do belong to σ, we define a function $f(x)$ by

$$f(0) = 0, \quad f(2\pi) = \pi a_0,$$

and $f(x)$ almost everywhere in $0 < x < 2\pi$ as the sum of (67). In either case, $f(2\pi) - f(0) = \pi a_0$. We have

$$\begin{aligned}\int_0^{2\pi} e^{ikn_p x}\,df &= f(2\pi) - f(0) - ikn_p \int_0^{2\pi} f(x)\,e^{ikn_p x}\,dx \\ &= \pi a_0 - ikn_p \int_0^{2\pi} \left[\tfrac{1}{2}a_0 x + \frac{a_{kn_p}}{kn_p}\sin kn_p x - \frac{b_{kn_p}}{kn_p}\cos kn_p x\right] e^{ikn_p x}\,dx \\ &= \pi(a_{kn_p} - ib_{kn_p}).\end{aligned}$$

Thus $f(x)$ satisfies the conditions of the preceding theorem.

THEOREM XXIX. *Let*

$$\tfrac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx) \tag{68}$$

be a trigonometric series with bounded coefficients. Let (68) *be summable by Poisson's method to zero in the intervals contiguous to a closed set* σ *of*

measure zero. Let m denote zero or a positive integer. Let there be a sequence $\lambda_p \to \infty$ such that, for each m,

$$\operatorname*{Max}_{(k)} (|a_{kn_p \pm m}| + |b_{kn_p \pm m}|)/\delta_p^2 \quad (1 \leqslant k \leqslant [\lambda_p/\delta_p^2]) \quad (69)$$

tends to zero as $p \to \infty$. Then

$$a_0 = 0, \quad a_n = 0, \quad b_n = 0 \quad (n = 1, 2, \ldots).$$

The condition (69) gives for $m = 0$, by the preceding theorem, $a_0 = 0$. We define the function $\phi(x)$ precisely as in Theorem XII, and we must show that $\phi(x)$, which, as in that theorem, is of class β, satisfies the inequality which $f(x)$ satisfies in the enunciation of Theorem XXVII. We find that

$$\int_0^{2\pi} e^{ikn_p x}\, d\phi = \frac{\pi k n_p}{2} \frac{a_{kn_p - m}}{kn_p - m} + \frac{\pi k n_p}{2} \frac{a_{kn_p + m}}{kn_p + m} + \frac{m\pi}{2} \frac{a_{kn_p + m}}{kn_p - m}$$

$$- \frac{i\pi k n_p}{2} \frac{b_{kn_p - m}}{kn_p - m} - \frac{i\pi k n_p}{2} \frac{b_{kn_p + m}}{kn_p + m} + \frac{im}{2} \frac{b_{kn_p - m}}{kn_p - m} - \frac{im}{2} \frac{b_{kn_p + m}}{kn_p + m},$$

so that, by (69), the conditions of Theorem XXVII are satisfied. This gives $a_m = 0$, and the proof that $b_m = 0$ is similar.

ON THE CONNECTION BETWEEN THE LIMITS OF OSCILLATION OF A SEQUENCE AND ITS CESÀRO AND RIESZ MEANS

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Introduction.

1. In our introduction we give a survey of the development of the problem before us, because some of the results under discussion are either not in print, or perhaps are not sufficiently known in the form that interests us.

Let

$$\sum_{n=0}^{\infty} u_n = u_0+u_1+u_2+\ldots+u_n+\ldots \tag{1}$$

be an infinite series of real terms. By s_n we denote the n-th partial sum $s_n = u_0+u_1+u_2+\ldots+u_n$, and by $s_n^{(1)}$ its first Cesàro mean, namely

$$s_n^{(1)} = \frac{s_0+s_1+\ldots+s_n}{n+1}.$$

As is well known, the existence of $\lim_{n\to\infty} s_n$ implies that of $\lim_{n\to\infty} s_n^{(1)}$, in which, admitting the values $+\infty$ and $-\infty$, we get two equal limits. In other words, the convergence of (1) involves its summability of the first order with like sum, while the unconditional divergence of (1) to $+\infty$ involves that of $\{s_n^{(1)}\}$ to $+\infty$. The converse is not true without restriction. But, if we suppose that $\lim_{n\to\infty} (s_n - s_n^{(1)}) = 0$, that is

$$\lim_{n\to\infty} \frac{u_1+2u_2+\ldots+nu_n}{n+1} = 0, \tag{2}$$

then, from

$$\lim_{n\to\infty} s_n^{(1)} = s \quad (-\infty \leqslant s \leqslant \infty), \tag{3}$$

follows

$$\lim_{n\to\infty} s_n = s. \tag{4}$$

In particular, the assumption

$$\lim_{n\to\infty} nu_n = 0 \tag{5}$$

implies the relation (2), and therefore in this case (3) implies (4).

Fejér† has proved, with a wider supposition than (5), namely

$$\lim_{n\to\infty} u_n = 0, \tag{6}$$

that, when (3) is true, we get an increasing sequence of numbers n_ν ($\nu = 1, 2, 3, \ldots$) such that

$$\lim_{\nu\to\infty} s_{n_\nu} = s. \tag{7}$$

2. In 1909 Hardy‡ showed that, if u_n tends to zero with $1/n$ in such a way that

$$|nu_n| \leqslant K \quad (0 < K < \infty), \tag{8}$$

then from (3) follows not merely (7), but the complete relation (4).

In the next year Landau§ showed that (3) still implies (4) in the case $-\infty < s < \infty$ when (8) is replaced by one of the conditions‖

$$nu_n \leqslant K, \tag{9a}$$

$$nu_n \geqslant -K. \tag{9b}$$

Likewise, with the help of Pólya's lemma¶, the above theorem of Fejér can be extended to the case when the condition (6) is stipulated for positive u_n only (or negative u_n only).

The results of Hardy and Landau can be expressed as follows. *If, for the series* (1) *which obeys* (8), *or else either* (9*a*) *or* (9*b*), *we have*

$$-\infty < \underline{\lim_{n\to\infty}}\, s_n < \overline{\lim_{n\to\infty}}\, s_n < \infty,$$

† *Comptes rendus*, 142 (1906), 501–503. This result is due to two facts: (*a*) when (6) holds, the limit points of $\{s_n\}$ are everywhere dense in its interval of oscillation, (*b*) $\lim s_n^{(1)}$ is contained in this interval.

‡ *Proc. London Math. Soc.* (2), 8 (1910), 302.

§ *Prac. Matematyczno-Fizyczynch*, 21 (1911), 97.

‖ Merely assuming the first condition we cannot include the case where $\lim s_n^{(1)} = \infty$, as appears from the following example: $s_n = \log(n/n_i)$ for $n_i \leqslant n < n_{i+1}$, where $n_i = 2^{2^i}$ ($i = 0, 1, 2, \ldots$). Under the second condition alone the case $\lim s_n^{(1)} = \infty$ must also be excluded, as is seen from the example $s_n = \log(n_{i+1}/n)$. The verification of these statements is left to the reader.

¶ *Rend. di Palermo*, 34 (1912), 108–109. This lemma plays here the same part as did previously (*a*) of foot-note †.

then also $\underline{\lim}_{n\to\infty} s_n^{(1)} < \overline{\lim}_{n\to\infty} s_n^{(1)}$†. This raises the question‡ whether, under (8), or (9*a*) or (9*b*), we can make an upper§ estimate of the length $l = \overline{\lim} s_n - \underline{\lim} s_n$ of the interval of oscillation of $\{s_n\}$ in terms of K and of the length $l^{(1)} = \overline{\lim} s_n^{(1)} - \underline{\lim} s_n^{(1)}$ of the interval of oscillation of its first Cesàro mean, *i.e.* whether in such case we can find a function $f(l^{(1)}, K)$, vanishing with $l^{(1)}$, for which we have always

$$l \leqslant f(l^{(1)}, K). \tag{10}$$

3. This question, first raised by Fejér, for the case (8), was in 1919 solved by him with a development of the arguments which Charlotte Nagy used in her Dissertation at the former's suggestion to obtain a new proof of Hardy's theorem. An alternative solution is afforded by Fejér's general results, obtained in 1921, in connection with "slowly oscillating" series. He calls the series (1) slowly oscillating, when, for all $\delta > 0$, a $N(\delta)$ and a $p(\delta) > 0$ can both be found such that

$$|u_{n+1} + u_{n+2} + \ldots + u_{n+q}| \leqslant \delta,$$

if $n > N(\delta)$, and q takes successively the values 1, 2, 3, ..., $[pn]$. For such series, according to Fejér, we have (assuming that $\delta < \frac{1}{2}l$, to avoid a trivial result)

$$l^{(1)} \geqslant \frac{p(\delta)}{p(\delta)+2}(l-2\delta).$$

In the special case (8) we can take $p(\delta) = \delta/K$, so that

$$l^{(1)} \geqslant \frac{\delta}{\delta+2K}(l-2\delta), \tag{11}$$

---

† Under condition (8) the last inequality is also valid for $-\infty \leqslant \underline{\lim} s_n < \overline{\lim} s_n \leqslant \infty$. This is not so, if only one of the conditions (9*a*) or (9*b*) is fulfilled, as the examples in footnote ||, page 489, show.

‡ The question may be put in a rather more general form. If $\overline{\lim} s_n = \begin{matrix} b \\ a \end{matrix}$ and $\overline{\lim} s_n^{(1)} = \begin{matrix} b^{(1)} \\ a^{(1)} \end{matrix}$, we inquire, when $b^{(1)}$ and $a^{(1)}$ are given, whether $l = b - a$ has an upper limit of the form $g(b^{(1)}, a^{(1)}, K)$, vanishing, in conformity with Hardy's convergency result, when $b^{(1)} = a^{(1)}$. If this is so, then, since l is the same for the sequence $s_n + c$, we must have

$$g(b^{(1)} + c, a^{(1)} + c, K) \equiv g(b^{(1)}, a^{(1)}, K),$$

indicating that $g(b^{(1)}, a^{(1)}, K)$ is of the form $f(b^{(1)} - a^{(1)}, K)$ or $f(l^{(1)}, K)$. In our work we actually obtain upper bounds for $a - a$ and $b - a$. The fact that the sum of the latter turns out to be the best obtainable upper bound for $b - a$ seems to be a matter of good fortune.

§ A lower estimate, namely $l \geqslant l^{(1)}$, is afforded by the known fact that $\underline{\lim} s_n \leqslant \underline{\lim} s_n^{(1)}$ and $\overline{\lim} s_n \geqslant \overline{\lim} s_n^{(1)}$ independently of any restriction on u_n. The same is true when $s_n^{(1)}$ is replaced by the Riesz means considered later.

which proves the existence of $f(l^{(1)}, K)$ with the properties above required†. Clearly the best lower estimate of $l^{(1)}$ obtainable from (11) is found by putting $\delta = \sqrt{(4K^2+Kl)}-2K$, whence

$$l^{(1)} \geqslant l-4\{\sqrt{(4K^2+Kl)}-2K\},$$

which leads to

$$l \leqslant l^{(1)}+4\sqrt{(l^{(1)}K)}. \tag{12}$$

4. Fekete, having learnt from Fejér the existence of a function $f(l^{(1)}, K)$ satisfying (10), has deduced the last inequality from the two inequalities

$$l^{(1)} \geqslant \frac{(a^{(1)}-a)^2}{4K} \quad (a = \underline{\lim}\, s_n,\ a^{(1)} = \underline{\lim}\, s_n^{(1)}), \tag{13}$$

$$l^{(1)} \geqslant \frac{(b-b^{(1)})^2}{4K} \quad (b = \overline{\lim}\, s_n,\ b^{(1)} = \overline{\lim}\, s_n^{(1)}), \tag{14}$$

which he had obtained in 1919 under the condition (8) by a method suggested from Nagy's work. In 1926 a second (as yet unpublished) paper by the latter came before his notice. This contained a new proof of Landau's extended‡ form of Hardy's theorem§ and also of its generalization‖ for Riesz's mean

$$t_n = u_0\left(1-\frac{\lambda_0}{\lambda_{n+1}}\right)+u_1\left(1-\frac{\lambda_1}{\lambda_{n+1}}\right)+\ldots+u_n\left(1-\frac{\lambda_n}{\lambda_{n+1}}\right) \tag{15}$$

$$(0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty;\ n \geqslant 0),$$

or

$$\lambda_{n+1}t_n = s_0(\lambda_1-\lambda_0)+s_1(\lambda_2-\lambda_1)+\ldots+s_n(\lambda_{n+1}-\lambda_n), \tag{15a}$$

with $[\lambda_n/(\lambda_n-\lambda_{n-1})]\,|u_n| \leqslant H$ $(n \geqslant 1)$, when λ_n is restricted by

$$\frac{\lambda_n-\lambda_{n-1}}{\lambda_n} \to 0, \quad \text{as} \quad n \to \infty. \tag{16}$$

Fekete then established (13) and (14), and consequently (12), in the case (9*a*) or (9*b*), and proved similar results for the limits of oscillation

† The results so far mentioned in § 3 are unpublished. We owe our acquaintance with them to the courtesy of Prof. Fejér.

‡ *Prac. Matematyczno Fizycznych*, 21 (1911), 97.

§ *Proc. London Math. Soc.* (2), 8 (1910), 302.

‖ *Loc. cit.*, 311.

of the Riesz mean (15), assuming the condition (16) and postulating either

$$\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\,u_n \leqslant H \tag{17a}$$

or

$$\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\,u_n \geqslant -H. \tag{17b}$$

The method applied here has proved adequate to cope with inequalities for the Riesz means corresponding to (13), (14), and (12), even when the drastic restriction (16) on λ_n is replaced by the supposition—previously implied by (16) with (17*a*) [or (17*b*)]—that the positive (or negative) u_n's tend to zero with $1/n$.

5. At this point the present authors started to consider whether it was possible in the case (9*a*) or (9*b*) to replace the number 4 in (12), (13), and (14) by a smaller number, and, by employing the above-mentioned method, arrived simultaneously and independently at the conclusion that (13) and (14) were valid with the number 2 instead of 4, and consequently (12) with $\sqrt{8}$ instead of 4.

Suitable examples indicated that a further reduction of these numbers was impossible†. But, by a further refinement of the method, Winn arrived at improved inequalities of a different kind. He showed that, if a, $a^{(1)}$, b, and $b^{(1)}$ have the meaning given above, u_n being subject to the condition (9*a*),

$$\frac{b^{(1)}-a^{(1)}}{K} \geqslant \frac{1}{2!}\left(\frac{a^{(1)}-a}{K}\right)^2+\frac{1}{3!}\left(\frac{a^{(1)}-a}{K}\right)^3+\frac{1}{4!}\left(\frac{a^{(1)}-a}{K}\right)^4+\dots \tag{18}$$

$$= e^{(a^{(1)}-a)/K}-\frac{a^{(1)}-a}{K}-1;$$

$$\frac{b^{(1)}-a^{(1)}}{K} \geqslant \frac{1}{2!}\left(\frac{b-b^{(1)}}{K}\right)^2+\frac{2}{3!}\left(\frac{b-b^{(1)}}{K}\right)^3+\frac{3}{4!}\left(\frac{b-b^{(1)}}{K}\right)^4+\dots‡. \tag{19}$$

† In § 2 it will appear that a sequence s_n with upper and lower limits b and a, satisfying $n(s_n-s_{n-1}) \leqslant K$, can be found to give equality in (18) and (20) simultaneously. Hence follows the existence of a sequence s_n with prescribed values for $\overline{\lim}\, s_n$ and $\overline{\lim}\, s_n^{(1)}$, satisfying $n(s_n-s_{n-1}) \leqslant K$ and giving equality in (18). Similar results hold for (20), if arbitrary upper limits are attached to s_n and $s_n^{(1)}$. Now the supposition that $l^{(1)} \geqslant \lambda(a^{(1)}-a)^2/K$ with $\lambda > \frac{1}{2}$ is impossible, since then $0 < (a^{(1)}-a)/K < 1-1/2\lambda$ is inconsistent with

$$\frac{l^{(1)}}{K} = e^{(a^{(1)}-a)/K}-\frac{a^{(1)}-a}{K}-1.$$

Similarly, on account of (20), the supposition that $l^{(1)} \geqslant \lambda(b-b^{(1)})^2/K$ is impossible with $\lambda > \frac{1}{2}$.

‡ Corresponding results are deducible from (9*b*).

6. By an entirely different method Winn established (18) anew, and replaced the relation (19) by the sharper† inequality

$$\frac{b^{(1)}-a^{(1)}}{K} \geqslant e^{-(b-a^{(1)})/K}+\frac{b-a^{(1)}}{K}-1, \tag{20}$$

and Fekete pointed out that this method could be applied to Riesz's mean in (15), provided only that the series (1) obeyed the condition (17 *a*), while λ_n was restricted by (16), so as to afford the following theorem:

THEOREM A. *If the positive terms of the series* (1) *satisfy the condition* (17 *a*), *where* $0=\lambda_0<\lambda_1<\ldots<\lambda_n\to\infty$, *and*

$$\frac{\lambda_n-\lambda_{n-1}}{\lambda_n}\to 0 \quad as \quad n\to\infty; \tag{16}$$

while for the partial sum s_n *and its Riesz mean* t_n *it is given that*‡ $\underline{\lim}\, s_n = a$, $\overline{\lim}\, s_n = b$, $\underline{\lim}\, t_n = a$, *and* $\overline{\lim}\, t_n = \beta$; *then*§

$$\frac{\beta-a}{H} \geqslant e^{(a-a)/H}-\frac{a-a}{H}-1 \tag{21}$$

and

$$\frac{\beta-a}{H} \geqslant e^{-(b-a)/H}+\frac{b-a}{H}-1. \tag{22}$$

7. From these two inequalities can clearly be deduced, as a special case, Hardy's theorem‖ of 1909 which states that the summability of (1) by Riesz means (15) implies its convergence, when (16) holds in conjunction with (17 *a*) and (17 *b*) together.

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† (20) is equivalent to $1-(b-b^{(1)})/K \geqslant e^{-(b-a^{(1)})/K}$, whence

$$\log\frac{1}{1-(b-b^{(1)})/K} \leqslant \frac{b-a^{(1)}}{K},$$

and therefore $\frac{b^{(1)}-a^{(1)}}{K} \geqslant \frac{1}{2}\left(\frac{b-b^{(1)}}{K}\right)^2+\frac{1}{3}\left(\frac{b-b^{(1)}}{K}\right)^3+\frac{1}{4}\left(\frac{b-b^{(1)}}{K}\right)^4+\ldots.$

‡ a, b are here finite. But (21), from which b is absent, is still obtainable if $b=\infty$. Similarly (22), from which a is absent, is still obtainable if $a=-\infty$.

§ Corresponding results are deducible from (17 *b*).

‖ *Loc. cit.*, 311.

This result was later† shown to be valid without the restriction (16), or when, retaining (16), only one of the conditions (17 *a*), (17 *b*) was imposed.

In order to include the first also of the foregoing results we were led to modify in part Theorem A above. And, in fact, we found that, to obtain (21), the condition (16) could here be replaced by the more general condition

$$\lim_{n\to\infty} (t_n - t_{n-1}) = 0. \tag{23}$$

In the proof of this result, which is incorporated in Theorem B of § I, we use a modification of the second method mentioned above. The remainder of Theorem A appears under Theorem C of the same paragraph.

8. The inequalities (21) and (22) are equivalent‡ to

$$e^{(\alpha-a)/H} - \frac{\beta-a}{H} \leqslant 1, \qquad e^{(\alpha-b)/H} - \frac{\beta-b}{H} \leqslant 1. \tag{24}$$

From § II it will appear that in general, under the suppositions of Theorem A, the number 1 can in neither relation be reduced§. For there we prove even more, namely:

THEOREM D. *Given any* λ_n *sequence* $(0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty)$ *which satisfies* (16), *we can find, for any prescribed* b, a, H, *subject to* $b > a$, $H > 0$, *a sequence* s_n^* *such that*

$$\overline{\lim}\, s_n^* = \frac{b}{a}, \qquad \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} (s_n^* - s_{n-1}^*) \leqslant H,$$

† Hardy, "An extension of a theorem on oscillating series", *Proc. London Math. Soc.* (2), 12 (1913), 174–180. *Cp.* Ananda Rau, "A note on a theorem of Mr. Hardy", *Proc. London Math. Soc.* (2), 17 (1918), 334–336. In a subsequent paper ["An example in the theory of summability by Riesz's typical means", *Proc. London Math. Soc.* (2), 30 (1930), 367–372], Ananda Rau has shown that the one-sided restriction upon u_n by one only of our conditions (17 *a*), (17 *b*), does not ensure the convergence of our series (1), which is supposed to be summable by the Riesz means (15), when λ_n does not satisfy (16). Nevertheless, it follows from Theorem B that for every finitely oscillating series (1) subject to (17 *a*), which is summable by (15) to a sum σ, we have, independently of (16), $\lim_{n\to\infty} s_n = \sigma$.

‡ These inequalities can be put in the form $f(a) \leqslant 0$, $f(b) \leqslant 0$, where

$$f(t) = e^{(\alpha-t)/H} + \frac{t-\beta}{H} - 1.$$

Now $f'(t) \gtrless 0$ according as $t \gtrless \alpha$. Hence, since $f(\alpha) \leqslant 0$, $f(\beta) \leqslant 0$, and $f(-\infty) = f(+\infty) = +\infty$, $f(t) = 0$ has exactly two roots, τ_1 and τ_2, such that $\tau_1 \leqslant \alpha \leqslant \beta \leqslant \tau_2$. *Thus it follows that* $\tau_1 \leqslant a \leqslant \alpha$, $\beta \leqslant b \leqslant \tau_2$.

§ If, however, we take $\lambda_n/(\lambda_n - \lambda_{n-1}) < K < \infty$, then for any s_n, with $\overline{\lim}\,(s_n - s_{n-1}) = 0$, it is fairly easy to show that $a = \alpha$ and $b = \beta$, which states more than (21) and (22), when $b > a$. For then instead of unity we can write $1 - (b-a)/H$ and $e^{-(b-a)/H}$ in the first and second relations of (24) respectively.

while the upper and lower limits of

$$\frac{1}{\lambda_{n+1}} \sum_{\nu=0}^{n} s_\nu^*(\lambda_{\nu+1}-\lambda_\nu), \quad as \quad n\to\infty,$$

are

$$\beta = b-H+\frac{b-a}{e^{(b-a)/H}-1}, \tag{25}$$

$$a = H\log\frac{b-a}{H(e^{-a/H}-e^{-b/H})}, \tag{26}$$

respectively†, *i.e.* each of the relations (24) holds with the sign of equality‡.

9. In the last two paragraphs, returning to the first method, we show by a suitable artifice that (21) and (22) can be obtained under the condition (17 a), without assuming (16) or (23), provided that the positive u_n's in (1) tend to zero§. These results are embodied in Theorems E and F, the latter of which clearly implies the earlier Theorem C. We leave as an open question the possible connection between Theorems B and E.

I.

1. THEOREM B. *If the positive terms of the series* (1) *satisfy the condition* (17 a), *where* $0=\lambda_0<\lambda_1<\ldots<\lambda_n\to\infty$, *and for the partial sum* s_n *and its Riesz mean* [*see* (15)] *we have given* $-\infty<\underline{\lim}_{n\to\infty} s_n=a$, $\underline{\lim}_{n\to\infty} t_n=a<\infty$, $\overline{\lim}_{n\to\infty} t_n=\beta$, $\overline{\lim}_{n\to\infty} s_n=b\leqslant\infty$, *then*‖

$$\frac{\beta-\alpha}{H}\geqslant e^{(\alpha-a)/H}-\frac{a-a}{H}-1, \tag{21}$$

provided that

$$\underline{\lim}_{n\to\infty}\ (t_n-t_{n-1})=0. \tag{23}$$

† It is worth noting that, under the assumptions of Theorem A, these values of β and α are the best obtainable lower and upper bounds for β and α in terms of b, a, H; and conversely, when β, a, the limits of the Riesz sum ($\beta>\alpha$), and H (>0) are given in advance, the solutions of (25) and (26) furnish the best upper and lower bounds for b and a.

‡ We observe that e^x-x-1 and $e^{-x}+x-1$ increase with positive x. Therefore, denoting the inverse functions by $\phi(x)$ and $\psi(x)$, the relations (21) and (22) are expressible as

$$(\alpha-a)/H\leqslant\phi\{(\beta-a)/H\} \quad \text{and} \quad (b-a)/H\leqslant\psi\{(\beta-a)/H\},$$

giving, on addition, $(b-a)/H\leqslant\chi\{(\beta-a)/H\}$. Since s_n^* gives equality in both (21) and (22), we obtain for it $(b-a)/H=\chi\{(\beta-a)/H\}$. (*Cp.* end of foot-note ‡, page 490.)

§ Corresponding results hold when the negative u_n's tend to zero.

‖ It should be observed that, on account of (35), (16) implies (23), when $a>-\infty$ and $\beta<\infty$; therefore this theorem involves the result of Theorem A (*Introduction*, 6) embodied in the inequality (21).

2. Since (21) is self-evident for $\beta = \infty$ or $\alpha = a$, we may take $\beta < \infty$ and $\alpha > a$ in proving Theorem B. Given a positive $\epsilon < \frac{1}{3}(\alpha - a)$, we can find a positive integer n for which we have simultaneously

$$a - \epsilon < s_n < a + \epsilon, \tag{27}$$

and, if $\nu \geqq n$,

$$\alpha - \epsilon < t_\nu < \beta + \epsilon, \tag{28}$$

$$t_\nu - t_{\nu-1} > -\frac{\epsilon^2}{H}. \tag{29}$$

Noting (17 a), we obtain from (15 a), when $p \geqq n$,

$$\begin{aligned}\lambda_{p+1}(t_p - s_n) - \lambda_{n+1}(t_n - s_n) &= \sum_{\nu=n+1}^{p} (\lambda_{\nu+1} - \lambda_\nu)(s_\nu - s_n) \\ &\leqq H \sum_{\nu=n+1}^{p} (\lambda_{\nu+1} - \lambda_\nu)\left(\frac{\lambda_{n+1} - \lambda_n}{\lambda_{n+1}} + \ldots + \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}\right) \\ &= H \sum_{\nu=n}^{p} (\lambda_{p+1} - \lambda_{\nu+1}) \frac{\lambda_{\nu+1} - \lambda_\nu}{\lambda_{\nu+1}},\end{aligned}$$

or

$$\begin{aligned}&\lambda_{p+1}(t_p - s_n) - \lambda_{n+1}(t_n - s_n) \qquad (30) \\ &\qquad \leqq H \left\{\lambda_{p+1} \sum_{\nu=n}^{p} \frac{\lambda_{\nu+1} - \lambda_\nu}{\lambda_{\nu+1}} - (\lambda_{p+1} - \lambda_n)\right\} \\ &\qquad < H \left\{\lambda_{p+1} \sum_{\nu=n}^{p} \int_{\lambda_\nu}^{\lambda_{\nu+1}} \frac{dx}{x} - (\lambda_{p+1} - \lambda_{n+1})\right\} \\ &\qquad = H \left\{\lambda_{p+1} \log \frac{\lambda_{p+1}}{\lambda_{n+1}} + \lambda_{p+1} \log \frac{\lambda_{n+1}}{\lambda_n} - (\lambda_{p+1} - \lambda_{n+1})\right\}.\end{aligned}$$

It follows that

$$t_p - s_n < \frac{\lambda_{n+1}}{\lambda_{p+1}}(t_n - s_n + H) + H\left(\log \frac{\lambda_{p+1}}{\lambda_n} - 1\right), \tag{31}$$

or, in consequence of (27) and (28),

$$\alpha - \epsilon - (a + \epsilon) < \frac{\lambda_{n+1}}{\lambda_{p+1}}\{\beta + \epsilon - (a - \epsilon) + H\} + H\left(\log \frac{\lambda_{p+1}}{\lambda_n} - 1\right),$$

whence, since $\lambda_{n+1} \leqq \lambda_{p+1}$, we get

$$\alpha - a < \frac{\lambda_{n+1}}{\lambda_{p+1}}(\beta - a + H) + H\left(\log \frac{\lambda_{p+1}}{\lambda_n} - 1\right) + 4\epsilon. \tag{32}$$

3. By a suitable choice of p in the last inequality we shall now derive (21). First we fix $p_0 \geqslant n$ so that, for $n \leqslant \nu \leqslant p_0$, we have

$$s_\nu < \alpha - 2\epsilon,$$

while

$$s_{p_0+1} \geqslant \alpha - 2\epsilon \tag{33}$$

(since $s_n < \alpha + \epsilon < \alpha - 2\epsilon$ and $\overline{\lim}_{n\to\infty} s_n = b \geqslant a$, the existence of p_0 is evident). Then we can choose an integer p_1, with $n \leqslant p_1 \leqslant p_0$, for which

$$\lambda_{p_1} < \lambda_n e^{(\alpha - a - 3\epsilon)/H} \leqslant \lambda_{p_1+1}. \tag{34}$$

For, by (33), (27), and (17 a),

$$\alpha - a - 3\epsilon < s_{p_0+1} - s_n = \sum_{\nu=n+1}^{p_0+1} u_\nu \leqslant H \sum_{\nu=n+1}^{p_0+1} \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu}$$

$$< H \sum_{\nu=n+1}^{p_0+1} \int_{\lambda_{\nu-1}}^{\lambda_\nu} \frac{dx}{x} = H \log \frac{\lambda_{p_0+1}}{\lambda_n};$$

hence, since $\epsilon < \frac{1}{3}(\alpha - a)$,

$$\lambda_{p_0+1} > \lambda_n e^{(\alpha - a - 3\epsilon)/H} > \lambda_n.$$

Now from (15) we have at once the identity

$$s_\nu - t_\nu = \frac{\lambda_\nu}{\lambda_{\nu+1} - \lambda_\nu}(t_\nu - t_{\nu-1}) \quad (\nu \geqslant 1). \tag{35}$$

Hence, in view of the choice of p_0 and (28) and (29), it follows that, when $n \leqslant \nu \leqslant p_0$,

$$\alpha - 2\epsilon - (\alpha - \epsilon) > \frac{\lambda_\nu}{\lambda_{\nu+1} - \lambda_\nu}\left(\frac{-\epsilon^2}{H}\right),$$

i.e.

$$\frac{\lambda_{\nu+1}}{\lambda_\nu} < 1 + \frac{\epsilon}{H}.$$

Thus, putting $p = p_1$, so that (34) is satisfied, in (32), we get

$$\alpha - a < \left(1 + \frac{\epsilon}{H}\right)(\beta - a + H) e^{-(\alpha - a - 3\epsilon)/H} + H \log\left(1 + \frac{\epsilon}{H}\right) + \alpha - a - 3\epsilon - H + 4\epsilon.$$

When $\epsilon \to 0$, this yields the equivalent of (21), namely

$$H \leqslant (\beta - a + H) e^{-(\alpha - a)/H}.$$

4. THEOREM C. *If the positive terms of the series* (1) *satisfy the condition*

$$\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} u_n \leqslant H, \tag{17a}$$

where $0 = \lambda_0 < \lambda_1 < \ldots < \lambda_n \to \infty$, *and for the partial sum* s_n *and its Riesz mean* t_n *we have given* $\underline{\lim}\, s_n = a \geqslant -\infty$, $\underline{\lim}\, t_n = a$, $-\infty < \overline{\lim}\, t_n = \beta < \infty$, $\overline{\lim}\, s_n = b$, *then*

$$\frac{\beta - a}{H} \geqslant e^{-(b-a)/H} + \frac{b-a}{H} - 1, \tag{22}$$

provided that

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1. \tag{36}$$

5. In proving (22) we discard the trivial case $b = \beta$, assuming that

$$b > \beta. \tag{37}$$

Our first step is to deduce an upper bound for b. Under the condition (17 a) we obtain from (15)

$$\begin{aligned}\lambda_{n+1}(s_n - t_n) &= u_1\lambda_1 + u_2\lambda_2 + \ldots + u_n\lambda_n \\ &\leqslant H(\lambda_1 - \lambda_0) + H(\lambda_2 - \lambda_1) + \ldots + H(\lambda_n - \lambda_{n-1}) = H\lambda_n < H\lambda_{n+1}.\end{aligned}$$

Hence $s_n < t_n + H$, and for the upper limits

$$b \leqslant \beta + H. \tag{38}$$

[The inequality (22) indicates that, if $a > -\infty$, $b < \beta + H$, since from (38) and $\beta < \infty$ we deduce that $b < \infty$.]

Since the truth of (22) for $a = -\infty$ follows from (38), we subsequently take $a > -\infty$.

Now, for any given $\epsilon > 0$, we can choose n so that, for $\nu \geqslant n$,

$$a - \epsilon < t_\nu < \beta + \epsilon, \tag{39}$$

$$\frac{\lambda_{\nu+1}}{\lambda_\nu} \leqslant 1 + \epsilon; \tag{40}$$

and afterwards $m > n$ so that

$$\lambda_{m+1} > \lambda_{n+1} e^{(b-a)/H}, \tag{41}$$

and

$$b - \epsilon < s_m < b + \epsilon. \tag{42}$$

When $n \leqslant q \leqslant m$, we obtain from (15 a), in conjunction with (17 a),

$$(43)\quad \lambda_{m+1}(s_m-t_m)-\lambda_{q+1}(s_m-t_q) = \sum_{\nu=q+1}^{m}(\lambda_{\nu+1}-\lambda_\nu)(s_m-s_\nu)$$

$$\leqslant H\sum_{\nu=q+1}^{m}(\lambda_{\nu+1}-\lambda_{q+1})\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}}$$

$$= H\left(\lambda_{m+1}-\lambda_{q+1}-\lambda_{q+1}\sum_{\nu=q+1}^{m}\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}}\right).$$

But

$$(44)\quad \sum_{\nu=q+1}^{m}\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}} = \sum_{\nu=q+1}^{m}\frac{\lambda_\nu}{\lambda_{\nu+1}}\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_\nu} > \sum_{\nu=q+1}^{m}\frac{\lambda_\nu}{\lambda_{\nu+1}}\int_{\lambda_\nu}^{\lambda_{\nu+1}}\frac{dx}{x}$$

$$\geqslant \operatorname*{Min}_{q+1\leqslant\nu\leqslant m}\frac{\lambda_\nu}{\lambda_{\nu+1}}\sum_{\nu=q+1}^{m}\int_{\lambda_\nu}^{\lambda_{\nu+1}}\frac{dx}{x} = \operatorname*{Min}_{q+1\leqslant\nu\leqslant m}\frac{\lambda_\nu}{\lambda_{\nu+1}}\log\frac{\lambda_{m+1}}{\lambda_{q+1}}.$$

Introducing (44) in (43), and using (40), we get the inequality

$$(45)\quad s_m-t_m-\frac{\lambda_{q+1}}{\lambda_{m+1}}(s_m-t_q) \leqslant H\left(1-\frac{\lambda_{q+1}}{\lambda_{m+1}}-\frac{\lambda_{q+1}}{\lambda_{m+1}}\log\frac{\lambda_{m+1}}{\lambda_{q+1}}\frac{1}{1+\epsilon}\right).$$

6. A suitable choice of q in the last inequality will enable us to obtain (22). In fact, we can find a q_0, with $n \leqslant q_0 < m$, such that

$$(46)\quad \lambda_{q_0+1} \leqslant \lambda_{m+1}e^{-(b-a)/H} < \lambda_{q_0+2}.$$

For, according to (41), we have $\lambda_{n+1} < \lambda_{m+1}e^{-(b-a)/H}$, while, by (37),

$$\lambda_{m+1} > \lambda_{m+1}e^{-(b-a)/H}.$$

Thus, putting $q = q_0$ in (45), and using (42), (39), (46), and (40), we obtain

$$b-\epsilon-\beta-\epsilon-e^{-(b-a)/H}(b+\epsilon-a+\epsilon)$$

$$< H\left(1-\frac{1}{1+\epsilon}e^{-(b-a)/H}-\frac{1}{(1+\epsilon)^2}e^{-(b-a)/H}\frac{b-a}{H}\right).$$

Hence, for $\epsilon \to 0$, it follows that

$$\frac{b-\beta}{H} \leqslant 1-e^{-(b-a)/H},$$

which is equivalent to (22).

II.

1. The present paragraph contains the proof of Theorem D, enunciated in the introduction.

In order to construct a sequence s_n^* with the desired properties we first note that, since a, b, and H satisfy the inequalities $b > a$, $H > 0$, and the λ_n sequence obeys (16), we can fix $n_0 \geqq 1$ so that, for $n \geqq n_0$, we have $\lambda_{n+1}/\lambda_n < e^{(b-a)/H}$. Since λ_n increases steadily to ∞, we can then obtain an increasing sequence† of positive integers $n_1, n_2, \ldots, n_i, \ldots$ defined by

$$\lambda_{n_{i+1}} \leqq \lambda_{n_i} e^{(b-a)/H} < \lambda_{n_i+1} \quad (i = 0, 1, \ldots). \tag{47}$$

With the help of these n_i's, we define the required sequence s_ν^* as follows:

$$s_\nu^* = 0 \quad \text{if} \quad \nu < n_0; \tag{48}$$

$$s_\nu^* = a \quad \text{if} \quad \nu = n_i \quad (i \geqq 0); \tag{48}$$

$$s_\nu^* = s_{\nu-1}^* + H\frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} \quad \text{if} \quad n_i < \nu < n_{i+1} \quad (i \geqq 0). \tag{49}$$

2. With reference to this type of sequence s_ν we establish the following lemma:

Suppose that $s_{n_i} = a$ $(i \geqq 0, n_i < n_{i+1})$ *and that* s_ν *increases as* ν *increases from* n_i *to* $n_{i+1}-1$. *If* t_ν *satisfies the difference equation*

$$t_\nu - t_{\nu-1} = \rho_\nu(s_\nu - t_\nu); \qquad \rho_\nu > 0 \ \textit{when} \ \nu \geqq n_0, \tag{50}$$

then $\overline{\lim_{i\to\infty}}\, t_{n_i-1} = \overline{\lim_{\nu\to\infty}}\, t_\nu$.

If $\underline{\lim}\, t_\nu = \overline{\lim}\, t_\nu$, the result is evident. So we take $\underline{\lim}\, t_\nu < \overline{\lim}\, t_\nu$ and first show that $t_{n_i} > s_{n_i} = a$, when $i \geqq i_0$. For, if this was not so, there would be an infinite number of i's such that $t_{n_i} \leqq a$. Now for these values of i we should have, by (50),

$$t_{n_i-1} = t_{n_i} - \rho_{n_i}(s_{n_i} - t_{n_i}) \leqq t_{n_i} \leqq a,$$

$$t_{n_i-2} = t_{n_i-1} - \rho_{n_i-1}(s_{n_i-1} - t_{n_i-1}) < t_{n_i-1} \quad (s_{n_i-1} > a).$$

Similarly $t_{n_i-3} < t_{n_i-2}$, etc. Thus, the values of n_i considered being arbitrarily large, t_ν would increase steadily, instead of oscillating. Hence we must have $t_{n_i} > s_{n_i}$ and so, by (50),

$$t_{n_i-1} > t_{n_i}. \tag{51}$$

Now, since t_ν oscillates, there is an index $q > n_i$ for which $t_q > t_{q-1}$, and therefore, in virtue of (50), $s_q > t_q$. Denoting the least such index by q_i, we conclude from (51) that q_i must lie *between* n_{j-1} and n_j $(j > i)$. Let us consider the behaviour of t_ν as ν increases from $n_i - 1$ to $n_j - 1$. By the

† For then $\lambda_{n_{i+1}+1} > \lambda_{n_i+1} \cdot (\lambda_{n_i}/\lambda_{n_i+1})\, e^{(b-a)/H} > \lambda_{n_i+1}$, and so $n_{i+1} > n_i$.

definition of q_i and from (51) we know that t_ν decreases as ν increases from n_i-1 to q_i-1. A further application of (50) shows that t_ν increases as ν increases from q_i to n_j-1. In fact,

$$t_{q_i+1}-t_{q_i} = \rho_{q_i+1}(s_{q_i+1}-t_{q_i+1}) > \rho_{q_i+1}(s_{q_i}-t_{q_i+1}) > \rho_{q_i+1}(t_{q_i}-t_{q_i+1});$$

and similarly for $t_{q_i+2}-t_{q_i+1}, \ldots, t_{n_j-1}-t_{n_j-2}$. Thus, when

$$n_i-1 < \nu < n_j-1,$$

t_ν is less than either t_{n_i-1} or t_{n_j-1}. Hence $\overline{\lim_{\nu\to\infty}}\, t_\nu = \overline{\lim_{i\to\infty}}\, t_{n_i-1}$.

3. Returning to the sequence s_n^* defined above, we shall first show that $\overline{\lim_{\nu\to\infty}}\, s_\nu^* = b$ and then that $\lim_{i\to\infty} t_{n_i}$ exists and is equal to $\overline{\lim_{\nu\to\infty}}\, t_\nu$. In establishing the existence of $\lim_{i\to\infty} t_{n_i}$ we derive inequalities for our example in addition to those of § I, 2, used in the proof of (21). The new relations enable us to get here also the inequality converse to (21) and so to deduce equality between its two members. Finally, from this result and the value obtained for $\overline{\lim_{\nu\to\infty}}\, t_\nu$, we attain equality also in (22).

4. To prove that $\overline{\lim_{\nu\to\infty}}\, s_\nu = b$, as $s_{n_{i+1}-1} > s_\nu$ for $n_i \leqslant \nu < n_{i+1}-1$, it will be sufficient to prove that $\lim_{i\to\infty} s_{n_{i+1}-1} = b$. Denoting by ϵ_i the smallest number for which

$$\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_\nu} \leqslant \frac{\epsilon_i}{H} \quad (\nu \geqslant n_i), \tag{52}$$

we obtain, on account of (16), a positive decreasing null-sequence ϵ_i $(i \geqslant 0)$. From (52) in conjunction with (47) we derive

$$e^{(b-a)/H}\left(1+\frac{\epsilon_i}{H}\right)^{-1} < \frac{\lambda_{n_{i+1}}}{\lambda_{n_i}} \leqslant e^{(b-a)/H}. \tag{53}$$

Now we have

$$s_{n_{i+1}-1}-a = s_{n_{i+1}-1}-s_{n_i} = \sum_{\nu=n_i+1}^{n_{i+1}-1} (s_\nu - s_{\nu-1}) = H \sum_{\nu=n_i+1}^{n_{i+1}-1} \frac{\lambda_\nu-\lambda_{\nu-1}}{\lambda_\nu}.$$

Hence, noting that $1-(1/x) < \log x < x-1$, for $x > 1$, and using (53), we get

$$0 < H \log \frac{\lambda_{n_{i+1}-1}}{\lambda_{n_i}} - (s_{n_{i+1}-1}-a) = H \sum_{\nu=n_i+1}^{n_{i+1}-1} \left(\log \frac{\lambda_\nu}{\lambda_{\nu-1}} - \frac{\lambda_\nu-\lambda_{\nu-1}}{\lambda_\nu}\right)$$

$$< H \sum_{\nu=n_i+1}^{n_{i+1}-1} \frac{\lambda_\nu-\lambda_{\nu-1}}{\lambda_\nu} \log \frac{\lambda_\nu}{\lambda_{\nu-1}} < \epsilon_i \log \frac{\lambda_{n_{i+1}-1}}{\lambda_{n_i}} < \epsilon_i \frac{b-a}{H}.$$

Since, by (53), (47), and (52),

$$e^{(b-a)/H} > \frac{\lambda_{n_{i+1}-1}}{\lambda_{n_i}} > e^{(b-a)/H}\frac{\lambda_{n_{i+1}-1}}{\lambda_{n_{i+1}+1}} > e^{(b-a)/H}\left(1+\frac{\epsilon_i}{H}\right)^{-2} > e^{(b-a-2\epsilon_i)/H},$$

it follows that

$$0 < b-a-(s_{n_{i+1}-1}-a) < 2\epsilon_i+\epsilon_i\frac{b-a}{H},$$

which yields $\lim_{i\to\infty} s_{n_{i+1}-1} = b$ and therewith $\overline{\lim_{\nu\to\infty}}\, s_\nu = b$.

5. To establish the existence of $\lim_{i\to\infty} t_{n_i}$ it will suffice to show that

$$\sigma_i = \frac{\lambda_{n_{i+1}+1}t_{n_{i+1}}-\lambda_{n_i+1}t_{n_i}}{\lambda_{n_{i+1}+1}-\lambda_{n_i+1}} \quad (i \geqslant 0)$$

tends to a limit, as $i\to\infty$. For, defining $\mu_i = \lambda_{n_i+1}$ and $\tau_{i-1} = t_{n_i}$, if $i \geqslant 1$ and $\mu_0 = 0$, $\tau_{-1} = 0$, we see that τ_i is the Riesz mean (15) of σ_i with respect to $0 = \mu_0 < \mu_1 < \ldots < \mu_i \to \infty$, and so, by the consistency theorem, tends to the same limit as σ_i.

In examining the existence of $\lim_{i\to\infty} \sigma_i$ we first observe that in § I, 2, when† $s_\nu - s_{\nu-1} = H\{(\lambda_\nu-\lambda_{\nu-1})/\lambda_\nu\}$ for $n < \nu \leqslant p$, the third member of the inequality (30) exceeds the first member by

$$H\lambda_{p+1}\left(\log\frac{\lambda_{n+1}}{\lambda_n} - \frac{\lambda_{n+1}-\lambda_n}{\lambda_{n+1}}\right) + H\lambda_{p+1}\sum_{\nu=n+1}^{p}\left(\int_{\lambda_\nu}^{\lambda_{\nu+1}}\frac{dx}{x} - \frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}}\right) + H(\lambda_{n+1}-\lambda_n)$$

$$< H\lambda_{p+1}\log\left(1+\frac{\epsilon}{H}\right) + H\lambda_{p+1}\sum_{\nu=n+1}^{p}\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}}\log\frac{\lambda_{\nu+1}}{\lambda_\nu}$$

$$< \epsilon\lambda_{p+1} + H\lambda_{p+1}\frac{\epsilon}{H}\log\frac{\lambda_{p+1}}{\lambda_{n+1}},$$

provided that $\lambda_{\nu+1}/\lambda_\nu < 1+\epsilon/H$ for $\nu \geqslant n$. Hence in the special case considered we derive, by comparing the first and last members of (30),

$$(\overline{31}) \qquad t_p - s_n > \frac{\lambda_{n+1}}{\lambda_{p+1}}(t_n - s_n + H) + H\log\frac{\lambda_{p+1}}{\lambda_{n+1}}\left(1-\frac{\epsilon}{H}\right) - H - \epsilon.$$

Now, in virtue of (48), (49) and (52), we can apply this result together with (31)—which holds generally, if (17 a) is valid—to the case where $s_n = s_n^*$, $p = n_{i+1}-1$, $n = n_i$, and $\epsilon = \epsilon_i$. Since $s_{n_i}^* = a$, we get

$$H\log\frac{\lambda_{n_i+1}}{\lambda_{n_i+1}}\left(1-\frac{\epsilon_i}{H}\right) - \epsilon_i < t_{n_{i+1}-1} - a + H - \frac{\lambda_{n_i+1}}{\lambda_{n_i+1}}(t_{n_i}-a+H) < H\log\frac{\lambda_{n_i+1}}{\lambda_{n_i}}.$$

† This ensures *equality* between the first two members of (30), as we see by examining the steps just before it.

But the middle term here is equal to

$$\frac{\lambda_{n_{i+1}+1}-\lambda_{n_i+1}}{\lambda_{n_{i+1}+1}}(\sigma_i-a+H)+t_{n_{i+1}-1}-t_{n_i+1}+(a-H-t_{n_i})\frac{\lambda_{n_i+1}}{\lambda_{n_i+1}}\left(1-\frac{\lambda_{n_{i+1}}}{\lambda_{n_{i+1}+1}}\right),$$

while, by (52) and (53), we have

$$\lim_{i\to\infty}\frac{\lambda_{n_{i+1}+1}}{\lambda_{n_i+1}}=\lim_{i\to\infty}\frac{\lambda_{n_i+1}}{\lambda_{n_i+1}}=\lim_{i\to\infty}\frac{\lambda_{n_i+1}}{\lambda_{n_i}}=e^{(b-a)/H}.$$

Moreover, in consequence of (16), (35) implies that

$$\lim_{\nu\to\infty}(t_\nu-t_{\nu-1}). \tag{23a}$$

Hence, noting (16), we obtain

$$\lim_{i\to\infty}\sigma_i=a-H+(b-a)(1-e^{(b-a)/H})^{-1}=\beta, \text{ say.}$$

Therefore $\lim_{i\to\infty} t_{n_i}=\beta$, and, by (23a), $\lim_{i\to\infty} t_{n_i-1}=\beta$.

Finally, since the above lemma is applicable in virtue of (35), we have

$$\overline{\lim_{\nu\to\infty}}\, t_\nu=\overline{\lim_{i\to\infty}}\, t_{n_i-1}=\beta.$$

6. Similarly we can apply ($\overline{31}$) to the case where $s_n=s_n{}^*$, $p=p_i$, with $n_i<p_i<n_{i+1}$, $n=n_i$, and $\epsilon=\epsilon_i$, obtaining

$$H\log\frac{\lambda_{p_i+1}}{\lambda_{n_i+1}}\left(1-\frac{\epsilon_i}{H}\right)-\epsilon_i<t_{p_i}-a+H-\frac{\lambda_{n_i+1}}{\lambda_{p_i+1}}(t_{n_i}-a+H)$$

$$\leqslant t_{p_i}-a+H-\frac{\lambda_{n_i+1}}{\lambda_{p_i+1}}(\beta-a+H)+|t_{n_i}-\beta|;$$

or, putting $\lambda_{p_i+1}/\lambda_{n_i+1}=x$,

$$t_{p_i}-a+\epsilon_i+\epsilon_i\log x+|t_{n_i}-\beta|\geqslant\frac{\beta-a+H}{x}+H(\log x-1)=f(x).$$

Now the minimum value of $f(x)$ for $x>1$ is easily found to be

$$H\log\left(1+\frac{\beta-a}{H}\right).$$

Hence, since $x\leqslant\lambda_{n_{i+1}+1}/\lambda_{n_i+1}\leqslant\{1+(\epsilon_i/H)\}e^{(b-a)/H}$ from (52) and (53), we have

$$t_{p_i}-a+\epsilon_i\left\{1+\log\left(1+\frac{\epsilon_i}{H}\right)+\frac{b-a}{H}\right\}+|t_{n_i}-\beta|\geqslant H\log\left(1+\frac{\beta-a}{H}\right).$$

But, since $n_i<p_i<n_{i+1}$, p_i assumes all integral values, as $i\to\infty$, apart from n_i. Consequently, from the last inequality, since $\epsilon_i\to 0$ and $t_{n_i}\to\beta$ as $i\to\infty$,

$$a-a\geqslant H\log\left(1+\frac{\beta-a}{H}\right),$$

where $\underline{\lim}_{\nu\to\infty} t_\nu = a$. This is equivalent to

$$\frac{\beta-\alpha}{H} \leqslant e^{(\alpha-a)/H} - \frac{\alpha-a}{H} - 1.$$

On the other hand, we have shown that $\lim_{\nu\to\infty}(t_\nu - t_{\nu-1}) = 0$, a relation which, together with (48) and (49), ensures the validity of (21) in accordance with Theorem B. Consequently, for our example,

$$\frac{\beta-\alpha}{H} = e^{(\alpha-a)/H} - \frac{\alpha-a}{H} - 1.$$

From the last result and the value obtained for β we can deduce

$$e^{(\alpha-a)/H} = \frac{\beta+H-a}{H} \quad \text{and} \quad e^{(b-\alpha)/H} = \frac{\beta+H-a}{\beta+H-b},$$

whence, on division, equality is also attained in (22).

III.

1. THEOREM E. *Replacing the condition* (23) *in Theorem B by*

$$\overline{\lim}\, u_n = 0, \tag{54}$$

and assuming the other conditions postulated there without any alteration, we have again†

$$\frac{\beta-\alpha}{H} \geqslant e^{(\alpha-a)/H} - \frac{\alpha-a}{H} - 1. \tag{21}$$

2. In order to prove Theorem E we may repeat the assumption that $\beta < \infty$ and $\alpha > a$. Then let p be an arbitrary integer greater than or equal to 2 and, for $0 \leqslant i \leqslant p$, put

$$x_i = a + \frac{i}{p}(\alpha - a).$$

Next, given a positive $\epsilon < \dfrac{\alpha-a}{4p(p+1)}$, on account of $\underline{\lim}\, s_n = a$, $\overline{\lim}\, t_n = \dfrac{\beta}{\alpha}$ and (54), we can choose n_0 so that

$$s_{n_0} < a + \epsilon, \tag{55}$$

† In consequence of (54), Theorems E and F are almost trivial when $\lambda_n/(\lambda_n - \lambda_{n-1})$ is bounded. For in that case they are deducible from the relations $b = \beta$ and $a = \alpha$, respectively, of foot-note ‡ on page 494.

and, when $n \geqslant n_0$,

$$\alpha-\epsilon < t_n < \beta+\epsilon, \tag{56}$$

$$u_n < 2\epsilon. \tag{57}$$

Now (55) and (57), together with the fact that $\overline{\lim}\, s_n \geqslant \alpha$, imply the existence of an index $n > n_0$ for which $x_i-\epsilon < s_n < x_i+\epsilon$ $(1 \leqslant i \leqslant p)$. Let n_i be the least n for which this holds†. Then $s_\nu \leqslant x-\epsilon$ for $n_0 < \nu < n_i$, and in particular $s_\nu \leqslant \alpha-\epsilon < t_\nu$ for $n_0 < \nu < n_p$. The last relation combined with (35) shows that t_ν decreases, as ν increases from $\nu = n_0$ to $\nu = n_p-1$. In fact, when $n_0 < \nu < n_p$,

$$t_\nu - t_{\nu-1} = \frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_\nu}(s_\nu - t_\nu) < 0.$$

Consequently, for $0 < i < p$, we get, noting (56),

$$\begin{aligned} t_{n_{i-1}} - t_{n_i} &= \sum_{\nu=n_{i-1}+1}^{n_i} (t_{\nu-1}-t_\nu) = \sum_{\nu=n_{i-1}+1}^{n_i} (t_\nu - s_\nu)\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_\nu} \\ &> \Big(\operatorname*{Min}_{n_{i-1}+1 \leqslant \nu \leqslant n_i} t_\nu - \operatorname*{Max}_{n_{i-1}+1 \leqslant \nu \leqslant n_i} s_\nu\Big) \sum_{\nu=n_{i-1}+1}^{n_i} \frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}} \\ &= (t_{n_i} - s_{n_i}) \sum_{\nu=n_{i-1}+2}^{n_i+1} \frac{\lambda_\nu - \lambda_{\nu-1}}{\lambda_\nu} \\ &> \{\alpha-\epsilon-(x_i+\epsilon)\} \sum_{\nu=n_{i-1}+2}^{n_i} \frac{\lambda_\nu-\lambda_{\nu-1}}{\lambda_\nu} = (\alpha-x_i-2\epsilon) \sum_{\nu=n_{i-1}+2}^{n_i} \rho_\nu, \end{aligned} \tag{58}$$

where $\rho_\nu = (\lambda_\nu - \lambda_{\nu-1})/\lambda_\nu$.

But using (17 a) and (57) we have

$$\begin{aligned} x_i-\epsilon-(x_{i-1}+\epsilon) < s_{n_i} - s_{n_{i-1}} &= \sum_{\nu=n_{i-1}+1}^{n_i} (s_\nu - s_{\nu-1}) \\ &< u_{n_{i-1}+1} + H \sum_{\nu=n_{i-1}+2}^{n_i} \frac{\lambda_\nu-\lambda_{\nu-1}}{\lambda_\nu} < H \sum_{\nu=n_{i-1}+2}^{n_i} \rho_\nu + 2\epsilon. \end{aligned}$$

Hence, since

$$4\epsilon < \frac{\alpha-a}{p(p+1)} < \frac{\alpha-a}{p},$$

$$\begin{aligned} t_{n_{i-1}} - t_{n_i} &> (\alpha-x_i-2\epsilon)(x_i-x_{i-1}-4\epsilon)\frac{1}{H} \\ &= \frac{1}{H}\left\{\frac{p-i}{p}(\alpha-a)-2\epsilon\right\}\left(\frac{\alpha-a}{p}-4\epsilon\right) \\ &> \frac{1}{H}\left\{\frac{p-i}{p}(\alpha-a)-\frac{\alpha-a}{p}\right\}\left\{\frac{\alpha-a}{p}-\frac{\alpha-a}{p(p+1)}\right\} \\ &= (p-i-1)\frac{\alpha-a}{pH}\frac{\alpha-a}{p+1} > \binom{p-i-1}{1}\frac{(\alpha-a)^2}{H(p+1)^2}. \end{aligned}$$

† Clearly $n_0 < n_1 < n_2 < \ldots < n_p$.

Therefore, when $0 \leqslant i < p-1$,

$$t_{n_i} - t_{n_{p-1}} = \sum_{\nu=i+1}^{p-1} (t_{n_{\nu-1}} - t_{n_\nu}) > \sum_{\nu=i+1}^{p-1} \binom{p-\nu-1}{1} \frac{(a-a)^2}{H(p+1)^2}$$

$$= \binom{p-i-1}{2} \frac{(a-a)^2}{H(p+1)^2},$$

whence, using (56),

$$t_{n_i} > a - \epsilon + \binom{p-i-1}{2} \frac{(a-a)^2}{H(p+1)^2}. \tag{59}$$

Employing the right member as lower bound for t_{n_i} in (58) instead of $a-\epsilon$, we get, for $0 < i < p-1$, the new estimate

$$t_{n_{i-1}} - t_{n_i} > (t_{n_i} - s_{n_i}) \sum_{\nu=n_{i-1}+2}^{n_i} \rho_\nu$$

$$> \left\{ \binom{p-i-1}{2} \frac{(a-a)^2}{H(p+1)^2} + a - x_i - 2\epsilon \right\} (x_i - x_{i-1} - 4\epsilon) \frac{1}{H}$$

$$> \left\{ \binom{p-i-1}{1} \frac{a-a}{p} + \binom{p-i-1}{2} \frac{(a-a)^2}{H(p+1)^2} \right\} \frac{a-a}{H(p+1)}$$

$$> \binom{p-i-1}{1} \frac{(a-a)^2}{H(p+1)^2} + \binom{p-i-1}{2} \frac{(a-a)^3}{H^2(p+1)^3}.$$

Summing as before, we get

$$t_{n_i} > a - \epsilon + \binom{p-i-1}{2} \frac{(a-a)^2}{H(p+1)^2} + \binom{p-i-1}{3} \frac{(a-a)^3}{H^2(p+1)^3} \quad (0 \leqslant i < p-2).$$

It follows by a repetition of this process that, when $r > 1$ and $i < p-r$,

$$t_{n_i} > a - \epsilon + \binom{p-i-1}{2} \frac{(a-a)^2}{H(p+1)^2} + \binom{p-i-1}{3} \frac{(a-a)^3}{H^2(p+1)^3}$$

$$+ \ldots + \binom{p-i-1}{r} \frac{(a-a)^r}{H^{r-1}(p+1)^r}.$$

Now put here $i = 0$ and $r = p-1$. Since $t_{n_0} < \beta + \epsilon$ by (56), we obtain

$$\frac{\beta - a + 2\epsilon}{H} > \binom{p-1}{2} \frac{(a-a)^2}{H^2(p+1)^2} + \ldots + \binom{p-1}{p-1} \frac{(a-a)^{p-1}}{H^{p-1}(p+1)^{p-1}}$$

$$= \left\{ 1 + \frac{a-a}{H(p+1)} \right\}^{p-1} - \frac{p-1}{p+1} \frac{a-a}{H} - 1.$$

When $p \to \infty$, and accordingly $\epsilon \to 0$, we deduce (21).

IV.

1. THEOREM F. *Replacing the condition* (36) *in Theorem C by*

$$\overline{\lim_{n\to\infty}}\, u_n = 0, \tag{54}$$

and assuming the other conditions postulated there without any alteration, we have again†

$$\frac{\beta-a}{H} \geqslant e^{-(b-a)/H} + \frac{b-a}{H} - 1. \tag{22}$$

2. If $\beta = b$ and $a \geqslant -\infty$, then the result is self-evident, while for the case $\beta < b$, $a = -\infty$ its truth follows from the relation

$$b \leqslant \beta + H, \tag{36}$$

which was seen to depend only on (17 a) (see § I, 5). Therefore we take $\beta < b$ and $a > -\infty$ in proving Theorem F. In this case, we can establish (22) with the help of the following

LEMMA. *Let b_1 and b_2 be subject to*

$$b \geqslant b_1 > b_2 \geqslant a; \tag{60}$$

further, let p be an integer ($\geqslant 2$) *and let*

$$0 < \epsilon < \frac{b_1 - b_2}{2p}. \tag{61}$$

Then, since $\underline{\lim}\, s_n = a \leqslant a$, *we can fix an index m_0 so that*

$$s_{m_0} < a + \epsilon, \tag{62}$$

and, in virtue of (54), *so that also*

$$u_n < 2\epsilon \quad \text{for} \quad n \geqslant m_0. \tag{63}$$

If we can find an index $n_0 > m_0$ for which we have simultaneously

$$s_{n_0} > b_1 - \epsilon, \tag{64}$$

$$t_{n_0} < b_2 + \epsilon, \tag{65}$$

then there is an index q between m_0 and n_0 for which

$$s_q > b_2 - \epsilon_1, \tag{66}$$

$$t_q < b_3 + \epsilon_1, \tag{67}$$

† See foot-note † on page 504.

where $$\epsilon_1 = \frac{C}{p} + Dp\epsilon,$$

C/H and D being positive numbers depending only on $\xi = (b_1 - b_2)/H$, namely $C/H = \xi^2(1+\xi)e^\xi$, $D = 1+4\xi e^\xi$, and

$$b_3 = b_2 - H\{1-(1-\xi)e^\xi\}.$$

3. Let us denote by ν_i (for $0 \leqslant i \leqslant p-1$) the least index between m_0 and n_0-1 inclusive such that

$$s_\nu \geqslant b_2 + \frac{i}{p}(b_1-b_2) + \epsilon \quad \text{for} \quad n_0 \geqslant \nu \geqslant \nu_i + 1. \tag{68}$$

The existence of ν_i follows from (64), (62), and (61); in fact, when $0 \leqslant i \leqslant p-1$,

$$s_{n_0} > b_1 - \epsilon = b_2 + \frac{p-1}{p}(b_1-b_2) + \epsilon + \left(\frac{b_1-b_2}{p} - 2\epsilon\right)$$

$$> b_2 + \frac{i}{p}(b_1-b_2) + \epsilon \geqslant b_2 + \epsilon \geqslant \alpha + \epsilon > s_{m_0}.$$

For this ν_i we have, by definition,

$$s_{\nu_i} < b_2 + \frac{i}{p}(b_1-b_2) + \epsilon, \tag{69}$$

while $s_{\nu_i} = s_{\nu_i+1} - u_{\nu_i+1}$ yields, with (63) and (68),

$$s_{\nu_i} > b_2 + \frac{i}{p}(b_1-b_2) + \epsilon - 2\epsilon = b_2 + \frac{i}{p}(b_1-b_2) - \epsilon. \tag{70}$$

Moreover, for $0 \leqslant i \leqslant p-2$, it is clear that $\nu_i < \nu_{i+1}$.

We now observe that t_ν decreases steadily as ν decreases from n_0 to ν_0. For, by (35), (64), (65), and (61),

$$\frac{\lambda_{n_0}}{\lambda_{n_0+1} - \lambda_{n_0}}(t_{n_0} - t_{n_0-1}) = s_{n_0} - t_{n_0} > b_1 - b_2 - 2\epsilon > 0,$$

whence $$t_{n_0-1} < t_{n_0} < b_2 + \epsilon;$$

further, as in the particular case $i = 0$, (68) yields

$$s_\nu \geqslant b_2 + \epsilon \quad \text{for} \quad n_0 \geqslant \nu \geqslant \nu_0 + 1, \tag{71}$$

we have

$$\frac{\lambda_{n_0-1}}{\lambda_{n_0} - \lambda_{n_0-1}}(t_{n_0-1} - t_{n_0-2}) = s_{n_0-1} - t_{n_0-1} > b_2 + \epsilon - (b_2 + \epsilon),$$

whence $t_{n_0-2} < t_{n_0-1} < b_2+\epsilon$, ...; finally, by (71),

$$\frac{\lambda_{\nu_0+1}}{\lambda_{\nu_0+2}-\lambda_{\nu_0+1}}(t_{\nu_0+1}-t_{\nu_0}) = s_{\nu_0+1}-t_{\nu_0+1} > b_2+\epsilon-(b_2+\epsilon),$$

i.e.
$$t_{\nu_0} < t_{\nu_0+1}.$$

Hence, for $1 \leqslant i \leqslant p-1$, we get, in virtue of (35) and (68),

$$(72)\qquad t_{\nu_{i+1}}-t_{\nu_i} = \sum_{\nu=\nu_i+1}^{\nu_{i+1}}(t_\nu-t_{\nu-1}) = \sum_{\nu=\nu_i+1}^{\nu_{i+1}}(s_\nu-t_\nu)\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_\nu}$$

$$> \Big(\operatorname*{Min}_{\nu_i+1\leqslant\nu\leqslant\nu_{i+1}} s_\nu - \operatorname*{Max}_{\nu_i+1\leqslant\nu\leqslant\nu_{i+1}} t_\nu\Big)\sum_{\nu=\nu_i+1}^{\nu_{i+1}}\frac{\lambda_{\nu+1}-\lambda_\nu}{\lambda_{\nu+1}}$$

$$> \Big\{b_2+\frac{i}{p}(b_1-b_2)+\epsilon-(b_2+\epsilon)\Big\}\sum_{\nu=\nu_i+2}^{\nu_{i+1}}\frac{\lambda_\nu-\lambda_{\nu-1}}{\lambda_\nu}$$

$$= \frac{i}{p}(b_1-b_2)\sum_{\nu=\nu_i+2}^{\nu_{i+1}}\rho_\nu = \frac{iH\xi}{p}\sum_{\nu=\nu_i+2}^{\nu_{i+1}}\rho_\nu,$$

where $\rho_\nu = (\lambda_\nu-\lambda_{\nu-1})/\lambda_\nu$, and ν_p means n_0.

But by (70), (64), and (69), followed by (63) and (17 a), we deduce that

$$\frac{\xi}{p}-\frac{2\epsilon}{H} < \frac{s_{\nu_{i+1}}-s_{\nu_i}}{H} = \frac{u_{\nu_i+1}}{H}+\sum_{\nu=\nu_i+2}^{\nu_{i+1}}\frac{s_\nu-s_{\nu-1}}{H} < \frac{2\epsilon}{H}+\sum_{\nu=\nu_i+2}^{\nu_{i+1}}\rho_\nu,$$

i.e.
$$\sum_{\nu=\nu_i+2}^{\nu_{i+1}}\rho_\nu > \frac{\xi}{p}-\frac{4\epsilon}{H},$$

which, together with (72), yields

$$(73)\qquad t_{\nu_{i+1}}-t_{\nu_i} > i\Big(H\frac{\xi^2}{p^2}-4\epsilon\frac{\xi}{p}\Big).$$

Therefore

$$t_{\nu_p}-t_{\nu_{i+1}} = \sum_{\kappa=i+1}^{p-1}(t_{\nu_{\kappa+1}}-t_{\nu_\kappa}) > \Big(H\frac{\xi^2}{p^2}-4\epsilon\frac{\xi}{p}\Big)\sum_{\kappa=i+1}^{p-1}\kappa,$$

or

$$(74)\qquad t_{\nu_p}-t_{\nu_{i+1}} > \Big(H\frac{\xi^2}{p^2}-4\epsilon\frac{\xi}{p}\Big)\Big\{\binom{p-i}{2}+i\binom{p-i-1}{1}\Big\},$$

whence, by (65),

$$-\operatorname*{Max}_{\nu_i+1\leqslant\nu\leqslant\nu_{i+1}} t_\nu = -t_{\nu_{i+1}}$$

$$> -(b_2+\epsilon)+\Big\{\binom{p-i}{2}+i\binom{p-i-1}{1}\Big\}\Big(H\frac{\xi^2}{p^2}-4\epsilon\frac{\xi}{p}\Big).$$

4. We now return to (72), employing there the last expression as lower bound for $-t_{\nu_{i+1}}$, instead of $-(b_2+\epsilon)$, in estimating $t_{\nu_{i+1}}-t_{\nu_i}$. The effect of this is to introduce on the right of (73) additional terms, *i.e.* the product of the right member of (74) and $(\xi/p)-(4\epsilon/H)$, and this exceeds

$$\left\{\binom{p-i}{2}+i\binom{p-i-1}{1}\right\}\left(H\frac{\xi^3}{p^3}-8\epsilon\frac{\xi^2}{p^2}\right).$$

Summing again as before, we obtain additional terms to the right of (74), amounting to at least

$$\left\{2\binom{p-i}{3}+i\binom{p-i-1}{2}\right\}\left(H\frac{\xi^3}{p^3}-8\epsilon\frac{\xi^2}{p^2}\right).$$

By a continuation of this process we get the $(r-1)$-th set of additional terms to the right of (74), *i.e.* at least†

$$\left\{r\binom{p-i}{r+1}+i\binom{p-i-1}{r}\right\}\left(H\frac{\xi^{r+1}}{p^{r+1}}-4r\epsilon\frac{\xi^r}{p^r}\right).$$

We lastly collate with the right of (74) all these extra terms, taking the special value $r=p-1$ and putting $i=0$. In this way we obtain

$$\frac{1}{H}(t_{\nu_p}-t_{\nu_1})>\binom{p}{2}\frac{\xi^2}{p^2}+2\binom{p}{3}\frac{\xi^3}{p^3}+\ldots+(p-1)\binom{p}{p}\frac{\xi^p}{p^p}$$
$$-\frac{4\epsilon}{H}\left\{1^2\binom{p}{2}\frac{\xi}{p}+2^2\binom{p}{3}\frac{\xi^2}{p^2}+\ldots+(p-1)^2\binom{p}{p}\frac{\xi^{p-1}}{p^{p-1}}\right\}.$$

Here the first part reduces to

$$1+\xi\left(1+\frac{\xi}{p}\right)^{p-1}-\left(1+\frac{\xi}{p}\right)^p,$$

† When $p\geqslant 2$, $1\leqslant\nu\leqslant p-2$, and $0\leqslant i\leqslant p-1$, we have

$$\sum_{\kappa=i+1}^{p-1}\left\{\nu\binom{p-\kappa}{\nu+1}+\kappa\binom{p-\kappa-1}{\nu}\right\}$$
$$=\sum_{\kappa=i+1}^{p-1}\left\{(\nu+1)\binom{p-\kappa}{\nu+1}+\kappa\binom{p-\kappa-1}{\nu}-\binom{p-\kappa}{\nu+1}\right\}$$
$$=\sum_{\kappa=i+1}^{p-1}\left\{p\binom{p-\kappa-1}{\nu}-\binom{p-\kappa}{\nu+1}\right\}=p\binom{p-i-1}{\nu+1}-\binom{p-i}{\nu+2}$$
$$=(p-i)\binom{p-i-1}{\nu+1}-\binom{p-i}{\nu+2}+i\binom{p-i-1}{\nu+1}=(\nu+1)\binom{p-i}{\nu+2}+i\binom{p-i-1}{\nu+1}.$$

which is greater than†

$$1-(1-\xi)e^{\xi}-\frac{\xi^2(1+\xi)e^{\xi}}{p},$$

whilst the factor of $-4\epsilon/H$ is less than $p\xi e^{\xi}$.

These results, in conjunction with (65), lead to the inequality,

$$t_{\nu_1}<b_2-H\{1-(1-\xi)e^{\xi}\}+\frac{H\xi^2(1+\xi)e^{\xi}}{p}+(1+4\xi e^{\xi})p\epsilon.$$

Finally, in virtue of (70) and (61), we have

$$s_{\nu_1}>b_2+\frac{b_1-b_2}{p}-\epsilon>b_2+\epsilon.$$

Thus the desired conditions (66) and (67) are fulfilled with $q=\nu_1$.

5. Applying the lemma just established by taking successively for b_1, b_2 the terms β_j, β_{j+1} of the decreasing‡ sequence

$$\beta_{-1}=b,\ \beta_0=\beta,\ \beta_1, \beta_2, \ldots,\ \beta_j, \beta_{j+1}, \ldots$$

defined by the recurrence formula

$$\beta_{j+1}=\beta_j-H\{1-(1-\xi_j)e^{\xi_j}\};\quad H\xi_j=\beta_{j-1}-\beta_j\quad(j\geqslant 0), \tag{75}$$

we shall show in due course that

$$\beta_j\geqslant\alpha\quad(j\geqslant 0). \tag{76}$$

Hence it will follow that

$$\xi_0<1, \tag{77}$$

and therefore [*cp.* (79)]

$$\xi_j<1\quad(j\geqslant 1). \tag{78}$$

† We know that, when p is a positive integer and $\xi>0$, we have $\{1+(\xi/p)\}^p<e^{\xi}$. Also

$$\left(1+\frac{\xi}{p}\right)^p=1+\xi+\sum_{n=2}^{\infty}\left(1-\frac{1}{p}\right)\left(1-\frac{2}{p}\right)\ldots\left(1-\frac{n-1}{p}\right)\frac{\xi^n}{n!}$$

$$>1+\xi+\sum_{n=2}^{\infty}\left\{1-\frac{n(n-1)}{2p}\right\}\frac{\xi^n}{n!}=e^{\xi}\left(1-\frac{\xi^2}{2p}\right).$$

Hence $$e^{-\xi}\left(1+\frac{\xi}{p}\right)^{p+1}-\xi e^{-\xi}\left(1+\frac{\xi}{p}\right)^p<\left(1-\xi+\frac{\xi^2+\xi^3}{p}\right)\left(1+\frac{\xi}{p}\right),$$

since $$1+\frac{\xi}{p}-\xi\left(1-\frac{\xi^2}{p}\right)<1-\xi+\frac{\xi+\xi^3}{p}+\frac{\xi^3+\xi^4}{p^2}.$$

‡ In virtue of our supposition $\beta<b$, we have $H\xi_0=\beta_{-1}-\beta_0>0$. Hence, by (79) and (75) respectively, it follows that, for $j\geqslant 1$, $\xi_j>0$ and $\beta_{j-1}-\beta_j>0$.

In fact from (38) we have $\xi_0 = (b-\beta)/H \leqslant 1$, while (75) leads to

$$1-\xi_{j+1} = (1-\xi_j)\, e^{\xi_j} \quad (j \geqslant 0). \tag{79}$$

Thus $\xi_0 = 1$ would make $\xi_j = 1$, and therewith

$$\beta_j = b - H \sum_{\nu=0}^{j} \xi_\nu = b-(j+1)\,H \quad (j \geqslant 0),$$

in contradiction to (76).

Now from (76), (77), and (78) the required inequality (22) is readily deducible. For (79) yields

$$\prod_{j=0}^{r} (1-\xi_{j+1}) = \prod_{j=0}^{r} (1-\xi_j)\, e^{\sum_{j=0}^{r} \xi_j},$$

i.e.

$$\sum_{j=0}^{r} \xi_j = \log \frac{1-\xi_{r+1}}{1-\xi_0}.$$

But, since β_r decreases to a limit (greater than or equal to α), as $r \to \infty$, we have

$$\xi_r = \frac{\beta_{r-1}-\beta_r}{H} \to 0 \quad (r \to \infty).$$

Hence

$$\alpha \leqslant \lim_{r\to\infty} \beta_r = \lim_{r\to\infty} \left(b - H \sum_{j=0}^{r} \xi_j\right) = b + H \log(1-\xi_0) = b + H \log\left(1-\frac{b-\beta}{H}\right),$$

as required.

6. We now return to the proof of (76). This inequality is true by definition, when $j = 0$. Let r be the least index, *if such there be*, for which

$$\beta_r < \alpha.$$

In this case we fix $\epsilon_0 > 0$ and $p_0 \geqslant 2$, $p_0^{2^{-r}}$ being an integer, so that the choice

$$p_j = p_0^{2^{-j}} \quad (j = 1, \ldots, r)$$

ensures the validity† of the inequalities

$$2p_j\,\epsilon_j < \beta_{j-1}-\beta_j \quad (j = 0, 1, \ldots, r-1), \tag{80}$$

$$2\epsilon_r < \alpha-\beta_r, \tag{81}$$

† By taking $\epsilon_0 = p_0^{-2}$ we have, for $0 \leqslant j \leqslant r$, $p_j \epsilon_j = K_j p_0^{-2^{-j}}$, where $K_0 = 1$ and $K_{j+1} = C_j + D_j K_j$. The relation is evident for $j = 0$; and, assuming it for j, we get

$$p_{j+1}\epsilon_{j+1} = C_j p_0^{-2^{-j-1}} + D_j K_j p_0^{-2^{-j}} \cdot p_0^{2^{-j-1}} = p_0^{-2^{-j-1}}(C_j + D_j K_j).$$

Since K_j is bounded for given H and $0 \leqslant j \leqslant r$, (80) and (81) are satisfied for large enough values of p_0.

where $$\epsilon_j = \frac{C_{j-1}}{p_{j-1}} + D_{j-1} p_{j-1} \epsilon_{j-1} \quad (j = 1, 2, \ldots, r),$$

with $$C_j = H(1+\xi_j)\,\xi_j^{\,2} e^{\xi_j}, \quad D_j = 1 + 4\xi_j e^{\xi_j} \quad (j = 0, 1, \ldots, r-1),$$

and so $$\epsilon_j > \epsilon_{j-1} \quad (j = 1, 2, \ldots, r).$$

We determine m_0 so that

$$s_{m_0} < \alpha + \epsilon_0,$$

while, for $\nu \geqslant m_0$,

(82) $$\alpha - \epsilon_0 < t_\nu < \beta + \epsilon_0,$$

$$u_\nu < 2\epsilon_0,$$

and fix $n_0 > m_0$ so that

$$s_{n_0} > b - \epsilon_0.$$

Then (remembering that $b > \beta > \beta_1 > \ldots > \beta_{r-1} \geqslant \alpha$) we deduce from the lemma the existence of integers n_j $(j = 1, 2, \ldots, r)$ satisfying the relations

$$n_0 > n_1 > n_2 > \ldots > n_r > m_0,$$

for which $$t_{n_j} < \beta_j + \epsilon_j \quad (j = 1, 2, \ldots, r).$$

Hence, in the particular case $j = r$, we have, with the help of (82),

$$\alpha - \epsilon_0 < t_{n_r} < \beta_r + \epsilon_r,$$

whence $$\alpha - \beta_r < \epsilon_0 + \epsilon_r < 2\epsilon_r,$$

which contradicts the inequality (81). Thus (76) must be true.

[*Added* 8 *June*, 1933. Recently Winn has considered the case where u_n is bounded *above and below* by

$$-K \leqslant \frac{\lambda_n}{\lambda_n - \lambda_{n-1}}\, u_n \leqslant H \quad (0 < H,\ K < \infty),$$

the λ_n's satisfying the relation (16). He has established inequalities, analogous to (21) and (22), between the limits of oscillation of s_n and t_n; see *Journal London Math. Soc.*, 8 (1933), 27–32.]

TWO CLASSES OF GENERALIZED POLYNOMIALS

By L. M. MILNE-THOMSON.

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Summary.

A certain general class of polynomials is defined and its properties are considered. By specializing the definition in one direction we are led to Bernoulli's polynomials of order n regarded as generalizations of x^ν, by specializing in another direction x^ν leads to Euler's polynomials of order n. The same methods are applied to Hermite's polynomials and thus two new types of polynomial are found.

1. *The ϕ polynomials.*

We define ϕ polynomials $\phi_\nu^{(n)}(x)$ of degree ν and order n by the relation

$$f(t, n)\, e^{xt+g(t)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!}\, \phi_\nu^{(n)}(x), \tag{1}$$

where $f(t, n)$ is a function of t and the integer n. We shall suppose n to be susceptible of negative integral values in cases where the expansion (1) remains valid. Writing $x = 0$ in (1) and $\phi_\nu^{(n)}(0) = \phi_\nu^{(n)}$, we have

$$f(t, n)\, e^{g(t)} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!}\, \phi_\nu^{(n)}. \tag{2}$$

We shall call the $\phi_\nu^{(n)}$ the ϕ numbers of order n. Putting $x+y$ for x in (1), we have

$$\Sigma \frac{t^\nu}{\nu!}\, \phi_\nu^{(n)}(x+y) = e^{xt} \Sigma \frac{t^\nu}{\nu!}\, \phi_\nu^{(n)}(y).$$

Equating coefficients of t^ν, we have

$$\phi_\nu^{(n)}(x+y) = \phi_\nu^{(n)}(y) + x\binom{\nu}{1} \phi_{\nu-1}^{(n)}(y) + x^2\binom{\nu}{2} \phi_{\nu-2}^{(n)}(y) + \ldots + x^\nu\binom{\nu}{\nu} \phi_0^{(n)}(y).$$

This leads to the symbolical equation

$$\phi_\nu^{(n)}(x+y) \doteqdot \{x+\phi^{(n)}(y)\}^\nu,$$

where, after expansion of the right-hand member, each index of $\phi^{(n)}(y)$, including zero, is to be replaced by the corresponding suffix. The sign $\doteqdot$ is used here to signify symbolic equivalence. If in this last result we put $y=0$, we have the fundamental property of the ϕ polynomials,

$$\phi_\nu^{(n)}(x) \doteqdot (x+\phi^{(n)})^\nu. \tag{3}$$

This result shows that, unless $\phi_0^{(n)}$ is zero, $\phi_\nu^{(n)}(x)$ is actually of degree ν. It thus appears that the ϕ polynomials are completely defined by (3) and by the ϕ numbers, so that it is only necessary for the purpose of their definition that (1) should be valid when $x=0$ for some value of $t \neq 0$.

As direct consequences of (3) we have

$$\frac{d}{dx}\phi_\nu^{(n)}(x) \doteqdot \nu(x+\phi^{(n)})^{\nu-1} = \nu\phi_{\nu-1}^{(n)}(x), \tag{4}$$

$$\int_a^x \phi_\nu^{(n)}(y)\,dy = \frac{\phi_{\nu+1}^{(n)}(x)-\phi_{\nu+1}^{(n)}(a)}{\nu+1}. \tag{5}$$

We have, from (4),

$$\frac{d}{dx}\frac{\phi_\nu^{(n)}(x)}{\nu!} = \frac{\phi_{\nu-1}^{(n)}(x)}{(\nu-1)!}, \tag{6}$$

so that $\phi_\nu^{(n)}(x)/\nu!$ is in a sense a generalization of $x^\nu/\nu!$, the simplest ϕ polynomial being x^ν. It may also be seen, although we shall make no use of the fact, that ϕ polynomials can be formed with an arbitrary finite or enumerably infinite sequence of ϕ numbers, by simply using (3) as the defining formula.

Operating on (1) with Δ, where $\Delta u(x) = u(x+1)-u(x)$, we have

$$\Sigma\frac{t^\nu}{\nu!}\Delta\phi_\nu^{(n)}(x) = (e^t-1)f(t,n)\,e^{xt+g(t)}. \tag{7}$$

Operating on (1) with ∇, where $\nabla u(x) = \frac{1}{2}\{u(x+1)+u(x)\}$, we have

$$\Sigma\frac{t^\nu}{\nu!}\nabla\phi_\nu^{(n)}(x) = \frac{e^t+1}{2}f(t,n)\,e^{xt+g(t)}. \tag{8}$$

The above results, (7) and (8), suggest two classes of polynomials suitable for particularization.

2. *The β polynomials.*

Equation (7) suggests considering

$$f(t, n) = \frac{t^n}{(e^t-1)^n}.$$

Accordingly we define the β polynomials by

$$\frac{t^n e^{xt+g(t)}}{(e^t-1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \beta_\nu^{(n)}(x), \tag{9}$$

so that

$$\sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \Delta\beta_\nu^{(n)}(x) = t \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \beta_\nu^{(n-1)}(x),$$

which gives

$$\Delta\beta_\nu^{(n)}(x) = \nu\beta_{\nu-1}^{(n-1)}(x), \tag{10}$$

the Δ analogue of (4).

We now make some applications of the β polynomials.

3. *Bernoulli's polynomials.*

Bernoulli's polynomials $B_\nu(x)$ may be defined by

$$\frac{t e^{xt}}{e^t-1} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} B_\nu(x), \tag{11}$$

so that we have from (9), with $g(t) = 0$,

$$\beta_\nu^{(1)}(x) = B_\nu(x), \tag{12}$$

$$\beta_\nu^{(0)}(x) = x^\nu.$$

That is to say, we regard x^ν as a β polynomial of order zero. Thus Bernoulli's polynomials are β polynomials of order 1, and, from (10),

$$\Delta B_\nu(x) = \nu x^{\nu-1}.$$

Bernoulli's polynomials of order n are then given by

$$\frac{t^n e^x}{(e^t-1)^n} = \Sigma \frac{t^\nu}{\nu!} B_\nu^{(n)}(x), \tag{13}$$

while Bernoulli's numbers of order n are given by

$$\frac{t^n}{(e^t-1)^n} = \Sigma \frac{t^\nu}{\nu!} B_\nu^{(n)}. \tag{14}$$

Also

$$B_\nu^{(n)}(x) \doteq (x+B^{(n)})^\nu \quad \text{from (3)}, \tag{15}$$

$$\Delta B_\nu^{(n)}(x) = \nu B_{\nu-1}^{(n-1)}(x) \quad \text{from (10)}. \tag{16}$$

These generalized Bernoulli polynomials were discovered by Nörlund*, who takes (16) and certain definitions of $B_\nu^{(n)}$ as the point of departure. They have interesting properties which flow naturally from the general properties of ϕ and β polynomials.

4. *Hermite's polynomials.*

Taking as the definition

$$H_\nu(x) = e^{\frac{1}{2}x^2}(-1)^\nu \frac{d^\nu}{dx^\nu} e^{-\frac{1}{2}x^2},$$

we easily obtain the recurrence relation

$$H_{\nu+1}(x) = xH_\nu(x) - \nu H_{\nu-1}(x), \tag{17}$$

with $H_0(x) = 1$, $H_1(x) = x$. Write

$$y(t) = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu(x),$$

and we obtain, by the use of (17),

$$\frac{dy}{dt} - xy + ty = 0,$$

whence $y(t) = e^{xt-\frac{1}{2}t^2}$ since $y(0) = 1$. Thus

$$e^{xt-\frac{1}{2}t^2} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu(x). \tag{18}$$

From (9), with $g(t) = -\frac{1}{2}t^2$, we see that, regarding Hermite's polynomials as β polynomials of order zero,

$$\beta_\nu^{(0)}(x) = H_\nu(x).$$

With this interpretation the polynomials of order n are given by

$$\frac{t^n e^{xt-\frac{1}{2}t^2}}{(e^t-1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n)}(x), \tag{19}$$

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* N. E. Nörlund, *Acta Math.*, 43 (1922), 121–196.

and Hermite's numbers of order n are given by

$$\frac{t^n e^{-\frac{1}{2}t^2}}{(e^t-1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n)}, \tag{20}$$

With regard to (20) we may note in passing that $H_\nu^{(n)}$ is a polynomial in n, so that, if we regard n as an unrestricted variable, we have a new type of polynomial defined by

$$\frac{t^x e^{-\frac{1}{2}t^2}}{(e^t-1)^x} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(x)}.$$

For Hermite's numbers of order zero we have

$$e^{-\frac{1}{2}t^2} = \Sigma \frac{t^\nu}{\nu!} H_\nu,$$

so that the numbers with odd suffixes vanish and

$$H_{2\nu} = (-1)^\nu (2\nu-1)(2\nu-3)\ldots 5\,.\,3\,.\,1,$$

which is an odd integer. Also

$$H_\nu(x) \doteqdot (x+H)^\nu,$$

so that Hermite's polynomials of zero order have the same parity as ν and can be written down individually without recourse to recurrence relations. The coefficients are all integers. For the general polynomial of order n, which is a ϕ polynomial,

$$H_\nu^{(n)}(x) \doteqdot (x+H^{(n)})^\nu,$$

$$\frac{d}{dx} H_\nu^{(n)}(x) = \nu H_{\nu-1}^{(n)}(x),$$

$$\int_a^x H_\nu^{(n)}(y)\,dy = \frac{H_{\nu+1}^{(n)}(x) - H_{\nu+1}^{(n)}(a)}{\nu+1},$$

and, since $H_\nu^{(n)}(x)$ is a β polynomial,

$$\Delta H_\nu^{(n)}(x) = \nu H_{\nu-1}^{(n-1)}(x). \tag{21}$$

To obtain a relation between polynomials of successive orders, differentiate (19) with respect to t and then multiply by t. This gives

$$\sum_{\nu=0}^{\infty} \frac{t^{\nu+1}}{\nu!} H_{\nu+1}^{(n)}(x) = (n+xt-t^2)\frac{t^n e^{xt-\frac{1}{2}t^2}}{(e^t-1)^n} - \frac{n\,t^{n+1} e^{(x+1)t-\frac{1}{2}t^2}}{(e^t-1)^{n+1}}$$

$$= (n+xt-t^2) \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n)}(x) - n \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} H_\nu^{(n+1)}(x+1),$$

Equating coefficients of t^ν, we have

$$nH_\nu^{(n+1)}(x+1) = (n-\nu)H_\nu^{(n)}(x) + \nu x H_{\nu-1}^{(n)}(x) - \nu(\nu-1)H_{\nu-2}^{(n)}(x).$$

From (21), $H_\nu^{(n+1)}(x+1) = H_\nu^{(n+1)}(x) + \nu H_{\nu-1}^{(n)}(x)$. Hence we have

$$H_\nu^{(n+1)}(x) = \left(1-\frac{\nu}{n}\right)H_\nu^{(n)}(x) + \nu\left(\frac{x}{n}-1\right)H_{\nu-1}^{(n)}(x) - \frac{\nu(\nu-1)}{n}H_{\nu-2}^{(n)}(x). \tag{22}$$

This determines the polynomials of order $n+1$ in terms of those of order n, $n \neq 0$.

Putting $x = 0$, we have the relation between Hermite's numbers of successive orders,

$$H_\nu^{(n+1)} = \left(1-\frac{\nu}{n}\right)H_\nu^{(n)} - \nu H_{\nu-1}^{(n)} - \frac{\nu(\nu-1)}{n}H_{\nu-2}^{(n)}. \tag{23}$$

Hermite's polynomials of order unity may be expressed in terms of Bernoulli's polynomials (of order unity) as follows:

$$\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}H_\nu^{(1)}(x) = e^{-\frac{1}{2}t^2}\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}B_\nu(x).$$

Equating the coefficients of t^ν, we have

$$H_\nu^{(1)}(x) = B_\nu(x) + \binom{\nu}{2}H_2 B_{\nu-2}(x) + \binom{\nu}{4}H_4 B_{\nu-4}(x) + \ldots,$$

since $H_1 = H_3 = H_5 = \ldots = 0$.

We can, of course, use the same process to express $H_\nu^{(n)}(x)$ in terms of Bernoulli's polynomials of order n.

5. *The η polynomials.*

A second method of generalization is suggested by (8). Putting

$$f(t, n) = \frac{2^n}{(e^t+1)^n},$$

we obtain a class of polynomials which we call η polynomials. They are defined by

$$\frac{2^n}{(e^t+1)^n}e^{xt+g(t)} = \sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}\eta_\nu^{(n)}(x), \tag{24}$$

whence

$$\nabla\eta_\nu^{(n)}(x) = \eta_\nu^{(n-1)}(x). \tag{25}$$

6. *Euler's polynomials.*

These may be defined by

$$\frac{2e^{xt}}{e^t+1} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} E_\nu(x), \tag{26}$$

so that Euler's polynomials are η polynomials of order 1. In fact, from (24), with $g(t) = 0$, $n = 1$,

$$\eta_\nu^{(1)}(x) = E_\nu(x),$$

so that

$$\nabla E_\nu(x) = \eta_\nu^{(0)}(x) = x^\nu. \tag{27}$$

Here we regard x^ν as an η polynomial of order zero.

The numbers which are usually called Euler's numbers are defined by $E_\nu = 2^\nu E_\nu(\frac{1}{2})$. In accordance with our general method we want the values of $E_\nu(0)$. To avoid confusion with the common nomenclature we write

$$F_\nu = E_\nu(0).$$

The generating function of the F numbers is then

$$\frac{2}{e^t+1} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} F_\nu. \tag{28}$$

Since

$$\frac{2}{e^t+1} - 1 = \frac{1-e^t}{1+e^t}$$

is an odd function, we have $F_0 = 1$, $F_{2\nu} = 0$, $\nu > 0$. We have then, by (3),

$$E_\nu(x) \doteq (x+F)^\nu = x^\nu + \binom{\nu}{1} F_1 x^{\nu-1} + \binom{\nu}{3} F_3 x^{\nu-3} + \ldots. \tag{29}$$

Since $F_1 = -\frac{1}{2}$, we see that

$$E_\nu(x) - x^\nu = -\tfrac{1}{2}\nu x^{\nu-1} + \binom{\nu}{3} F_3 x^{\nu-3} + \ldots$$

is an odd function if ν is even, and an even function if ν is odd.

From (27) we have

$$(x+F+1)^\nu + (x+F)^\nu \doteq 2x^\nu,$$

so that, putting $x = 0$, the F numbers are determined successively from

$$(F+1)^\nu + F^\nu \doteq 0 \quad (\nu > 0). \tag{30}$$

Putting $x = \frac{1}{2}$ in (29) we have, for Euler's numbers,

$$E_\nu = 2^\nu E_\nu(\tfrac{1}{2}) \doteq (2F+1)^\nu,$$

so that

$$F \doteq \tfrac{1}{2}(E-1).$$

Substituting in (30) we have

$$(E+1)^\nu + (E-1)^\nu \doteq 0 \quad (\nu > 0),$$

which is the recurrence relation for E_ν, and, from (29),

$$E_\nu(x) \doteq \left(x + \frac{E-1}{2}\right)^\nu.$$

For Euler's polynomials of order n we have the generating function

$$\frac{2^n e^{xt}}{(e^t+1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} E_\nu^{(n)}(x),$$

and for the F numbers of order n

$$\frac{2^n}{(e^t+1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} F_\nu^{(n)},$$

so that, from (25),

$$\nabla E_\nu^{(n)}(x) = E_\nu^{(n-1)}(x), \tag{31}$$

$$E_\nu^{(n)}(x) \doteq (x+F^{(n)})^\nu.$$

Equation (31), together with certain definitions of $E_\nu^{(n)}$, form the starting point of Nörlund's investigations* of the polynomials $E_\nu^{(n)}(x)$.

7. *Hermite's polynomials of the second kind.*

If we regard $H_\nu(x)$ as an η polynomial of zero order we can define Hermite's polynomial of order n of the second kind by means of

$$\frac{2^n e^{xt-\frac{1}{2}t^2}}{(e^t+1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \text{𝔎}_\nu^{(n)}(x). \tag{32}$$

The generating function of the 𝔎 numbers is then

$$\frac{2^n e^{-\frac{1}{2}t^2}}{(e^t+1)^n} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} \text{𝔎}_\nu^{(n)}. \tag{33}$$

We have also

$$\text{𝔎}_\nu^{(n)}(x) \doteq (\text{𝔎}^{(n)}+x)^\nu,$$

$$\nabla \text{𝔎}_\nu^{(n)}(x) = \text{𝔎}_\nu^{(n-1)}(x). \tag{34}$$

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* N. E. Nörlund, *Differenzenrechnung* (1924), chap. 6.

To obtain the relation between polynomials of successive orders differentiate (32) with respect to t. This gives

$$(x-t)\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}\,\mathfrak{U}_\nu^{(n)}(x)-\tfrac{1}{2}n\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}\,\mathfrak{U}_\nu^{(n+1)}(x+1)=\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}\,\mathfrak{U}_{\nu+1}^{(n)}(x).$$

Equating coefficients of t^ν and using (34) we obtain

$$x\,\mathfrak{U}_\nu^{(n)}(x)-\nu\,\mathfrak{U}_{\nu-1}^{(n)}(x)-n\{\mathfrak{U}_\nu^{(n)}(x)-\tfrac{1}{2}\mathfrak{U}_\nu^{(n+1)}(x)\}=\mathfrak{U}_{\nu+1}^{(n)}(x).$$

Thus we have

$$\tfrac{1}{2}n\,\mathfrak{U}_\nu^{(n+1)}(x)=\mathfrak{U}_{\nu+1}^{(n)}(x)+(n-x)\,\mathfrak{U}_\nu^{(n)}(x)+\nu\,\mathfrak{U}_{\nu-1}^{(n)}(x).$$

The recurrence relation between numbers of successive orders is obtained by putting $x=0$. The polynomials of the first order of the second kind can be expressed in terms of Euler's polynomials as follows:

$$\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}\,\mathfrak{U}_\nu^{(1)}(x)=e^{-\frac{1}{2}t^2}\sum_{\nu=0}^{\infty}\frac{t^\nu}{\nu!}\,E_\nu(x),$$

so that

$$\mathfrak{U}_\nu^{(1)}(x)=E_\nu(x)+\binom{\nu}{2}H_2\,E_{\nu-2}(x)+\binom{\nu}{4}H_4\,E_{\nu-4}(x)+\ldots.$$

A similar relation between $\mathfrak{U}_\nu^{(n)}(x)$ and Euler's polynomials of order n can be easily obtained in the same way.

SOLUTIONS INSIDE THE SPHERE AND CYLINDER ON EINSTEIN'S THEORY

By A. J. CARR.

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Introduction.

Let the surface of a solid of revolution be mapped out by x_2, x_3; and consider the ground-form,

$$ds^2 = -e^{\lambda} dx_1^2 - e^{\mu} dx_2^2 - e^{\nu} dx_3^2 + e^{\rho} dx_4^2, \tag{1}$$

λ, μ, ν, ρ being functions of x_1, x_2 only. Also let $\kappa = 4\pi\gamma/c^2$, where γ is the constant of gravitation. The "identification" equations* are

$$H_{\alpha}^{\beta} \equiv (G_{\alpha}^{\beta} - \tfrac{1}{2} g_{\alpha}^{\beta} G) + 2\kappa T_{\alpha}^{\beta} = 0 \tag{2}$$

with
$$(H_{\alpha}^{\beta})_{\beta} \equiv (G_{\alpha}^{\beta} - \tfrac{1}{2} g_{\alpha}^{\beta} G)_{\beta} + 2\kappa (T_{\alpha}^{\beta})_{\beta} \equiv G_{(\alpha)} + 2\kappa T_{(\alpha)} = 0, \tag{3}$$

since $G_{(\alpha)}$, $T_{(\alpha)}$ are separately zero. But by (1) the only surviving $G_{\alpha\beta}$ are G_{11}, G_{22}, G_{33}, G_{44}, G_{12}. Hence, from (2), $T_{\alpha}^{\beta} = 0$ $(\alpha \neq \beta)$ except $T_1^2 (= T_2^1)$, so that the right-hand members of the relations

$$0 = T_{(\alpha)} \equiv \partial \{T_{\alpha}^{\beta} \sqrt{(-g)}\} / \sqrt{(-g)}\, \partial x_{\beta} - \{\alpha\beta,\ \delta\}\, T_{\delta}^{\beta} \tag{4}$$

vanish identically for $\alpha = 3, 4$, while the remaining equations of (4) furnish two conditions among the five quantities T_1^1, T_2^2, T_3^3, T_4^4, T_1^2, thus showing that only three can be prescribed arbitrarily.

Also, using (1), the four equations $G_{(\alpha)} = 0$ are satisfied for $\alpha = 3, 4$. Hence

$$(H_1^{\beta})_{\beta} = 0, \quad (H_2^{\beta})_{\beta} = 0.$$

These two relations between the five extant H_{α}^{β} entail that only three are independent, so we have only *three* unconnected equations of (2) to determine *four* $g_{\alpha\beta}$. The same result occurs if $T_{\alpha}^{\beta} = 0$. *It is thus at our*

* Eddington, *Mathematical theory of relativity* (1923), 120.

disposal, whatever T_α^β, to assign at will any condition we please between the four $g_{\alpha\beta}$.* The simplest and most obvious is

$$a\lambda+b\mu+c\nu+d\rho=\phi(x_1, x_2), \tag{6}$$

a, b, c, d being arbitrary constants and $\phi(x_1, x_2)$ an arbitrary function of x_1, x_2.

PART I. *The sphere.*

Let now λ, μ, ρ be functions of x_1 only, while ν is a function of x_1 and x_2; suppose, too, that T_α^β behave like $g_{\alpha\beta}$, *i.e.* let T_1^1, T_2^2, T_4^4 be functions of x_1 only, T_3^3 of x_1 and possibly x_2, while T_1^2 is zero. Then, from (2), $G_{12}=0$, while, for $\alpha=2$, equation (4) gives

$$\nu_2(T_2^2-T_3^3)=0. \tag{7}$$

(*a*) For $\nu_2 \neq 0$ we get the cylindrical problem to be discussed later.

(*b*) For
$$T_3^3=T_2^2 \tag{8}$$

we have the spherical problem in which (8) shows that T_3^3 is *not* a function of x_2. If we write λ_α for $\partial\lambda/\partial x_\alpha$, etc., equation (4) for $\alpha=1$ is now

$$\partial T_1^1/\partial x_1+\tfrac{1}{2}(\mu_1+\nu_1)(T_1^1-T_2^2)+\tfrac{1}{2}\rho_1(T_1^1-T_4^4)=0. \tag{9}$$

This, with (8), gives *two* relations between the *four* T_α^β, leaving only two T_α^β arbitrary. This result will shortly be seen to have its counterpart in the $G_{\alpha\beta}$ and $g_{\alpha\beta}$.

Written out in full the equations (2) are, by (8),

$$\mu_{11}+\nu_{11}+\rho_{11}+\tfrac{1}{2}[\mu_1^2+\nu_1^2+\rho_1^2-\lambda_1(\mu_1+\nu_1+\rho_1)] = 2\kappa e^\lambda(T_1^1-2T_2^2-T_4^4), \tag{10}$$

$$\mu_{11} \quad +\tfrac{1}{2}\mu_1(\mu_1+\nu_1+\rho_1-\lambda_1)+e^{\lambda-\mu}(\nu_{22}+\tfrac{1}{2}\nu_2^2) = -2\kappa e^\lambda(T_1^1 \quad +T_4^4), \tag{11}$$

$$\nu_{11} \quad +\tfrac{1}{2}\nu_1(\mu_1+\nu_1+\rho_1-\lambda_1)+e^{\lambda-\mu}(\nu_{22}+\tfrac{1}{2}\nu_2^2) = -2\kappa e^\lambda(T_1^1 \quad +T_4^4), \tag{12}$$

$$\rho_{11}+\tfrac{1}{2}\rho_1(\mu_1+\nu_1+\rho_1-\lambda_1) = -2\kappa e^\lambda(T_1^1+2T_2^2-T_4^4), \tag{13}$$

$$\nu_{12}+\tfrac{1}{2}\nu_2(\nu_1-\mu_1)=0. \tag{14}$$

Differentiating (13) with respect to x_2, we have

$$\tfrac{1}{2}\rho_1\nu_{12}=0. \tag{15}$$

* *Loc. cit.* (2 ed.), Supp. Note 5.

Hence, since ρ is not constant, $\nu_{12}=0$. Substituting this in (14), we have

$$\tfrac{1}{2}(\nu_1-\mu_1)\,\nu_2=0.$$

Hence, by (a), $$\nu_1=\mu_1. \tag{16}$$

Thus, if $f=f(x_2)$, we get $$\nu=\mu+f.$$

Equations (11) and (12) now both become

$$e^{\mu-\lambda}[\mu_{11}+\tfrac{1}{2}(2\mu_1{}^2+\mu_1\rho_1-\lambda_1\mu_1)]+2\kappa e^{\mu}(T_1{}^1+T_4{}^4)=-(f_{22}+\tfrac{1}{2}f_2{}^2). \tag{17}$$

Each side must thus be a constant $2k$, and hence

$$e^f=(A/k)\sin^2\sqrt{k}(x_2-B). \tag{18}$$

It is curious to find this relation dropping naturally out of equations (2), for Schwarzschild* assumes it, with

$$x_1=r,\quad x_2=\theta,\quad x_3=\phi,\quad x_4=ct,\quad A=k=1,\quad B=0. \tag{19}$$

There are now but *three* $g_{\alpha\beta}$ and three $G_{\alpha\beta}$, since, by (16), (11), and (12), $G_{22}=G_{33}$, the counterpart of $T_2{}^2=T_3{}^3$. Also $(H_2{}^\beta)_\beta=0$ is now satisfied identically, so that, with $(H_1{}^\beta)_\beta=0$, equations (10)–(13) reduce to *two* independent equations. Hence in equation (6) we must put $c=0$, giving

$$a\lambda+b\mu+d\rho=\phi(x_1). \tag{6 a}$$

For $T_\alpha{}^\beta=0$ investigators have used the following particular cases of (6 a):

(i)† $\lambda=0$, (ii)‡ $\mu=2\log r$, (iii)§ $\mu=\lambda+2\log r$, (iv)‖ $\lambda+2\mu+\rho=4\log r$.

With $T_\alpha{}^\alpha\neq 0$ the only manageable cases seem to be for (i) μ any function of x_1; (ii) $\mu=\lambda+2\log x_1$.

(i) μ *prescribed.* Write (17) in the form

$$\mu_{11}+\tfrac{1}{2}(2\mu_1{}^2-\lambda_1\mu_1+\mu_1\rho_1)-2e^{\lambda-\mu}k=-2\kappa e^{\lambda}(T_1{}^1+T_4{}^4). \tag{20}$$

Put $\nu_1=\mu_1$ in (10) and take equations (10)–(13); then

$$2\mu_{11}+\tfrac{1}{2}(2\mu_1{}^2-2\lambda_1\mu_1-2\mu_1\rho_1)=4\kappa e^{\lambda}(T_1{}^1-T_4{}^4). \tag{21}$$

Add to this twice (20); then

$$4\mu_{11}+\tfrac{1}{2}(6\mu_1{}^2-4\lambda_1\mu_1)-4ke^{\lambda-\mu}=-8\kappa e^{\lambda}T_4{}^4. \tag{22}$$

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* *Berliner Sitzungsberichte* (1916), 189–196.

† J. Droste, *Proc. Amsterdam Acad.*, 19 (1916), 197–215.

‡ Schwarzschild, *loc. cit.*

§ F. W. Hill and G. B. Jeffery, *Phil. Mag.* (6), 41 (1921), 823–826, and an approximate solution by de Sitter, *Proc. Amsterdam Acad.*, 19 (1916), 367–381.

‖ Einstein, *Berliner Sitzungsberichte* (1915), 844–847.

Integrating, we have

$$e^{\lambda} = \tfrac{1}{4}\mu_1^{\,2} e^{\mu}\{k - 2e^{-\frac{1}{2}\mu}[m(x_1)+c]\}^{-1}, \tag{23}$$

where

$$m(x_1) = \tfrac{1}{2}\kappa \int T_4^{\,4}\mu_1 e^{\frac{3}{2}\mu} dx_1. \tag{24}$$

Equation (21) now gives, on substituting (23), (24) and integrating once,

$$e^{\rho} = D\{k - 2e^{-\frac{1}{2}\mu}[m(x_1)+c]\} \exp\phi(x_1), \tag{25}$$

where

$$\phi(x_1) = \int \frac{\kappa(T_4^{\,4} - T_1^{\,1})\mu_1 e^{\mu} dx_1}{k - 2e^{-\frac{1}{2}\mu}[m(x_1)+c]}. \tag{26}$$

Also

$$e^{\nu} = e^{\mu}(A/k)\sin^2\sqrt{k}(x_2 - B). \tag{27}$$

Equations (10) and (11) [or (12)] are now satisfied; also (13) is satisfied in consequence of (9). Use (19), put $e^{\frac{1}{2}\mu} = \chi(r)$, and substitute for r in terms of χ in $T_a^{\,a}$; then

$$e^{\lambda} dr^2 = \{1 - 2[m(\chi)+c]\chi^{-1}\}^{-1} d\chi^2, \tag{23a}$$

$$M(\chi) = \kappa \int^{\chi} T_4^{\,4}(\xi)\,\xi^2 d\xi, \tag{24a}$$

$$e^{\rho} = D\{1 - 2[M(\chi)+c]\chi^{-1}\} \exp\Phi(\chi), \tag{25a}$$

$$\Phi(\chi) = 2\kappa \int^{\chi} \frac{T_4^{\,4}(\xi) - T_1^{\,1}(\xi)}{1 - 2[M(\xi)+c]\xi^{-1}}\, \xi\, d\xi, \tag{26a}$$

$$e^{\nu} = \chi^2 \sin^2\theta \quad (A = 1,\ \theta = x_2 - B). \tag{27a}$$

With $\chi = r$, $D = 1$, this is J. T. Combridge's solution* for the spherical shell. It is to be noted that his conclusions involve implicitly

(i) that the coefficients of the line-element are nowhere zero or infinite inside the cavity of the shell,

(ii) that these coefficients are continuous across the boundary surfaces $r = r_1$, $r = r_2 (< r_1)$.

(iii) $\Phi(r_2) = \Phi(r_1)$†.

(ii) $\lambda = \mu - 2\log x_1$. (*a*) Here (22) becomes

$$4\mu_{11} + \mu_1^{\,2} + 4\mu_1/x_1 - 4k/x_1^{\,2} = -8\kappa e^{\mu} T_4^{\,4}/x_1^{\,2}. \tag{28}$$

Put $e^{\frac{1}{2}\mu} = v$, then (28) is

$$x_1^{\,2} v_{11} + x_1 v_1 - \tfrac{1}{4}kv = -\tfrac{1}{2}\kappa v^5 T_4^{\,4}. \tag{29}$$

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* *Phil. Mag.* (7), 1 (1926), 278.

† This shows that $T_4^{\,4}(r) - T_1^{\,1}(r)$ vanishes at some concentric spherical surface $r = r_3$ ($r_1 > r_3 > r_2$) and then changes its sign.

Now $\kappa = 9{\cdot}3\times10^{-28}$ cm./gm. Hence, as far as the first power of κ with $x_1 = r$, $k = 1$,

$$e^{\lambda} = r^{-1}e^{\mu}$$

$$= E + Fr^{-1} + \tfrac{1}{2}\kappa\int^{r} T_4^4(\xi)(E+F\xi^{-1})^5(\xi^2 r^{-1}-\xi)\,d\xi. \qquad (30)$$

This being the isotropic solution, we find by continuity of the $g_{\alpha\beta}$ at the boundaries of the shell $r = r_1$, $r = r_2$, and by using the Hill-Jeffery solution,

$$e^{\lambda} = 1 + 2\kappa\left\{\frac{r_1[v(r_1)-v(r)]+r_2[v(r)-v(r_2)]}{r_1-r_2} + \frac{u(r)-u(r_2)}{r} - \frac{r_1 r_2[v(r_1)-v(r_2)]}{r(r_1-r_2)}\right\}, \qquad (31)$$

where

$$u(r) = \int^{r} T_4^4(\xi)\,\xi^2\,d\xi, \qquad (32)$$

$$v(r) = \int^{r} T_4^4(\xi)\,\xi\,d\xi, \qquad (33)$$

$$m(r) = \kappa[u(r)-u(r_2)] \quad [m = m(r_1)]. \qquad (34)$$

For a sphere without cavity a simple investigation proves that near $r = 0$, $T_4^4 = ar^n (n > -2)$, and $\lim_{r_2\to 0} u(r_2)$ $(= -c)$, $\lim_{r_2\to 0} r_2 v(r_2)$ are zero. Then (31) is

$$e^{\lambda} = 1 + 2\kappa[v(r_1)-v(r)] + 2m(r)/r. \qquad (35)$$

Also, by (21),

$$e^{\rho} = [1-m(r)/r]^2 \exp\psi(r), \qquad (36)$$

$$\psi(r) = 2\kappa\int_0^{r}[T_4^4(\xi)-T_1^1(\xi)]\,\xi\,d\xi,$$

where

$$\psi(r_1) = 0.$$

(*b*) If $T_4^4 = \rho_0$ (a constant), $\sigma = \kappa\rho_0$, $k = 1$, $x_1 = r$, (29) has the exact solution

$$v = Er^{\frac{1}{2}}(1+\tfrac{1}{6}\sigma E^4 r^2)^{-\frac{1}{4}} \quad (E \text{ constant}).$$

Thus

$$e^{\lambda} = G(1+\tfrac{1}{6}\sigma G r^2)^{-2} \qquad (37)$$

$$= G\,[1+\tfrac{1}{2}Gm(r)/r)^{-2},$$

where

$$G = E^4, \qquad m(r) = \kappa\int_0^{r} T_4^4(\xi)\,\xi^2\,d\xi = \tfrac{1}{3}\sigma r^3.$$

Also, by using the Hill-Jeffery solution and by continuity of $g_{\alpha\beta}$ for $r = r_1$, we get

$$\tfrac{1}{2}\alpha(m/r_1)^2 G = 1-\alpha(m/r_1)-(1-2\alpha m/r_1)^{\frac{1}{2}}$$

$$\alpha = (1+m/2r_1)^4,$$

or, correct to the fourth power of m/r (as is the Hill-Jeffery solution),

$$G = 1+3\frac{m}{r_1}+\tfrac{27}{4}\left(\frac{m}{r_1}\right)^2+\tfrac{67}{4}\left(\frac{m}{r_1}\right)^3+\tfrac{709}{16}\left(\frac{m}{r_1}\right)^4.$$

Also $$e^{\rho} = \frac{1+\frac{1}{6}\sigma G r^2}{1-\frac{1}{6}\sigma G r^2}\exp\left\{-2\kappa G\int_0^r \frac{T_1{}^1(\xi)}{1-\sigma^2 G^2\xi^4/36}\,\xi\,d\xi\right\}, \quad [(21)] \quad (39)$$

provided that $T_1{}^1(r) = o(r^{-2+s})$ $(s>0)$ as $r\to 0$.

Having found λ, μ, ν, ρ, we derive $T_2{}^2$ from (13) in each case.

PART II. *The cylinder: approximate solutions.*

In this section we use method (ii) of Part I in dealing with (*a*) (p. 524), viz. $\nu_2 = 0$. Equation (9) is now*

$$dT_1{}^1/dx_1+\tfrac{1}{2}(T_1{}^1-T_2{}^2)\mu_1+\tfrac{1}{2}(T_1{}^1-T_3{}^3)\nu_1+\tfrac{1}{2}(T_1{}^1-T_4{}^4)\rho_1 = 0.$$

Written out in full the equations of (2) become

$$\mu_{11}+\nu_{11}+\rho_{11}+\tfrac{1}{2}[\mu_1{}^2+\nu_1{}^2+\rho_1{}^2-\lambda_1(\mu_1+\nu_1+\rho_1)] = -2\kappa e^{\lambda}(2T_1{}^1-T), \quad (40)$$

$$\mu_{11} \quad +\tfrac{1}{2}\mu_1(\mu_1+\nu_1+\rho_1-\lambda_1) \quad = -2\kappa e^{\lambda}(2T_2{}^2-T), \quad (41)$$

$$\nu_{11} \quad +\tfrac{1}{2}\nu_1(\mu_1+\nu_1+\rho_1-\lambda_1) \quad = -2\kappa e^{\lambda}(2T_3{}^3-T), \quad (42)$$

$$\rho_{11}+\tfrac{1}{2}\rho_1(\mu_1+\nu_1+\rho_1-\lambda_1) \quad = -2\kappa e^{\lambda}(2T_4{}^4-T). \quad (43)$$

On subtracting (40) from the sum of (41), (42), (43), we replace (40) by

$$\nu_1\rho_1+\rho_1\mu_1+\mu_1\nu_1 = 8\kappa e^{\lambda}T_1{}^1. \quad (44)$$

We have also (6) in the form

$$a\lambda+b\mu+c\nu+d\rho = \phi(x_1); \quad (45)$$

this gives infinite scope, but the analysis is only of the four following cases, viz. (i) $\lambda = 0$, (ii) $\lambda = \mu$, (iii) $\nu = \lambda+2\log x_1$, (iv) $\nu = \mu+2\log x_1$, where $(x_1, x_2, x_3, x_4) \equiv (r, z, \phi, ct)$, and the procedure adopted is first to make κ (or $T_{\alpha}{}^{\beta}$) zero in (41) to (44) giving (with $\mu' = \mu_1, \ldots$)

$$\mu_0''+\tfrac{1}{2}\mu_0'(\mu_0'+\nu_0'+\rho_0'-\lambda_0') = 0, \quad (46)$$

$$\nu_0''+\tfrac{1}{2}\nu_0'(\mu_0'+\nu_0'+\rho_0'-\lambda_0') = 0, \quad (47)$$

$$\rho_0''+\tfrac{1}{2}\rho_0'(\mu_0'+\nu_0'+\rho_0'-\lambda_0') = 0, \quad (48)$$

$$\nu_0'\rho_0'+\rho_0'\mu_0'+\mu_0'\nu_0' = 0. \quad (49)$$

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* For greater symmetry we write $ds^2 = e^{\lambda}dx_1{}^2+e^{\mu}dx_2{}^2+e^{\nu}dx_3{}^2+e^{\rho}dx_4{}^2$.

We solve for the $(g_{\alpha\beta})_0$, then substitute

$$\lambda' = \lambda_0' + \kappa\lambda_1', \quad \mu' = \mu_0' + \kappa\mu_1', \quad \nu' = \nu_0' + \kappa\nu_1', \quad \rho' = \rho_0' + \kappa\rho_1' \tag{50}$$

in the left-hand side of (41) to (44) and expand as far as the first power of κ; then, using (46)–(49) and cancelling κ, we get

$$\mu_1'' + \tfrac{1}{2}\mu_1'(\Sigma\mu_0' - \lambda_0') + \tfrac{1}{2}\mu_0'(\Sigma\mu_1' - \lambda_1') = -2(2T_2^2 - T)\exp\lambda_0, \ \ldots \tag{51}$$

$$\Sigma\mu_1'(\nu_0' + \rho_0') = 8T_1^1\exp\lambda_0. \tag{52}$$

The solutions of these four linear differential equations vary in general with the ten selections of three of the five data T_1^1, T_2^2, T_3^3, T_4^4, T (the invariant density). Only the simplest examples are taken.

(i) $\lambda = 0$. Then $\lambda_0 = 0 = \lambda_1$.

Put
$$\tau' = \mu' + \nu' + \rho', \tag{53}$$

then
$$\exp(\tfrac{1}{2}\tau_0) = B + Ax_1, \tag{54}$$

$$(\mu_0', \nu_0' \rho_0') = 2(A_1, A_2, A_3)(B + Ax_1)^{-1}, \tag{55}$$

where, by addition,
$$\Sigma A_1 = A, \tag{56}$$

and, by (49),
$$\Sigma A_2 A_3 = 0, \tag{57}$$

the A's and B being constants.

If T, T_1^1, T_2^2 are known, we get, by adding equations (51),

$$\tau_1' = 2(B + Ax_1)^{-2}\int (T + 2T_1^1)(B + Ax_1)^2 dx_1. \tag{58}$$

(51) now gives

$$\mu_1'(B + Ax_1) = -A_1\tau_1 - 2\int (2T_2^2 - T)(B + Ax_1)dx_1, \tag{59}$$

while, by (52), (53),

$$(A_2 - A_3)\nu_1' = (A_1 + A_2)\tau' + (A_3 - A_1)\mu_1' - 4T_1^1(B + Ax_1), \tag{60}$$

$$(A_2 - A_3)\rho_1' = -(A_1 + A_3)\tau_1' + (A_1 - A_2)\mu_1' + 4T_1^1(B + Ax_1). \tag{61}$$

The second of (51) then gives T_3^3; and then

$$T_4^4 = T - T_1^1 - T_2^2 - T_3^3. \tag{62}$$

(ii) $\lambda = \mu$. Put
$$\eta' = \nu' + \rho', \tag{63}$$

then
$$\exp(\tfrac{1}{2}\eta_0) = Ax_1 + B,$$

$$(\mu_0', \nu_0', \rho_0') = 2(A_1, A_2, A_3)(Ax_1 + B)^{-1}, \tag{64}$$

where, by (63),
$$A_2+A_3=A, \tag{65}$$

and, by (49),
$$\Sigma A_2A_3=A_2A_3+AA_1=0. \tag{66}$$

Again,
$$\mu'=\lambda'=2A_1(Ax_1+B)^{-1}+\kappa\lambda_1';\ \exp\lambda_0=C(Ax_1+B)^{2A_1/A}. \tag{67}$$

With given T, T_1^1, T_2^2, we get from (51)
$$\eta_1'=4C(Ax_1+B)^{-2}\int(T_1^1+T_2^2)(Ax_1+B)^{2(A_1/A+1)}dx_1. \tag{68}$$

Substituting (67) in the first of (51) and integrating, we get
$$\lambda_1'(Ax_1+B)=-A_1\eta_1-2C\int(2T_2^2-T)(Ax_1+B)^{2A_1/A+1}dx_1. \tag{69}$$

Also, from (63), (52), and (66),
$$(A_2-A_3)\nu_1'=A\lambda_1'+(A_1+A_2)\eta_1'-4CT_1^1(Ax_1+B)^{2A_1/A+1}, \tag{70}$$
$$(A_2-A_3)\rho_1'=-A\lambda_1'-(A_3+A_1)\eta_1'+4CT_1^1(Ax_1+B)^{2A_1/A+1}. \tag{71}$$

T_3^3 is got from the second of (51) and T_4^4 from (62).

If, however, we were given T, T_3^3, T_4^4, we should use (68) and the integrals of the second and third of (51) in the forms
$$(Ax_1+B)^2\eta_1'=-4C\int(Ax_1+B)^{2(A_1/A+1)}(T_3^3+T_4^4-T)\,dx_1, \tag{72}$$
$$(Ax_1+B)\nu_1'=-A_2\eta_1-2C\int(Ax_1+B)^{2A_1/A+1}(2T_3^3-T)\,dx_1, \tag{73}$$
$$(Ax_1+B)\rho_1'=-A_3\eta_1-2C\int(Ax_1+B)^{2A_1/A+1}(2T_4^4-T)\,dx_1. \tag{74}$$

Integrating (72) by parts proves (73) and (74) to be consistent. To find λ, T_1^1, T_2^2 we have, from (52) and the derivative of (69),
$$(A\lambda_1'+A_1\eta_1'+A_3\nu_1'+A_2\rho_1')(Ax_1+B)^{-1}=4T_1^1C(Ax_1+B)^{2A_1/A}, \tag{75}$$
$$\lambda_1''+(A\lambda_1'+A_1\eta_1')(Ax_1+B)^{-1}$$
$$=-2C(T-2T_1^1-2T_3^3-2T_4^4)(Ax_1+B)^{2A_1/A}. \tag{76}$$

Subtract (75) from (76); then
$$\lambda_1''=(A_3\nu_1'+A_2\rho_1')(Ax_1+B)^{-1}-2C(T-2T_3^3-2T_4^4)(Ax_1+B)^{2A_1/A},$$
whence λ is obtained by two integrations. Then, substituting for λ_1' in (75), we get T_1^1; and T_2^2 is obtained from (62).

(iii) $\nu=\lambda+2\log x_1$. Here
$$\nu'=\lambda'+2/x_1. \tag{77}$$

Let $$\zeta' = \mu' + \rho'. \tag{78}$$

Then $$\exp(\tfrac{1}{2}\zeta_0) = A \log x_1 + B$$

and $$(\mu_0', \nu_0', \rho_0') = 2(A_1, A_2, A_3) x^{-1}(A \log x_1 + B)^{-1}, \tag{79}$$

with $$A = A_1 + A_3, \tag{80}$$

and, by (49), $$\Sigma A_2 A_3 = A_3 A_1 + A A_2 = 0, \tag{81}$$

whence $$\exp \lambda_0 = C(A \log x_1 + B)^{2A_2/A} x_1^{-2}.$$

We now have

$$\zeta_1'(A \log x_1 + B)^2 x_1 = 4C \int u(A \log x_1 + B)^{2(A_2/A+1)} x_1^{-1} dx_1, \tag{82}$$

where $$u = T_1^{\,1} + T_3^{\,3} = T - T_2^{\,2} - T_4^{\,4}. \tag{83}$$

If T, $T_1^{\,1}$, $T_3^{\,3}$ are known, then (51) and $\nu_1' = \lambda_1'$ [from (77)] give us

$$\lambda_1' x_1 (A \log x_1 + B) = -A_2 \zeta_1 + 2C \int (2T_3^{\,3} - T)(A \log x_1 + B)^{2A_2/A+1} x_1^{-1} dx_1. \tag{84}$$

Finally, by (52) and (78) with (82), (83), and (84),

$$(A_3 - A_1)\mu_1' = -A\lambda_1' - (A_1 + A_2)\zeta_1' + 4T_1^{\,1} C x_1^{-1}(A \log x_1 + B)^{2A_2/A+1}, \tag{85}$$

$$(A_3 - A_1)\rho_1' = A\lambda_1' + (A_2 + A_3)\zeta_1' - 4T_1^{\,1} C x_1^{-1}(A \log x_1 + B)^{2A_2/A+1}.$$

If T, $T_2^{\,2}$, $T_4^{\,4}$ are given, by integrating (51) and using (82), (83), we get

$$\mu_1' x_1 (A \log x_1 + B) = -A_1 \zeta_1 - 2C \int (2T_2^{\,2} - T) x_1^{-1}(A \log x_1 + B)^{2A_2/A+1} dx_1, \tag{86}$$

$$\rho_1' x_1 (A \log x_1 + B) = -A_3 \zeta_1 - 2C \int (2T_4^{\,4} - T) x_1^{-1}(A \log x_1 + B)^{2A_2/A+1} dx_1. \tag{87}$$

Then, by a method similar to that adopted with equations (75), (76),

$$\frac{d(\lambda_1' x_1)}{dx_1} = \frac{A_3 \mu_1' + A_1 \rho_1'}{A \log x_1 + B} - \frac{2}{x_1} C(T - 2T_2^{\,2} - 2T_4^{\,4})(A \log x_1 + B)^{2A_2/A}.$$

Equations (85) and (83) now give $T_1^{\,1}$ and $T_3^{\,3}$ respectively.

(iv) $\nu = \mu + 2 \log x_1$. With

$$\xi' = 2\mu' + \rho', \tag{88}$$

we can obtain $$\lambda_0' = \xi_0' = A/x_1 \tag{89}$$

and $$(\mu_0', \nu_0', \rho_0') = (A_1, A_2, A_3) x_1^{-1}, \tag{90}$$

where $$A = 2A_1 + A_3 \tag{91}$$

and $$\Sigma A_2 A_3 = 0. \tag{92}$$

Putting $$\exp \lambda_0 = Bx_1{}^A, \tag{93}$$

we now derive $$\xi_1 = \lambda_1 - 4B \int x_1^{A+1} (T_3{}^3 - T_2{}^2)\, dx_1, \tag{94}$$

if T, $T_2{}^2$, $T_3{}^3$ are given. Also, from the first of (51) after an integration,

$$\mu_1' x_1 = -2B \int x_1^{A+1} [(2T_2{}^2 - T) + A_1 (T_2{}^2 - T_3{}^3)]\, dx_1. \tag{95}$$

By taking the first and third of (51), using (93) and (94) and eliminating $T_1{}^1$ between the result and (52), we have, using (90) and (91),

$$\xi_1'' - A_1 \xi_1' x_1^{-1} = \tfrac{1}{2}\mu_1' x_1^{-1}(2A_3 - A_1 - A_2) - 2Bx_1{}^A [(2+A)(T_2{}^2 - T_3{}^3) - T], \tag{96}$$

whence, by (95), ξ_1' and ξ_1 are easily found. Then λ_1 is got from (94) and ρ_1' from (95) and (88) reduced. Substituting in (52) and the third of (51), we obtain $T_1{}^1$, $T_4{}^4$.

But, if $T_1{}^1$, $T_4{}^4$ are given, then, by differentiating (52),

$$\mu_1'' + \frac{\mu_1'}{x_1} = -\frac{A_1 + A_2}{A_1 + A_2 + 2A_3} \left(\rho_1'' + \frac{\rho_1'}{x_1}\right) + 8Bx_1{}^A \frac{[(2+A) T_1{}^1 + x_1 (T_1{}^1)']}{A_1 + A_2 + 2A_3}. \tag{97}$$

By adding twice the first to the third of (51), we get

$$2(\mu_1'' + \mu_1' x_1^{-1}) + \rho_1'' + \rho_1' x_1^{-1} = -2Bx_1{}^A (2+A)(T_2{}^2 - T_3{}^3) + 2Bx_1{}^A (2T_1{}^1 + T).$$

The third of (51) is

$$\rho_1'' + \rho_1' x_1^{-1} = -2Bx_1{}^A (2T_4{}^4 - T) - BA_3 x_1{}^A (T_2{}^2 - T_3{}^3).$$

Eliminating $T_2{}^2 - T_3{}^3$ from these and substituting from (97) we derive an equation of the form

$$\rho_1'' + \rho_1' x_1^{-1} = f'(x_1) x_1^{-1},$$

whence $$\rho_1' x_1 = f(x_1). \tag{98}$$

Put this value in (97) to get μ_1'; then, since, by (88),

$$\xi_1' = 2\mu_1' + \rho_1',$$

equation (96) gives $T_2{}^2 - T_3{}^3$; the insertion of this in the derivative of (95) gives $T_2{}^2$. From these values $T_3{}^3$ is found and ultimately ν_1' $(= \mu_1')$ and λ_1 [from (94)] are obtained.

The reason for adopting the forms (i)–(iv) above is that, if

$$ds^2 = dl^2 + e^{\rho} dx_4^2,$$

the most practicable forms for dl^2 appear to be

(i) $$dr^2 + e^{\mu} dz^2 + e^{\nu} d\phi^2,$$

(ii) $$e^{\mu}(dr^2 + dz^2) + e^{\nu} d\phi^2,$$

(iii) $$e^{\nu}(dr^2 + r^2 d\phi^2) + e^{\mu} dz^2,$$

(iv) $$e^{\nu}(dz^2 + r^2 d\phi^2) + e^{\lambda} dr^2.$$

(v) However, since for weak internal fields the $g_{\alpha\beta}$ might not differ appreciably from Euclidean values, it seems equally feasible to take

$$dl^2 = dr^2 + dz^2 + r^2 d\phi^2,$$

and to put $\lambda_0' = \mu_0' = \rho_0' = 0$, $\nu_0' = 2x_1^{-1}$. We could deal with cases (i) to (iv) on these lines, but only (ii) is given here. In this, if we know T, T_1^1, T_2^2, the only change up to (69) is to make A_1, A_3, $B = 0$; $A_2 = A = 1$. (52) is

$$\mu_1' + \rho_1' = 4x_1 T_1^1. \tag{99}$$

Then $$\eta_1' x_1^2 = 4\int (T_1^1 + T_2^2) x_1^2 dx_1 \tag{100}$$

and $$\mu_1' = \lambda_1' = -2x_1^{-1} \int (2T_2^2 - T) x_1 dx_1. \tag{101}$$

From (99), (101) we get ρ, and finally ν_1 from

$$\nu_1' = \eta_1' - \rho_1'.$$

*Wilson's rod.** W. Wilson used

$$ds^2 = e^{\lambda} dr^2 + e^{\mu} dz^2 + r^2 d\phi^2 + e^{\rho} dx_4^2 \tag{102}$$

for the exterior line-element of an infinite rod. Applying to this form the present type of analysis, put

(vi) $\nu = 2 \log x_1$. Then

$$(\mu_0', \rho_0') = (A_1, A_3) x_1^{-1}, \tag{103}$$

$$\lambda_0' = \mu_0' + \rho_0' = (A_1 + A_3) x_1^{-1}, \tag{104}$$

where, by (49), $$0 = 2A_1 + 2A_3 + A_1 A_3. \tag{105}$$

Put $$\mu' + \rho' - \lambda' = \chi' = \kappa \chi_1' \quad \text{by (50).} \tag{106}$$

* *Phil. Mag.* (6), 40 (1920), 703–712.

Then $\chi_0 = 0$ and

$$\chi_1' = -4Bx_1^{A_1+A_3+1}(2T_3^3 - T) \quad \text{by (104).} \tag{107}$$

From the first of (51),

$$\mu_1' = -\tfrac{1}{2}A_1\chi_1 x_1^{-1} - 4Bx_1^{-1}\int x_1^{A_1+A_3+1}(2T_2^2 - T)\,dx_1. \tag{108}$$

Knowing T, T_2^2, T_3^3, we can add (107), (52), and the first and third of (51), giving

$$d[\rho_1'(x_1-2-A_1)+\mu_1'(x_1-2-A_3)+\tfrac{1}{2}(A_1+A_3+2)\chi_1]/dx_1 = 4Bx_1^{A_1+A_3+1}T, \tag{109}$$

from which, by (107), (108), we find ρ_1'; then T_1^1 is got from (52) and finally, by (106), the equation

$$\lambda_1 = \mu_1 + \rho_1 - \chi_1 \tag{110}$$

yields λ_1, while the third of (51) gives T_4^4. Clearly the same methods apply if T_2^2, T_4^4 are interchanged. If T_1^1 is given instead of T_2^2, we still get χ_1 from (107), then we have μ_1', ρ_1' from (52) and the integral of (109), λ_1 from (110), and T_2^2, T_4^4 from the second of (51) and (52) respectively.

If T, T_1^1, and one of T_2^2, T_4^4 (say T_2^2) are given, we get, by eliminating χ_1 between the first of (51) and the sum of (107) and the third of (51), and integrating the result,

$$(A_3+2)x_1\mu_1' - A_1x_1\rho_1' = -4B\int x_1^{A_1+A_3+1}[(A_3+2)(2T_2^2-T)+2A_1(T_1^1+T_2^2)]\,dx_1$$

and by (52),

$$(A_3+2)x_1\mu_1' + (A_1+2)x_1\rho_1' = 8Bx_1^{A_1+A_3+1}T_1^1.$$

Thus μ_1', ρ_1' are obtained; T_4^4 is deduced from the third of (51); T_3^3 from (62), from (107), and lastly λ_1 from (110). The problem is therefore solved.

PART III. *The cylinder: exact solutions.*

A large number of cases can be obtained by varying the relation

$$a\lambda + b\mu + c\nu + d\rho = \phi(x_1), \tag{45}$$

and choosing certain selections of the four T_α^β and T. Only a classification is here made and some forms for T_α^β, T are suggested in order to facilitate solution of equations (40) to (44). No interpretation is attempted.

Adding (41), (42), (43) we get, if

$$v' = \mu' + \nu' + \rho', \tag{111}$$

$$v'' + \tfrac{1}{2}v'^2 - \tfrac{1}{2}\lambda' v' = 2\kappa e^{\lambda}(2T_1{}^1 + T). \tag{112}$$

(I) $a \neq 0$, $b = c = d = 0$. This is the same as assigning λ arbitrarily. Given $T_1{}^1$, $T_2{}^2$, if the Riccati equation (112) can then be solved for v, (41) integrated gives

$$\mu' e^{\frac{1}{2}(v-\lambda)} = A_1 - 2\kappa \int e^{\frac{1}{2}(v+\lambda)}(2T_2{}^2 - T)\, dx_1. \tag{113}$$

Substituting in (44) for ρ' from (111), we get a quadratic for ν', then (111) gives ρ', (42) and (43) yield $T_3{}^3$, $T_4{}^4$.

(i) If we put $$v' = 2u'/u, \tag{114}$$

(112) becomes

$$d[e^{-\frac{1}{2}\lambda} u']/dx_1 = \kappa(2T_1{}^1 + T)\, u e^{\frac{1}{2}\lambda}.$$

We have Legendre's equation if

$$\exp(-\tfrac{1}{2}\lambda) = x_1{}^2 - 1, \quad \kappa(2T_1{}^1 + T) = n(n+1)(x_1{}^2 - 1). \tag{115}$$

(ii) Writing $$e^{\nu} = e^{\sigma} x_1{}^2, \tag{116}$$

$$w' = \mu' + \sigma' + \rho' = v' - 2/x_1,$$

(112) is

$$dw'/dx_1 + \tfrac{1}{2}w'^2 + (2/x_1 - \tfrac{1}{2}\lambda')\, w' = \lambda'/x_1 + 2\kappa e^{\lambda}(T + 2T_1{}^1). \tag{117}$$

Equate each side of (117) to zero, then

$$e^{-\lambda} = A + 2\kappa \int x_1(T + 2T_1{}^1)\, dx_1, \tag{118}$$

$$\frac{e^{\frac{1}{2}\lambda}}{w' x_1{}^2} = A_0 + \tfrac{1}{2} \int \frac{e^{\frac{1}{2}\lambda}}{x_1{}^2}\, dx_1. \tag{119}$$

(iii) Put $$e^{\lambda} = x_1{}^4, \tag{120}$$

(a) $$2\kappa(T + 2T_1{}^1) = m x_1^{-6}, \tag{121}$$

then $$w' = \{1 \pm \sqrt{(9 + 2m)}\}\, x_1^{-1}; \tag{122}$$

(b) $$2\kappa x_1{}^6(T + 2T_1{}^1) = -4 + D x_1^{2\epsilon(2i+\epsilon)^{-1}}, \quad \epsilon = \pm 1, \tag{123}$$

then (117) is integrable in finite terms.

(II) $b = c = d$. Write (111) and (45) in the form

$$v' = \mu' + \nu' + \rho' = m\lambda' + \phi'(x_1), \tag{124}$$

then (112) becomes

$$m\lambda''+\tfrac{1}{2}m(m-1)\lambda'^2+\tfrac{1}{2}(2m-1)\phi'\lambda'+\phi''+\tfrac{1}{2}\phi'^2=2\kappa e^{\lambda}(T+2T_1{}^1). \quad (125)$$

(i) $m=0$. Equation (125) now gives, on integration,

$$\phi'^2 e^{\phi-\lambda}=A+4\kappa\int(T+2T_1{}^1)\phi' e^{\phi}\,dx_1. \quad (126)$$

(ii) Equation (126) breaks down if $\phi'=0$; then, by (125),

$$T=-2T_1{}^1. \quad (127)$$

We can now assign an extra relation such as (45).

(ii a) λ *an arbitrary function.* We can here use (I) (i), (ii), or (iii). In particular for (i) with $\lambda=0$ we get from (113) the surprisingly simple result

$$\mu'=A_1-2\kappa\int(2T_2{}^2-T)\,dx_1. \quad (128)$$

(ii b) *Data.* $\nu=k\mu$, $v'=0$, T, $T_2{}^2$. Writing (44) as

$$\nu'^2+\nu'(\mu'-v')+\mu'^2-v'\mu'+8\kappa e^{\lambda}T_1{}^1=0, \quad (129)$$

we get, by (ii b),

$$(k^2+k+1)\mu'^2=-8\kappa e^{\lambda}T_1{}^1=4\kappa e^{\lambda}T \quad \text{by (127).} \quad (130)$$

Differentiate, then

$$\tfrac{1}{2}\lambda'=\mu''/\mu'-\tfrac{1}{2}d(\log T)/dx_1, \quad (131)$$

so that (41) becomes

$$2\kappa e^{\lambda}(2T_2{}^2-T)=\tfrac{1}{2}\mu'\lambda'-\mu''=-\tfrac{1}{2}\mu' T^{-1}dT/dx_1. \quad (132)$$

Eliminating λ between (130), (132), and integrating, we get

$$\mu=A+\int\frac{T'\,dx_1}{(k^2+k+1)(T-2T_2{}^2)}. \quad (133)$$

λ is now found from (130), ν from (ii b), ρ' from (124), $T_3{}^3, T_4{}^4$ from (42), (43).

(ii c) *Data.* $v'=0$, T, $T_2{}^2$, $\alpha(2T_2{}^2-T)+\beta(2T_3{}^3-T)=0$ (α, β *constants*), $T_1{}^1=-\tfrac{1}{2}T$. Hence

$$2T_4{}^4-T=(\alpha-\beta)(2T_2{}^2-T)/\beta. \quad (134)$$

Eliminate T, $T_3{}^3$ from (42), (41) by (ii c) and integrate, then

$$e^{\lambda}=A(\alpha\mu'+\beta\nu')^2. \quad (135)$$

Since $\rho' = -(\mu'+\nu')$, (44) gives $\nu' = \gamma\mu'$, where

$$\gamma = \frac{8\kappa TA\alpha\beta - 1 \pm \sqrt{\{16\kappa AT(\alpha^2-\alpha\beta+\beta^2)-3\}}}{2(1-4\kappa TA\beta^2)}. \tag{136}$$

Thus, from (135),

$$e^\lambda = A\mu'^2(\alpha+\beta\gamma)^2 = A\mu'^2\delta^2, \tag{137}$$

where, by (136),

$$\delta = \frac{2\alpha-\beta \pm \beta\sqrt{\{16\kappa TA(\alpha^2-\alpha\beta+\beta^2)-3\}}}{2(1-4\kappa TA\beta^2)}. \tag{138}$$

Eliminating λ between (137) and (41), using $v' = 0$, then integrating, we get

$$\mu = B - \frac{1}{4\kappa A\delta^2(2T_2{}^2-T)} + \frac{1}{4\kappa A}\int \frac{dx_1}{\delta^2}\frac{d}{dx_1}(2T_2{}^2-T)^{-1}. \tag{139}$$

(137) now gives $$e^\lambda = \delta'^2/4\kappa^2 A\delta^4(2T_2{}^2-T)^2. \tag{140}$$

Then $\nu' = \gamma\mu' = (\delta-\alpha)\mu'/\beta$, $\rho' = (\alpha-\beta-\delta)\mu'/\beta$ are found.

(iii) $m \neq 0$. Put

$$\lambda = -\phi/m + 2\log u/(m-1); \tag{141}$$

then (125) is

$$u'' + \phi' u'/2m = \kappa(1-m^{-1})e^{-\phi/m}u^{(m+1)/(m-1)}(T+2T_1{}^1). \tag{142}$$

(iii a) *Data.* (124), $T+2T_1{}^1 = 0$, $2T_2{}^2-T$, $2T_3{}^3-T$. (141), (142) give

$$\left.\begin{aligned} u &= B + A\int \exp(-\tfrac{1}{2}\phi/m)\,dx_1, \\ e^\lambda &= \exp(-\phi/m)\Big[B + A\int \exp(-\tfrac{1}{2}\phi/m)\,dx_1\Big]^{2/(m-1)} \end{aligned}\right\}. \tag{143}$$

For $m = 1$, the substitution (141) is invalid, but, for (127), (125) is easily integrable, giving

$$\lambda = -\phi + B + A\int \exp(-\tfrac{1}{2}\phi)\,dx_1. \tag{144}$$

v' is got by (124), then μ', ν' from (41), (42), $\rho'(= v'-\mu'-\nu')$, $T_1{}^1$ from (44), and $T(= -2T_1{}^1)$ from (iii a).

(iii b) *Data.* $m = -1$, T, $T_1{}^1$, $T_2{}^2$. (142) gives, on integration,

$$u' \exp(-\tfrac{1}{2}\phi) = A + 2\kappa\int (T+2T_1{}^1)\exp\tfrac{1}{2}\phi\,dx_1;$$

then, by (141),

$$e^{-\lambda} = e^{-\phi}\Big[B + A\int \exp\tfrac{1}{2}\phi\, dx_1 + 2\kappa\int \exp\tfrac{1}{2}\phi\Big\{\int \exp\tfrac{1}{2}\phi(T+2T_1^{\,1})\,dx_1\Big\}dx_1\Big]. \quad (145)$$

(124) gives v' and (41) gives μ'. Put $\rho' = v' - \mu' - \nu'$ in (44) to get ν', then ρ' is obtained, and $T_3^{\,3}$, $T_4^{\,4}$ from (43), (44).

(iii c) If ϕ is zero or a constant and

$$\kappa(T+2T_1^{\,1})\exp(-\phi/m) = A, \quad (146)$$

where A is a constant, (142) can be integrated, giving

$$u'^2 = B + \{(m-1)/m\}^2 A\, u^{2m/(m-1)}. \quad (147)$$

If $m = \frac{1}{2}$, $\phi = 0$, $\qquad e^{\lambda} = u^{-4} = B^2\{-A + (Bx_1 + C)^2\}^{-2}$. (148)

(iii d) $m = \frac{1}{2}$, $\phi' = 2k$, $2T_1^{\,1} + T = a$, a constant. From (125),

$$k\exp(-\tfrac{1}{2}\lambda) = C + \sqrt{(C^2 - \kappa a)}\cosh(2kx_1 + B). \quad (149)$$

(III) *Data.* $\lambda = 0$, $2T_1^{\,1} + T = 0$, T, $T_2^{\,2}$. From (112),

$$\exp\tfrac{1}{2}v = Ax_1 + B; \quad (150)$$

then, by (41),

$$\mu'(Ax_1 + B) = A_1 - 2\kappa\int (Ax_1 + B)(2T_2^{\,2} - T)\,dx_1, \quad (151)$$

an equation similar to (59) [$\tau_1 = 0$ by (54) and (150)]. Substitute $\rho' = v' - \mu' - \nu'$, $2T_1^{\,1} = -T$ in (44) to get ν', then ρ' is obtained and $T_3^{\,3}$, $T_4^{\,4}$ are then obtainable from (42), (43).

(IV) *Data.* $\lambda = \mu$, $T_2^{\,2} = -T_1^{\,1}$, $2\kappa(2T_2^{\,2} - T)\,(= \beta)$, $2\kappa(T_3^{\,3} - T)(= \gamma)$. Add (42), (43), and integrate. Then

$$\nu' + \rho' = 2(x_1 + B/A)^{-1} = 2X^{-1} \quad \text{(say)}. \quad (152)$$

(41) thus becomes $\qquad X\mu'' + \mu' = -e^{\mu}\beta X.$

If $\beta = 2n^2 CDX^{n-2}$, $\qquad e^{\mu} = (C + DX^n)^{-2}$, (153)

and, integrating (42), we get

$$X\nu' = E - \int \gamma X e^{\mu}\, dX. \quad (154)$$

ρ' is got from (152), (154), λ from (IV). T is got from (40) after $T_1^{\,1}$ from (44); $T_2^{\,2}(= -T_1^{\,1})$, $T_4^{\,4}$ from (43)*.

* If $B = 0$, we get the case discussed for empty space (with $g_{\mu\nu}$ functions of two variables) by Levi-Civita, Weyl, and others.

(V) *Data.* $\nu = \lambda + 2\log x_1$, $T_3{}^3 = -T_1{}^1$, $2\kappa(2T_2{}^2 - T)$, $2\kappa(2T_3{}^3 - T)$ $(= \gamma)$. Add (41) to (43), and integrate, then

$$\mu' + \rho' = 2x_1^{-1}(A + \log x_1)^{-1}. \tag{155}$$

Substitute (155) in (42), which becomes

$$\lambda'' + \lambda'\left[\frac{1}{x_1(A+\log x_1)} + \frac{1}{x_1}\right] = -\gamma e^{\lambda}. \tag{156}$$

This can be integrated if γ has either of the two forms

(i) $(-B^2)/\{2x_1{}^2(A+\log x_1)^2\}$; (ii) $(-m)/\{Bx_1^{m+2}(A+\log x_1)^{n+1}\}$;

giving

(i) $4c/B \exp(-\frac{1}{2}\lambda) = D(A+\log x_1)^{-c} - D^{-1}(A+\log x_1)^c$;

(ii) $e^{\lambda} = Bx_1{}^m(A+\log x_1)^n$.

μ' is now got from (41), ρ' from (155), ν from (V). Adding (41), (42), (43), (40) we get T; $T_1{}^1$ from (44), $T_3{}^3 (= -T_1{}^1)$ and $T_4{}^4$ from (43).

(VI) *Data.* $\nu = \mu + 2\log \dot{x}_1$, T, $2T_1{}^1 = -T$, $2\kappa(T_2{}^2 - T_3{}^3) = C$. Then

$$e^{-\lambda} = Cx_1{}^2 \log x_1, \quad \nu' = \mu' + \nu' + \rho' = (x_1 \log x_1)^{-1}, \tag{157}$$

$$3\mu'^2 + 2\mu'(3x_1^{-1} - 2\nu') + 2\nu' x_1^{-1} - 4x_1^{-2} = -8\kappa e^{\lambda} T_1{}^1. \tag{158}$$

ν is now got from (VI), ρ' from (157), $T_2{}^2$, $T_3{}^3$, $T_4{}^4$ from (41), (VI), (62).

There are about a dozen other integrable cases corresponding to various modifications of (45) and conditions on the $T_\alpha{}^\beta$, but the cases above selected and catalogued are the least eccentric and most interesting.

SOME FORMULAE FOR PRIMALS IN FOUR DIMENSIONS

By L. ROTH.

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The formulae of this paper are concerned with the singularities of a primal in [4] whose space section is a surface containing a double curve of general type; such a primal, therefore, has a double surface* on which there is a triple curve containing a number of quadruple points. It is thus a particular case of the primal to be considered in developing the theory of reciprocal polars in four dimensions (which would necessitate the additional assumption of a cuspidal surface) and, in fact, the equations given in § 3 are the first step towards such a theory.

The formulae referred to are obtained by considering the coincidences of various correspondences on one or more curves; in every case the terms which appear in the correspondences are obvious enough, but the method of evaluating the multiplicities of the coincidences *ab initio* (by Zeuthen's rule) would be a difficult matter. However, the difficulty is avoided by making use of the results already found for primals generated by planes or lines†; most of the formulae required for our present purpose have been given in these investigations, and any others that are necessary are readily determined by the same methods. The process that has been adopted in the sequel is to insert in the correspondence equations the formulae for the terms which occur, and then to determine their multiplicities by inspection of the equations, which thus reduce to identities.

The correspondence formulae employed are of two familiar types, as follows.

I. *The general correspondence theorem.* This is conveniently stated in two forms:

(i) If on a scroll of order c there are two curves of orders α, α' meeting an arbitrary generator in x and x' points respectively, then the number of intersections of the curves is $x\alpha'+x'\alpha-c$.

* See Roth, *Proc. Camb. Phil. Soc.*, 28 (1932), 300.

† "On plane-forms in four dimensions", *Proc. London Math. Soc.* (2), 33 (1931), 115; "Ruled forms in four dimensions", *ibid.*, 35 1933), 380.

(ii) In the correspondence $(x-1, x-1)$ set up on the first curve by the generators, the number of united points is

$$2(x-1)a-c.$$

In the applications which follow, the scroll is invariably a cone.

II. *Chasles' correspondence theorem.* If the points of a line are in (x, x') correspondence, the number of united points is $x+x'$.

The general correspondence theorem is used exclusively to obtain the equations in § 3; Chasles' theorem is employed in §§ 5, 7. In the subsidiary equations of § 4 it is again obvious which terms will enter into the expression; their multiplicities may then be determined, as has been already explained.

There is another problem of some interest and importance which the results of this paper enable us to solve; given a primal with a double surface of the prescribed type, how many characters of the primal are to be chosen at will? For a surface containing a double curve with pinch-points and triple points it is known that four characters define the surface completely; these may be taken to be the order n, the order b and rank μ_1 of the double curve, and the number t of its triple points; or, if we retain the additive characters only, the last three are replaced by the rank a and class m of the surface, and the number j of its pinch-points.

For the primal it will be shown that there are seven independent characters; besides those already described above there are the ranks r_j, r_t of the curves of pinch-points and triple points, and the number ξ of quadruple points on the triple curve. For the two latter we may substitute additive characters, as in the case of the surface; we use, in fact, the characters w, j_1, and j_0 (§ 1). One object of the present paper is to show how this may be accomplished.

The nature of the singularities of the primal will depend upon the generality which it is assumed to possess. Now a surface in [4] has, in general, a finite number of improper nodes, whence it might be supposed that such nodes would occur on the double surface considered in the present investigation. We shall, however, assume that the given primal has the characteristics of the projection, from an arbitrary vertex, of a general threefold in higher space. In this case there are no improper nodes on the double surface. For consider the projection of a general threefold in [5] from an external point O; the double surface arises from the chords drawn from O to the threefold, the triple curve and the quadruple points from the trisecants and quadrisecants passing through O. If in the projection the double surface had an improper node, this could only arise

from an isolated chord drawn to the apparent double surface, and no such chords exist. The singularities of the double surface are due to the trisecants and quadrisecants above mentioned, and there can be no others*.

1. *Singularities of the primal.* Let the double surface B be of order b (or μ_0), rank μ_1, class μ_2, with ν_2 apparent pinch-points. The primal F contains a triple curve C_t of order t, rank r_t, and classes t_1, t_0 of immersion in F and B respectively; at a point of this curve there are, in general, three tangent spaces to F, so that C_t is triple on B as well as on F. There is, however, a number τ of points at which two of the tangent spaces coincide; these stationary points are *actual* pinch-points of B, so that in the projection of B on ordinary space there are $\tau+\nu_2$ pinch-points. Further, there is on C_t a number ξ of quadruple points, at which there are four tangent spaces to F; such points are quadruple for F and sextuple for B, since B has six tangent planes there. B has an apparent double curve of order $2k$, cut out by the cone of chords passing through an arbitrary point P; this curve contains the points ξ and ν_2 mentioned above, and also the points in which B is met by the T trisecants which may be drawn from P; these project into the triple points on the projection of B.

There is a pinch-curve C_j of order j, at points of which there are two coincident tangent spaces to F; this curve lies on B and on every surface of contact of tangents drawn to F from an arbitrary point; it also passes through the τ stationary points, but has no other intersections with C_t.

The complete list of characters is as follows:

The primal F: order n, rank a, first class m, and class w.

The double surface: order μ_0, rank μ_1, class μ_2, and ν_2 apparent pinch-points; apparent double curve of order $2k$ and T apparent triple points.

The triple curve: order t, rank r_t, classes of immersion t_1 in F and t_0 in B; containing ξ quadruple points and τ stationary points.

The pinch-curve: order j, rank r_j, classes of immersion j_1 in F and j_0 in B. The curve is without singularities and is simple on B; it meets the triple curve at the points τ.

Notation. The first polar of an arbitrary point P meets F in the double surface B and the surface A of contact of tangents drawn to F from P.

* A similar example is provided by the surface in [3], which is the projection of the apparent contour of a primal in [4]. In general, the projection of a surface in [4] has a number of pinch-points on the double curve, but in this case it is seen that there can be no such points.

The majority of the formulae to be given are concerned with some particular type of such tangents, or secants to B, or secants to B which are elsewhere tangent to F. When there is a single infinity of such tangents or secants, we denote the curve of contact or secancy by C_x, where x is the order of the curve in question. The number of these lines which are elsewhere tangent to F is denoted by $[xa]$ and the number of those which are elsewhere secant to B is represented by $[xb]$. In all the results which may be obtained from an arbitrary space section of F we use the notation of Cayley's formulae, which we now proceed to give; the results are obtained by considering an arbitrary space section of the figure through P.

2. *The Cayley formulae.* From the intersection of F with the first polar of P we have

$$a+2\mu_0 = n(n-1) = a+2b. \tag{1}$$

Next consider the curve of intersection of A with the second polar of P; this consists of C_κ, the curve of contact of inflexional tangents passing through P and C_ρ, that of tangents through P which touch F in points of B. Thus

$$a(n-2) = \kappa+\rho. \tag{2}$$

Similarly, from the intersection of B with the second polar,

$$b(n-2) = \rho+3t. \tag{3}$$

Next consider the residual intersections with F of the tangents from P; the coincidences arise from the curve $C_{2\delta}$ of contact of bitangents to F and the curve $C_{[ab]}$ in which F is met by the tangents which are elsewhere secant to B. Hence

$$a(n-2)(n-3) = 2\delta+2[ab]. \tag{4}$$

Also, from the residual intersections with F of the secants to B we obtain

$$b(n-2)(n-3) = 4k+[ab]. \tag{5}$$

Since A and B meet in C_j and touch along C_ρ we deduce that

$$[ab] = ab-2\rho-j. \tag{6}$$

It remains to connect these results with m, which is the rank of A. For this purpose we note that the first polar of another point P', with surface of contact A', meets A in C_ρ and C_j, along which A and A' touch, and the curve C_m of contact of tangent planes to F passing through PP'. Hence

$$a(n-1) = \rho+2j+m. \tag{7}$$

By means of these formulae we may express the remaining characters of B in terms of n, a, j, and m, the additive characters*. The rank of B is then given by

$$\mu_1 = b(b-1)-2k-6t. \tag{8}$$

By means of (3), (5), (6), and (7) this reduces to

$$\mu_1 = an-a-\tfrac{5}{2}j-m. \tag{9}$$

Also from (3) we obtain

$$t = \binom{n}{3} - \tfrac{1}{2}an + \tfrac{1}{3}(2a+2j+m). \tag{10}$$

From (4), (6), and (7),

$$\delta = \tfrac{1}{2}a^2+a-3j-2m. \tag{11}$$

Finally, from (7) and (8), we have the relation

$$2(\rho-\mu_1) = j. \tag{11}'$$

For the next group of formulae it is necessary to specify the intersections of the curves already introduced with the singular curves of F.

The curve C_κ of contact of inflexional tangents to F contains the contacts of the μ_4 4-point tangents which may be drawn from P, and the inflexional contacts of the μ_{3-2} inflexional tangents which touch F elsewhere; it meets C_j in the points j_1 of contact of tangent spaces through P which touch F in points of C_j, and it meets C_ρ in the ρ_1 contacts of inflexional tangents in points of B.

The curve C_ρ of contact of tangents to F in points of B contains these points ρ_1, and also contains the τ stationary points and the ν_2 contacts of the tangents to B which pass through P. In addition it meets C_j in the points j_1 and C_t in the points t_1 of contact of tangent spaces through P which touch F in points of C_t.

The curve $C_{2\delta}$ of contact of bitangents to F contains the points μ_4, the simple and inflexional contacts of the tangents μ_{3-2}, and the contacts of the ${}_3\mu_2$ tritangents passing through P. It meets C_j in the points Q such that PQ touches F elsewhere, and meets C_ρ in the points ν_2 and the points Q' such that PQ' touches F elsewhere.

The cone of chords to B through P meets B in a curve C_{2k}; this contains the points of section of the T trisecants to B; it meets C_t in the points t_1 and ξ, and C_j in the $[jb]$ points Q such that PQ meets B elsewhere.

* See Roth, *Proc. Camb. Phil. Soc.*, 26 (1930), 43.

It intersects C_ρ in the points ν_2 and in the $[\rho b]$ points Q' such that PQ' meets B again elsewhere.

The actual multiplicities of these intersections do not at the moment concern us; they will be determined later by means of the equations.

3. *The coincidence formulae for F.* We now consider the coincidences of the correspondences* on the curves already obtained.

(i) The cone $P(C_j)$ has a residual intersection with F of order $j(n-2)$; the correspondence $(n-2, 1)$ between the two curves has for united points the τ stationary points and the j_1 points of contact of tangents to F at points of C_j, the latter counted twice; hence

$$j(n-2) = \tau + 2j_1. \tag{12}$$

The correspondence $(n-3, n-3)$ between points on the residual intersection has coincidences arising from the generators of the cone which touch F elsewhere and those which are elsewhere secant to B. Thus

$$j(n-2)(n-3) = [ja] + 2[jb]. \tag{13}$$

(ii) Similarly, dealing with the cone $P(C_t)$, we obtain

$$t(n-3) = t_1 + 4\xi, \tag{14}$$

$$t(n-3)(n-4) = [ta] + 2[tb]. \tag{15}$$

(iii) The residual intersection with F of the cone $P(C_\kappa)$ meets C_κ in the μ_4 points of contact of the flecnodal tangents, and the ρ_1 inflexional tangents in points of B; thus

$$\kappa(n-3) = \mu_4 + \rho_1. \tag{16}$$

The correspondence on the residual intersection has coincidences arising from the μ_{3-2} inflexional tangents which touch F elsewhere, and the inflexional tangents which meet B elsewhere; hence

$$\kappa(n-3)(n-4) = \mu_{3-2} + 2[\kappa b]. \tag{17}$$

(iv) The residual intersection of the cone $P(C_{2\delta})$ meets $C_{2\delta}$ in the inflexional contacts of the inflexional tangents which are elsewhere tangent to F, so that

$$2\delta(n-4) = \mu_{3-2} + [\rho a]. \tag{18}$$

The coincidences on the curve of residual intersection arise from the

* Not all of the equations to be given are independent; there will be various relations between them in virtue of the Cayley formulae.

${}_3\mu_2$ tritangents to F and the bitangents elsewhere secant to B; thus

$$\delta(n-4)(n-5) = {}_3\mu_2 + 2\,[\delta b]. \tag{19}$$

The involutory correspondence between the points of contact of the bitangents gives

$$2\delta = \mu_4 + 2\nu_2. \tag{20}$$

(v) The residual intersection of the cone $P(C_\rho)$ meets C_ρ in the points ρ_1, the points t_1, and the ν_2 contacts of the tangents to B; hence

$$\rho(n-3) = \rho_1 + 2t_1 + 2\nu_2. \tag{21}$$

The coincidences on the residual intersection give

$$\rho(n-3)(n-4) = [\rho a] + 2\,[\rho b]. \tag{22}$$

(vi) Consider the cone $P(C_k)$. The correspondence between the points of section with B has for united points the ν_2 contacts above-mentioned, the ξ quadruple points, and the t_1 contacts of the tangents to F in points of C_t. We thus have

$$2k = 2t_1 + 6\xi + \nu_2. \tag{23}$$

The coincidences on the residual section have united points due to the $[ka]$ chords which touch F elsewhere, and the T apparent triple points of B, so that

$$k(n-4)(n-5) = [ka] + 6T. \tag{24}$$

The correspondence between the curves of section and residual intersection gives

$$2k(n-4) = [\rho b] + 3\,[tb]. \tag{25}$$

(vii) Consider the cone $P(C_{[ab]})$ of secants to B which are elsewhere tangent to F, the curves of contact and secancy, C_1 and C_2, say, each being of order $[ab]$. The correspondence between C_1 and the curve C' of residual intersection gives the equation

$$[ab](n-4) = [\kappa b] + [\rho b]. \tag{26}$$

The correspondence between C_2 and C' gives

$$[ab](n-4) = [\rho a] + 3\,[ta]. \tag{27}$$

The correspondence between C_1 and C_2 gives

$$[ab] = \rho_1 + t_1. \tag{28}$$

And the correspondence on C' yields

$$[ab](n-4)(n-5) = 2\,[\delta b] + 4\,[ka]. \tag{29}$$

4. It remains to express the bracket symbols, which represent apparent intersections, in terms of the characters of F, wherever this is possible; this is accomplished by observing at which points the apparent intersections give place to actual intersections. Thus, to evaluate $[\kappa b]$, we note that C_κ must meet B at points common to A and B, and that these lie on C_ρ and C_j (§ 1). Hence we obtain

$$[\kappa b] = \kappa b - 3\rho_1 - 3j_1. \tag{30}$$

From a similar remark we deduce that

$$[\delta b] = \delta b - 2\,[\rho a] - 2\,[ja] - 4\nu_2. \tag{31}$$

From the intersections of C_{2k} with A we have

$$[ka] = ak - 2\,[\rho b] - [jb] - 2\nu_2 - 2t_1. \tag{32}$$

Since C_t must meet A in points of C_ρ or C_j,

$$[ta] = at - 2t_1 - \tau. \tag{33}$$

To find $[jb]$ we project B from P, so that $[jb]$ is the number of intersections of the projection of C_j with the double curve on the projection of B. We thus derive

$$[jb] = j(b-1) - 2\tau - j_0. \tag{34}$$

Similarly, to determine $[tb]$ we use the same projection. From the intersections of the projection of C_t with the third polar of an arbitrary point with regard to the projection of B, we have

$$[tb] = t(b-3) - 12\xi - t_0. \tag{35}$$

5. The relation between the classes t_0 and t_1 is found as follows. Consider the correspondence $(2t_1, 2t_1)$ set up on a line l by the tangent spaces to F at points of C_t; the united points are due to the t_0 tangent planes to B, at points of C_t, which meet l, and the coincident tangent spaces at the τ stationary points. Hence

$$4t_1 = 2t_0 + \tau. \tag{36}$$

To determine r_t we again consider the projection of B, B_0 say; the tangent planes to B_0 at points of its triple curve define a correspondence $(2t_0, 2t_0)$ on a given line l, and the united points are due to the τ stationary points and the r_t tangents to the triple curve which meet l. We thus derive the equation

$$2t_0 - 4r_t = \tau. \tag{37}$$

From (36) we therefore deduce that

$$2t_1 - 2r_t = \tau. \tag{38}$$

By means of (14) we deduce that

$$\tfrac{1}{2}\tau = t(n-3) - 4\xi - r_t,$$

a result analogous to that obtained by eliminating ρ between (3) and (11)′.

6. We may find the rank r_j of C_j very easily from the fact that it is an additive character of F and is, therefore, expressible as a linear function of the remaining fundamental additive characters n, a, m, w, j, j_0, and j_1; but, since it vanishes for any primal having no singularities, for which the last characters are zero, only these latter can appear in the expression for r_j. We therefore assume that

$$r_j = \lambda j + \mu j_0 + \nu j_1,$$

and determine the coefficients from three special cases. The plane-forms for which $n = 3$, $a = 4$ and $n = 4$, $a = 6$ are immediately available; for these we have $j = 2$, $j_0 = 0$, $j_1 = 1$, $r_j = 2$ and $j = 4$, $j_0 = 4$, $j_1 = 4$, $r_j = 6$ respectively*. For the third case we use the quartic primal with a double quadric, for which $j = 4$, $j_0 = 4$, $j_1 = 4$, $r_j = 8$. Hence we obtain

$$r_j = 4j_1 - j_0 - j. \tag{39}$$

7. We have now to connect the class μ_2 of B with the characters previously given, and for this purpose we use the result that the number of tangent planes to B which meet two given lines l, l' is $\mu_2 + \nu_2$. Consider then the tangent planes to B which meet l; their curve of contact C is of order μ_1 and meets C_t in t_0 points and C_j in j_0 points. Its class ζ of immersion in F is the number of its intersections with the second polar of an arbitrary point, excluding the t_0 points referred to; we thus obtain

$$\zeta = \mu_1(n-2) - t_0. \tag{40}$$

Next consider the correspondence (ζ, ζ) set up on l' by the pairs of tangent planes to B at points of C. The united points arise from the stationary points τ, the j_0 points above mentioned, and the $\mu_2 + \nu_2$ tangent planes meeting l and l'. Hence the class of B is given by

$$2\{\mu_1(n-2) - t_0\} = 2(\mu_2 + \nu_2) + 4\tau + j_0. \tag{41}$$

* See Roth, *Proc. London Math. Soc.* (2), 33 (1931), 123.

8. From this formula we can find the number T of apparent triple points of B in terms of its remaining characters. We substitute in (24) the value of $[ak]$ given in (32) and in this result eliminate $[jb]$ and j_0 by means of (34) and (41). We then substitute for $[\rho b]$ from (25) and for $[bt]$ from (35) and eliminate t_1 and t_0 by means of (37) and (38). The process is as follows. We have

$$\begin{aligned}
6T &= k(n-4)(n-5)-ak+2[\rho b]+[jb]+2\nu_2+2t_1 \\
&= k(2b-8n+20)+4k(n-4)-6\,[bt]+j(b-1)-2\tau-j_0+2\nu_2+\tau+2r_t \\
&= 2bk-4k(n-1)-6\,[bt]+2r_t+j(b-1)+3\tau+2(\mu_2+2\nu_2) \\
&\qquad -2\mu_1(n-2)+2t_0, \text{ by (41)}, \\
&= 2bk-6\,[bt]+2r_t+3\tau+2(\mu_2+2\nu_2)+4r_t+\tau \\
&\qquad -2n(2k+\mu_1)+4k+4\mu_1+j(b-1).
\end{aligned}$$

Now write

$$2k+\mu_1 = b(b-1)-6t,$$

$$\begin{aligned}
j(b-1) &= 2(\rho-\mu_1)(b-1) \quad (\S\,2) \\
&= 2(b-1)\{b(n-2)-3t-\mu_1\}.
\end{aligned}$$

On substituting and reducing we then obtain

$$T = \binom{b}{3} - \tfrac{1}{2}\mu_1 b + \tfrac{1}{3}(2\mu_1+2\nu_2+2\tau+\mu_2) - t(3b-8)+20\xi+5r_t+\tfrac{3}{2}\tau. \quad (42)$$

9. It remains to express the class w in terms of the other characters of F. Now w is the class of A and may be found by means of Cayley's equations* for the projection A_0 of A; this is of order a, rank m, with a double curve of order δ containing ${}_3\mu_2$ triple points and a cuspidal curve of order κ, meeting the double curve in μ_4 points which are stationary on the cuspidal curve and in μ_{3-2} points which are stationary on the double curve. For such a surface the class is given by

$$w = 3_3\mu_2 - 3\binom{a}{3} + \tfrac{3}{2}am - 2m + 8\mu_4 + \tfrac{9}{2}\mu_{3-2}.$$

We evaluate this in terms of n, a, j, m, ξ, j_1, and j_0. Or alternatively, supposing that w is given we may obtain an expression for ξ. The process of substitution is indicated below.

* See Salmon-Rogers, *Analytic geometry of three dimensions*, 2 (1915), 318.

From (19),

$$3_3\mu_2 = \delta(n-4)(n-5)-2[\delta b]$$
$$= \delta(n-4)(n-5)-2\{\delta b-2[\rho a]-[ja]-4\nu_2\} \quad \text{by (31).}$$

From (20), $\mu_4 = 2\delta-2\nu_2.$

From (18), $\mu_{3-2} = 2\delta(n-4)-[\rho a].$

Inserting these values we obtain, after simplifying,

$$w = \tfrac{3}{2}am-2m-3\binom{a}{3}+\delta(n^2-2b)-8\nu_2-\tfrac{1}{2}[\rho a]+2[ja].$$

Now, from (27) and (33),

$$[\rho a] = [ab](n-4)-3(at-2t_1-\tau).$$

From (13) and (34),

$$[ja] = j(n-2)(n-3)-2\{j(b-1)-2\tau-j_0\}.$$

Substituting for τ and t_1 from (12) and (14) respectively and putting

$$\nu_2 = 2k-6\xi-2t_1, \quad \text{by (23),}$$
$$= 2k-2t(n-3)+2\xi, \quad \text{by (14),}$$

we have w in the required form; finally, we express k, δ, $[ab]$, and t in terms of n, a, j, and m, as in § 2. The formula is then

$$\xi = \binom{n}{4}-\tfrac{1}{2}a\binom{n}{2}+\tfrac{1}{8}a^2+\tfrac{1}{3}n(2a+2j+m)-\tfrac{1}{4}(3a+5j+3m+w+13j_1)+j_0. \quad (43)$$

This is the four-dimensional analogue of (10).

By means of (14) and (23) we may now express ν_2 in terms of the same characters. We find, by using the above formula,

$$\nu_2 = \tfrac{1}{2}(a^2-m-w-13j_1)+2j_0. \quad (44)$$

Hence, by (41), we have, for the class μ_2,

$$\mu_2 = \tfrac{1}{2}a^2+mn+2j-\tfrac{1}{2}(3m+3w+33j_1-11j_0). \quad (45)$$

Similarly, the remaining characters may be expressed in terms of the seven characters n, a, j, m, w, j_1, j_0.

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RATE OF PUBLICATION OF PROCEEDINGS.

It has been decided by the Council of the London Mathematical Society that the number of parts of the *Proceedings* of the Society published annually shall be increased forthwith (beginning with Vol. 36) from about 11 to about 14.

It is proposed not to make any change, either in the Annual Subscription of Members of the Society, or in the price per volume to non-members; also it is proposed not to make any change in the number of pages per part or per volume of the *Proceedings*. Thus, in future, Members will receive about 2 volumes (instead of about $1\frac{1}{2}$ volumes) annually, in addition to the *Journal*, in return for their subscription of £2. 2*s*. 0*d*., while non-members will continue to be able to purchase the *Proceedings* at the price of £1. 15*s*. 0*d*. per volume of about 560 pages, but such payments will fall due more frequently.

The increasing interval between the receipt of papers intended for the *Proceedings* and their publication (due to the growth in the number of papers communicated to the Society which are accepted for publication) has for some time been a cause of concern to the Council. It is anticipated that the decision to publish parts of the *Proceedings* more frequently will, in due course, result in this interval being substantially reduced.

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